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## **On the existence of resolvents**

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ON THE EXISTENCE OF RESOLVENTS <sup>1)</sup>

J.C. Taylor

Introduction. In [6] a measure-theoretic theorem is proved which gives a sufficient condition for a proper kernel  $V$  on a measure space  $(E, \underline{E})$  to be the kernel  $V_0$  defined by a sub-Markovian resolvent.

The theorem of Lion is an immediate corollary of this result. It states that, for  $E$  locally compact and  $\sigma$ -compact, if  $V$  satisfies the complete maximum principle and maps  $\underline{C}_c(E)$  into  $\underline{C}_0(E)$  then there is a sub-Markovian resolvent  $(V_\lambda)$  on  $\underline{C}_0(E)$  with  $V = V_0$  on  $\underline{C}_c(E)$ .

F. Hirsch in [2], by using different methods from those of Lion in [3], has shown that the theorem of Lion is valid for arbitrary locally compact spaces. The purpose of this note is to obtain a generalisation of theorem 2 in [6] which has as corollary 3.1 the above result of Hirsch. In addition, it has as corollary 3.3 the result of Mokobodzki and Sibony in [5] (without the restriction, imposed in [6], that  $E$  be  $\sigma$ -compact) and as corollary 3.2 an extension of Hirsch's result to kernels satisfying the domination principle.

The generalised theorem. Let  $E$  be a set and let  $\underline{E}$  be a  $\sigma$ -ring of subsets of  $E$  (i.e.  $\underline{E}$  is closed under countable unions and relative complements). A function  $f : E \rightarrow \bar{\mathbb{R}}^+$  is measurable if for all  $\alpha > 0$ ,  $\{f > \alpha\} \in \underline{E}$ . It will be said to be locally measurable if for each  $X \in \underline{E}$ ,  $f|X$  is measurable in the usual

<sup>1)</sup> This work was done while the author was a visiting professor at the Mathematisches Institut der Universität Erlangen-Nürnberg.

sense with respect to the  $\sigma$ -field of sets of the form  $A \cap X$ ,  $A \in \underline{E}$ . A kernel on  $(E, \underline{E})$  is an additive, increasing, positively homogeneous map  $V : \underline{E}^+ \rightarrow \underline{E}^+$  such that if  $(f_n) \subset \underline{E}^+$  is decreasing to  $f$  then  $(Vf_n) \xrightarrow{in}$  decreases to  $Vf$  (here  $\underline{E}^+$  denotes the convex cone of non-negative measurable functions).

Let  $E$  be locally compact. In [1] the  $\sigma$ -ring  $\underline{B}_0$  of Baire sets is defined to be the smallest  $\sigma$ -ring containing  $\underline{H}$ , the class of compact  $G_\delta$ -subsets of  $E$ .

It is shown in [1] that, every  $A \in \underline{B}_0$  is a subset of a countable union of sets from  $\underline{H}$ .

Remark. It is not hard to see that if  $K_1, K_2 \in \underline{H}$  then  $K_1 \setminus K_2$  is a countable union of sets from  $\underline{H}$  i.e. is in  $\underline{H}_\sigma$ . Consequently, the class of  $\underline{H}$ -Borelian sets  $\underline{B}(\underline{H})$  coincides with  $\underline{B}_0(\underline{B}(\underline{H}))$  is the smallest class containing  $\underline{H}$  and closed under countable unions and countable intersections).

A convex cone  $\underline{C} \subset \underline{C}^+(E)$  is said to be adapted if for each  $f \in \underline{C}$  there exists  $g \in \underline{C}$  with  $f \in o(g)$  i.e. for all  $\epsilon > 0$  there exists a compact  $K = K(\epsilon)$  with  $f(x) \leq \epsilon g(x)$  if  $x \notin K$ .

Proposition 1. Let  $E$  be locally compact and denote by  $M$  a positive linear map of  $\underline{C}_0(E)$  into  $\underline{C}(E)$  with  $M(\underline{C}_0^+(E)) \subset \underline{C}$ , where  $\underline{C} = \underline{C}_0^+(E)$  or is an adapted cone. Then there is a unique kernel  $N$  on  $(E, \underline{B}_0)$  such that  $N$  agrees with  $M$  on  $\underline{C}_0(E)$ .

The kernel  $N$  satisfies the principle of domination (respectively, the complete maximum principle) if for all  $\varphi, \psi \in \underline{C}^+(E)$ ,

$M \varphi \geq M \psi$  on  $\{\psi > 0\}$  implies  $M \varphi \geq M \psi$  and for all  $x \in E$  there exists  $\varphi \in C_c(E)$  with  $M \varphi(x) \neq 0$  (respectively,  $1 + M \varphi \geq M \psi$  on  $\{\psi > 0\}$  implies  $1 + M \varphi \geq M \psi$ ).

**Proof:** If  $\underline{C}$  is an adapted cone then each  $f \in \underline{C}$  vanishes outside of a countable union of compact sets. Hence,  $f = \sum_n \varphi_n$ ,  $(\varphi_n) \subset \underline{C}^+(E)$ . Consequently,  $M \varphi \in \underline{B}_0^+$  if  $\varphi \in \underline{C}^+(E)$ .

Define the Radon measure  $\mu_x$  by setting  $\langle \mu_x, \varphi \rangle = M \varphi(x)$ ,  $\varphi \in \underline{C}(E)$ . If  $f \in \underline{B}_0^+$  define  $N(x, f) = \langle \mu_x, f \rangle$ .

Let  $K \in \underline{H}$  and let  $\underline{F}$  be the  $\sigma$ -field of sets of the form  $A \cap K$ ,  $A \in \underline{B}_0$ . Denote by  $\underline{M}$  the vector space of differences of non-negative bounded Baire measurable functions. Then  $\underline{F}_b$  can be naturally identified with a subspace of  $\underline{M}$ .

Consider  $\underline{K} = \{f \in \underline{F}_b \mid Nf \in \underline{M}\}$ . This a subspace of  $\underline{F}_b$  closed under monotone limits and containing 1 (since  $K \in \underline{H}$ ). Assume  $(f_n) \subset \underline{K}$  and that  $f_n$  tends to  $f$  uniformly. There is a set  $X \in \underline{B}_0$  such that, for all  $n$ ,  $\{|Nf_n| > 0\} \subset X$  and the functions  $Nf_n$  are all measurable in the usual sense, when viewed on  $X$ , with respect to the  $\sigma$ -field on  $X$  induced by  $\underline{B}_0$ . Hence,  $Nf_n$  (which equals  $\lim_{n \rightarrow \infty} Nf_n$ ) is in  $\underline{M}$  since when viewed on  $X$  it is measurable. It follows from IT20 in [4] that  $\underline{K} = \underline{F}_b$ . Furthermore, if  $f \in \underline{F}_b^+$  then  $Nf \in \underline{B}_0^+$ . This follows since  $Nf \geq 0$  is locally measurable and  $\{Nf > 0\} \in \underline{B}_0^+$ .

If  $f \in \underline{B}_0^+$  then  $\{f > 0\} \subset \bigcup_n K_n$ ,  $(K_n) \subset \underline{H}$ , increasing. Hence,  $Nf = \lim_{n \rightarrow \infty} N(f \cdot 1_{K_n})$  and so is in  $\underline{B}_0^+$ .

The last two statements follow by the argument used to prove XT4 in [4].

Let  $N$  be a kernel on  $(E, \underline{E})$ . A set  $A \in \underline{E}$  will be said to be bounded if  $N1_A$  is finite and will be said to be  $\sigma$ -bounded if it is a countable union of bounded sets.

A locally measurable function  $u$  will be said to be supermedian if, for all  $f$  and  $g \in \underline{E}^+$ ,

$$u + Nf \geq Ng \text{ on } \{g > 0\} \text{ implies } u + Nf \geq Ng.$$

A supermedian function  $u$  is said to vanish at the boundary if there exists an increasing sequence  $(A_n)$  of bounded sets with  $\inf_n R_{A_n}^s u = 0$  (note: it is no longer required as in [6] that  $\bigcup_n A_n = E$ ).

Proposition 2. The following conditions are equivalent:

- (1) every finite potential  $Nf$ ,  $f \in \underline{E}^+$ , vanishes at the boundary; and
- (2) the potential  $N1_A$  of every bounded set  $A$  vanishes at the boundary.

Proof: Let  $f \in \underline{E}^+$  have a finite potential  $Nf$  and let  $\{f > 0\} = \bigcup_n B_n$ , with each  $B_n$  bounded and  $(B_n)$  increasing. It can be assumed further that  $f$  is bounded on each set  $B_n$ .

Let  $N1_{B_n}$  vanish at the boundary relative to  $(A_m^n)$ . Then, if  $A_p = \bigcup_{n=1}^p \bigcup_{m=1}^p A_m^n$ , each  $N1_{B_n}$  vanishes at the boundary relative to  $(A_p)$ .

Hence, each  $N(f 1_{B_n})$  vanishes at the boundary relative to  $(A_p)$  and so, by the lemma preceding proposition 1 in [6],  $Nf$  vanishes at the boundary relative to  $(A_p)$ .

Theorem 3. Let  $N$  be a kernel on  $(E, \underline{E})$  that satisfies the domination principle and is such that the following conditions are satisfied:

- (1) every set  $A \in \underline{E}$  is  $\sigma$ -bounded;
- (2) every finite potential  $Nf$ ,  $f \in \underline{E}^+$ , vanishes at the boundary ; and
- (3) if  $A \in \underline{E}$  there exists a finite supermedian function  $u$  which is strictly positive on  $A$ .

Then there exists a unique resolvent  $(N_\lambda)_{\lambda > 0}$  of kernels  $N_\lambda$  on  $(E, \underline{E})$  with  $N_0 = N$ . Further, this resolvent is sub-Markovian if  $N$  satisfies the complete maximum principle.

Proof: If  $g \in \underline{E}^+$  there exists a set  $X \in \underline{E}$  with the following properties:

- (1)  $\{g > 0\} \subset X$ ;
- (2)  $X = \bigcup_n A_n$ , each  $A_n$  bounded; and
- (3) for each  $n$ ,  $N1_{A_n}$  vanishes on  $\bar{C}X$  and vanishes at the boundary relative to  $(A_n)$ . First note that if  $(B_n)$  is any sequence of bounded sets then there exist sequences  $(B'_n)$  and  $(B''_n)$  of bounded sets such that (a) each  $N1_{B'_n}$  vanishes at the boundary relative to  $(B'_n)$  (see the proof of proposition 2) and (b) for each  $n$ ,  $\{N1_{B'_n} > 0\} \subset \bigcup_m B''_m$ . Consequently, there exists a sequence  $(C_n)$  of bounded sets such that, for all  $n$ ,  $N1_{C_n}$  vanishes on  $(\bigcup_n C_n)$  and vanishes at the boundary relative to some subsequence (depending on  $n$ ) of  $(C_n)$ . Further,  $(C_n)$  can be assumed to contain  $\{g > 0\}$  in its union.

Let  $X = \bigcup_n C_n$  and  $A_n = \bigcup_{i=1}^n C_i$ .

Let  $\underline{X}$  be the  $\sigma$ -field of sets of the form  $A \cap X, A \in \underline{E}$ .  
 If  $f \in \underline{E}^+$  and  $\{f > 0\} \subset X$  then (3) implies  $\{Nf > 0\} \subset X$ . Hence,  
 $N$  induces a proper kernel  $R$  on  $(X, \underline{X})$  which satisfies the  
 domination principle.

Let  $u_0$  be a finite supermedian function on  $E$  which  
 is strictly positive on  $X$ . Define  $Vf = (1/u_0) R(fu_0)$ ,  
 $f \in \underline{X}^+$ .

Then the kernel  $V$  satisfies the hypotheses of  
 theorem 2 in [6] since  $u$  supermedian implies  $u/u_0$  restricted  
 to  $X$  is supermedian relative to  $V$ . Let  $(V_\lambda)_{\lambda > 0}$  be the  
 sub-Markovian resolvent on  $(X, \underline{X})$  with  $V = V_0$ . Then, if  
 $R_\lambda h = u_0 V_\lambda (h/u_0)$ ,  $h \in \underline{X}^+$ ,  $(R_\lambda)_{\lambda > 0}$  is the resolvent of  
 kernels on  $(X, \underline{X})$  with  $R_0 = R$ . Hence,  $Ng = (I + \lambda N) R_\lambda g$ .

Denote by  $X'$  another set in  $E$  satisfying (1), (2)  
 and (3) and let  $R'$  be the kernel induced on  $(X', \underline{X}')$  by  $N$ .  
 Let  $(R'_\lambda)_{\lambda > 0}$  be the resolvent on  $(X', \underline{X}')$  with  $R'_0 = R'$ .  
 Since  $Ng = (I + \lambda N) R'_\lambda g$  it then follows from XI 7 in [4]  
 that, for all  $\lambda > 0$ ,  $R_\lambda g = R'_\lambda g$ .

Define  $N_\lambda g$  to be  $R_\lambda g$ , where  $(R_\lambda)_{\lambda > 0}$  is the resolvent  
 defined by a set  $X \in \underline{E}$  satisfying (1), (2) and (3) and the  
 kernel  $R$  induced by  $N$ .

It follows that (i) each  $N_\lambda$  is a kernel, (ii)  $(N_\lambda)_{\lambda > 0}$   
 is a resolvent family and (iii)  $N = N_0$ .

Application to locally compact spaces. Let  $E$  be a locally  
 compact and denote by  $\underline{B}_0$  the  $\sigma$ -ring of Baire subsets of  $E$ .

**Corollary 3.1.** (F. Hirsch [2]). Let  $V$  be a positive linear map of  $\underline{C}_c(E)$  into  $\underline{C}_0(E)$  such that for all  $\varphi, \psi \in \underline{C}_c^+(E)$

$$1 + V \varphi \geq V \psi \text{ on } \{\psi > 0\} \text{ implies } 1 + V \varphi \geq V \psi.$$

Then there is a resolvent family  $(V_\lambda)_{\lambda > 0}$  of sub-Markovian operators  $V_\lambda$  on  $\underline{C}_0(E)$  with  $V \varphi = \lim_{\lambda \rightarrow 0} V_\lambda \varphi$ , for all  $\varphi \in \underline{C}_c(E)$ .

**Proof:** From proposition 1 and theorem 3 it follows that there is a sub-Markovian resolvent  $(V_\lambda)_{\lambda > 0}$  of sub-Markovian kernels  $V_\lambda$  on  $(E, \underline{B}_0)$  with  $V_0 = V$ . Note that by the lemma preceding proposition 1 in [6] every finite potential  $Vf$  vanishes at the boundary because  $\{f > 0\} \subset \bigcup_n K_n$ ,  $(K_n) \subset \underline{H}$  and each  $V 1_{K_n}$  clearly vanishes at the boundary.

Let  $\varphi \in \underline{C}_0^+(E)$ . Then there exists an open set  $X \in \underline{B}_0$  such that (1)  $\{\varphi > 0\} \subset X$ ; (2)  $X = \bigcup_n K_n$ ,  $(K_n) \subset \underline{H}$ ; and (3) for each  $n$ ,  $V 1_{K_n}$  vanishes on  $X$  and vanishes at the boundary relative to  $(K_n)$ . It suffices to note that the sets  $B_n$ ,  $B_n'$  and  $B_n''$  in the proof of the theorem can all be assumed to be open and relatively compact under the hypotheses of this corollary.

Consequently, there exists a  $a \in \underline{C}_c^+(E)$  with (1)  $X = \{a > 0\}$  and (2)  $Va$  bounded. The argument given in the remark following corollary 2.4 in [6] then implies  $V_\lambda \varphi$  is continuous for all  $\lambda > 0$ . Hence, each  $V_\lambda$  leaves  $\underline{C}_0(E)$  invariant.

**Corollary 3.2.** Let  $M$  be a positive linear map of  $\underline{C}_c(E)$  into  $\underline{C}_0(E)$  such that, for all  $\varphi, \psi \in \underline{C}_c^+(E)$

$$M \varphi \geq M \psi \text{ on } \{\psi > 0\} \text{ implies } M \varphi \geq M \psi.$$



Assume  $M$  is non-degenerate, i.e. for all  $x \in E$  there exists  $\varphi \in \underline{C}_c(E)$  with  $M \varphi(x) \neq 0$  and that there is a supermedian function  $u$  with  $\overline{\{u < 1\}}$  compact. Then there is resolvent family  $(M_\lambda)_{\lambda > 0}$  of unbounded operators  $M_\lambda$  on  $\underline{C}_0(E)$  such that  $M \varphi = \lim_{\lambda \rightarrow 0} M_\lambda \varphi$ , for all  $\varphi \in \underline{C}_c(E)$ .

**Proof:** Let  $N$  be the kernel on  $(E, \underline{B}_0)$  determined by  $M$ . It satisfies the domination principle since  $M$  is non-degenerate. If  $X \in \underline{B}_0$  then there exists  $(\varphi_n) \subset \underline{C}_c^+(E)$  with  $u = \sum_n M(\varphi_n) \in \underline{C}^+(E)$  and  $\{u > 0\} \supset X$ . The argument of proposition 3 in [6] shows that  $u$  is a supermedian function.

Because there exists a supermedian function  $u$  with  $\overline{\{u < 1\}}$  compact, every finite potential  $Nf$  vanishes at the boundary and so there is a resolvent  $(N_\lambda)_{\lambda > 0}$  of kernels  $N_\lambda$  on  $(E, \underline{B}_0)$  with  $N_0 = N$ .

Because each  $X \in \underline{E}$  is contained in  $\{u > 0\}$ , for some  $u$  continuous and supermedian, it follows that, if  $\varphi \in \underline{C}_c^+(E)$  and  $X \in \underline{E}$  is open and satisfies the conditions in the proof of corollary 3.1, then the kernel  $V$  on  $(X, \underline{X})$  induced by  $N$  and  $u$  maps  $\underline{C}_c(X)$  into  $\underline{C}_0(X)$ . Hence  $R_\lambda \varphi \in \underline{C}_0(X)$  and so  $N_\lambda$  leaves  $\underline{C}_0(E)$  invariant. Define  $M_\lambda \varphi = N_\lambda \varphi$ , if  $\varphi \in \underline{C}_0(E)$ .

#### Application to adapted cones

Let  $E$  be locally compact and denote by  $\underline{C} \subset \underline{C}^+(E)$  an adapted cone. Let  $M$  be a positive linear map of  $\underline{C}_c(E)$  into  $\underline{C}(E)$  with  $M(\underline{C}_c^+(E)) \subset \underline{C}$ . Assume the following conditions satisfied:

- $A_2)$  for each  $x \in E$  there exists  $u \in \underline{C}$ ,  $u(x) > 0$ ; and  
 $A_4)$  if  $u \in \underline{C}$  and  $\varphi \in \underline{C}_c^+(E)$  then

$$u \geq M \varphi \text{ whenever } u \geq M \varphi \text{ on } \{\varphi > 0\}.$$

Corollary 3.2 (Mokobodzki-Sibony [5]). Let  $N$  be the kernel on  $(E, \underline{B}_0)$  determined by  $M$ . Then there is a resolvent  $(N_\lambda)_{\lambda > 0}$  of kernels on  $(E, \underline{B}_0)$  with  $N_0 = N$ .

Proof: Let  $\underline{C}_\sigma$  denote the set of continuous functions  $u$  on  $E$  of the form  $u = \sum_n u_n$ ,  $(u_n) \subset \underline{C}$ . Each  $u \in \underline{C}_\sigma$  is supermedian (see proposition 3 in [6]) and for each  $X \in B_0$  condition  $A_2$ ) implies that there exists  $u \in \underline{C}_\sigma$  with  $\{u > 0\} \supset X$ .

If  $\varphi \in \underline{C}_c^+(E)$  the fact that  $\underline{C}$  is adapted and contains only supermedian functions implies  $N \varphi$  vanishes at the boundary. Hence, as in the proof of corollary 3.1, each finite potential  $Nf$  vanishes at the boundary. The result then follows immediately.

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