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DAVID C. HEATH

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SKOROKHOD STOPPING VIA POTENTIAL THEORY

by David Heath *

I. Introduction. We present here a potential-theoretic viewpoint of the construction of Skorokhod given in [2] for the proof of the following result:

If μ is a probability measure on $\mathbb{R}$ with $\int x^2 d\mu < \infty$ and $\int x d\mu = 0$ then there exists a (randomized) stopping time T for Brownian motion $(X_t, t \geq 0)$ starting at 0 such that the distribution of X_T is μ and $E(T) = \int x^2 d\mu$.

The construction of [2] consists essentially of finding a monotone collection $(I(s), s \in [0,1])$ of intervals in $\mathbb{R}$ such that the required stopping time may be defined as

$$T = \inf \{ t > 0 : X_t \notin I(S) \}$$

where S is a random variable independent of the Brownian motion, with distribution uniform on $[0,1]$. (Actually, in [2] the intervals are parameterized differently so that S has distribution μ .) Here is a proof which differs slightly from that of Skorokhod. To simplify the notation we restrict our attention (as did [2]) to the case in which μ has a continuous distribution function.

It is easy to see that under this hypothesis one can find a family of intervals $(I(s), s \in [0,1])$ of the form $I(s) = (x_1(s), x_2(s))$ for which

$$\begin{aligned} \text{a) } & \mu(I(s)) = s \\ \text{and b) } & \int_{I(s)} x d\mu = 0. \end{aligned}$$

Clearly x_1 and x_2 are strictly monotone functions, and they are essentially unique. Define T as above, and let ν be the distribution of X_T . We

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wish to show that $\nu = \mu$. It is obvious that conditions a) and b) remain true if μ is replaced by ν .

Let f be any bounded continuous function on $\mathbb{R}$ and let σ be any probability measure satisfying conditions a) and b). Define

$J_s = \int_{I(s)} f d\sigma$. Clearly J is absolutely continuous; moreover, except on the countable set $\{s: x_1 \text{ or } x_2 \text{ is not continuous at } s\}$, J is differentiable and one can easily check that

$$J'_s = \frac{-x_1(s)}{x_2(s)-x_1(s)} f(x_2(s)) + \frac{x_2(s)}{x_2(s)-x_1(s)} f(x_1(s)).$$

Since $J_0 = 0$ and $J_1 = \int f d\sigma$, we see that

$$\int f d\sigma = \int_0^1 J'_s ds = \int_0^1 P_{I'(s)} f(0) ds$$

where $I'(s)$ is the complement of $I(s)$ and $P_{I'(s)}$ is the corresponding hitting operator for Brownian motion. From this we conclude that $\int f d\sigma$ depends only upon the family $(I(s))$ and hence $\int f d\nu = \int f d\mu$, so that $\nu = \mu$. The equation for $E(T)$ is easily obtained.

II. Another Definition Of $(I(s))$. The proof given above relies on the geometry of $\mathbb{R}$ both for selecting the intervals and for identifying ν with μ . We seek now another characterization of these intervals. Suppose now that μ has support in some bounded interval E and let $U\mu$ be the potential of μ in E given by

$$U\mu(x) = -2 \int_0^x F(y) dy + h(x)$$

where F is the distribution function of μ and h is a harmonic (i.e. linear) function chosen to make $U\mu$ vanish at the endpoints of E . Fix $s \in [0,1]$ and let $x_1 = x_1(s)$ and $x_2 = x_2(s)$. Define

$$t_i(x) = U\mu(x_i) + (x-x_i) (U\mu)'(x_i) \quad \text{for } i=1,2.$$

It is easy to check that because of the choice of $I(s)$, $t_1(0) = t_2(0)$.

Further, one can check that the function g defined by

$$g(x) = \begin{cases} U\mu(x) & \text{if } x \notin (x_1, x_2) \\ t_1(x) & \text{if } x \in (x_1, 0) \\ t_2(x) & \text{if } x \in [0, x_2) \end{cases}$$

is the potential of a probability measure with mass s at 0, mean 0, and agreeing with μ on $(-\infty, x_1) \cup (x_2, \infty)$. Notice that we have then

$$g = R(U\mu - sU\epsilon_0) + sU\epsilon_0,$$

where ϵ_0 is the probability measure assigning mass one to $\{0\}$ and Rf is the infimum of all supermedian functions dominating f . Also,

$$I(s) = \{x: R(U\mu - sU\epsilon_0) > U\mu - sU\epsilon_0\}.$$

The purpose of the next two sections is to show that the above structure exists in at least slightly more generality.

III. A Theorem Of Mokobodzki. The result of this section is due to G. Mokobodzki (private communication) and is more general than we shall use. We suppose given a (sub-) Markovian semigroup; 'excessive' and 'supermedian' are with respect to this semigroup. As above, if f is any function we let Rf be the infimum of all supermedian functions dominating f and $R_A f = R(fI_A)$.

LEMMA. Suppose Rf is everywhere finite and $A = \{f > (1-\epsilon)Rf\}$ for some ϵ in $(0,1)$. Then $R_A Rf = Rf$.

PROOF. Set $g = R_A Rf$. Both on A and off A it is clear that $f \leq (1-\epsilon)Rf + \epsilon g$; hence this supermedian function dominates Rf . Since $g \leq Rf$, it is also dominated by Rf ; hence $(1-\epsilon)Rf + \epsilon g = Rf$, so $g = Rf$. $\square$

Now let a and b be excessive functions with a everywhere finite. For $t \in [0,1]$, let $\bar{t} = 1-t$ and define:

$$v_t = a - \bar{t}b, \quad V_t = Rv_t, \\ A_t^\varepsilon = \{ v_t > V_t - \varepsilon \bar{t}a \}, \quad h_t^\varepsilon = R_{A_t^\varepsilon} b.$$

Clearly $V_t \leq a$, so $A_t^\varepsilon \supseteq \{ v_t > (1 - \varepsilon \bar{t}) V_t \}$; thus according to the lemma $R_{A_t^\varepsilon} V_t = V_t$. Moreover, for $\lambda \varepsilon [0, 1]$ we have $v_{\lambda s + \bar{\lambda} t} = \lambda v_s + \bar{\lambda} v_t$; the subadditivity of R then shows that $V_\cdot$ is convex.

Fix x and consider the graphs of $V_t(x)$ (convex) and $v_t(x) + \varepsilon \bar{t}a(x)$ (linear). These functions are equal at $t=1$; hence $\{ t: V_t(x) < v_t(x) + \varepsilon \bar{t}a(x) \}$ is of the form $[C, 1]$. This means that A_t^ε increases with t . As $\varepsilon \searrow 0$, $A_t^\varepsilon \searrow$, so $h_t^\varepsilon \searrow$; call its limit h_t . We then have the following:

THEOREM. $a - R(a-b) = \int_0^1 h_t dt$.

PROOF. Since $v_t = v_s + (t-s)b$, the subadditivity of R gives (for $s < t$) $V_t \leq V_s + (t-s)b$; apply $R_{A_t^\varepsilon}$ to get $V_t \leq V_s + (t-s) h_t^\varepsilon$.

In the other direction, on A_s^ε we have $v_s > V_s - \varepsilon \bar{t}a$, so that on A_s^ε , $V_t \geq v_t = v_s + (t-s)b > V_s - \varepsilon \bar{t}a + (t-s)b$. Thus $(V_t + \varepsilon \bar{t}a) I_{A_s^\varepsilon} \geq V_s I_{A_s^\varepsilon} + (t-s) I_{A_s^\varepsilon} b$. Since $R_{A_s^\varepsilon}$ is additive on supermedian functions, we obtain

$$V_t + \varepsilon \bar{t}a \geq R_{A_s^\varepsilon}(V_t + \varepsilon \bar{t}a) \geq R_{A_s^\varepsilon} V_s + (t-s) R_{A_s^\varepsilon} b = V_s + (t-s) h_s^\varepsilon.$$

Combining these, we obtain

$$h_s^\varepsilon - \frac{\varepsilon \bar{t}a}{(t-s)} \leq \frac{V_t - V_s}{(t-s)} \leq h_t^\varepsilon.$$

After letting $\varepsilon \searrow 0$, we see that the right derivative of $V_\cdot$ lies between $h_\cdot$ and $h_{\cdot+}$. Therefore $V_1 - V_0 = \int_0^1 h_t dt$ as desired. $\square$

IV. Skorokhod Stopping In $\mathbb{R}^N$. Let E be a bounded ball about 0 in $\mathbb{R}^1$ or $\mathbb{R}^2$ or $E = \mathbb{R}^N$ for $N \geq 2$. Let $(X_t, t \geq 0)$ be standard Brownian motion in $\mathbb{R}^N$ starting at 0 and killed when it leaves E , and let u and U be the associated potential kernel, as in Blumenthal and Gettoor [1], p. 253. Let ε_0 be as before.

THEOREM. Let μ be a probability measure with support in E satisfying

$U\mu \leq U\epsilon_0$ and $U\mu(0) < \infty$. There is then a monotone collection
 $(A(s), s \in [0, 1])$ of subsets of E such that if S is independent of $(X_t, t \geq 0)$
with distribution uniform on $[0, 1]$ and $T = \inf \{t > 0: X_t \in A(S)\}$, then
 X_T has distribution μ .

PROOF. According to exercise (1.27) of Chapter VI of [1], $(X_t, t \geq 0)$ is in duality with itself relative to Lebesgue measure restricted to E . Set $a = U\mu$ and $b = U\epsilon_0$; because of the regularity of Brownian motion we can set $\epsilon=0$ in the proof of the previous theorem. Letting $A(s) = A_s^{0+}$ we obtain $a = \int_0^1 R_{A(s)} b \, ds$. It follows from Theorem (6.12) of Chapter II of [1] that $R_{A(t)} b = P_{A(t)} b$ almost everywhere (Lebesgue measure) for each $t \in [0, 1]$. Using Fubini's theorem we can then conclude that $\int_0^1 R_{A(s)} b \, ds = \int_0^1 P_{A(s)} b \, ds$ almost everywhere. Since $P_{A(s)} b$ is a monotone function of s , its integral can be expressed as the limit of an increasing sequence of excessive functions and is therefore excessive. Since this integral is almost everywhere equal to a , an excessive function, by Proposition (1.3) of Chapter VI of [1], they must be equal, i.e.,

$$a = \int_0^1 P_{A(s)} b \, ds.$$

Suppose now that T is defined as in the statement of the theorem, and let ν be the distribution of X_T . Clearly $\nu(\cdot) = \int_0^1 P_{A(s)}(0, \cdot) \, ds$. Now $U\nu(x) = \int_0^1 \int_E u(x, z) P_{A(s)}(0, dz) \, ds = \int_0^1 P_{A(s)} u(x, 0) \, ds = \int_0^1 P_{A(s)} b(x) \, ds = a(x)$, where the second equality follows from Theorem (1.16) of Chapter VI of [1]. Thus μ and ν have the same potential; by Proposition (1.15) of Chapter VI of [1], they must be equal. $\square$

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