MURALI RAO

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DOOB DECOMPOSITION AND BURKHOLDER INEQUALITIES

Murali Rao

Let $X_0, \ldots, X_N$ be a martingale relative to $\sigma$-fields $F_0, \ldots, F_N$. Let $X_i = X_0 + \sum_{j=1}^{i-1} X_j - X_{j-1}$ for $i=1, \ldots, N$ so that $X_N = \frac{n}{\sigma} x_i$.

Let $|\mathcal{F}_i| \leq 1$, $0 \leq i \leq N$ and $\mathcal{F}_i$-measurable for $i=0, \ldots, N-1$.

Put $g_n = \sum_{i=0}^{n} \mathcal{F}_i x_i$, $n=0, \ldots, N$, $g_N^* = \max_{n} |g_n|$ and $S_N = S_N(X) = (\sum_{i=1}^{N} x_i)^{1/2}$.

In [1] Burkholder proved the following remarkable inequalities:

For $a > 0$,

\begin{align*}
(1) & \quad \mathbb{P}[\max_{n} |g_n| > a] \leq 52 \mathbb{E}[|X_N|] \\
(2) & \quad \mathbb{P}[\max_{n} S_N > a] \leq 52 \mathbb{E}[|X_N|].
\end{align*}

In [2] Gundy, making use of his decomposition for $L'$-bounded martingales obtains inequalities for "class B mappings" which include (1) and (2) above. In this note we exploit Doob decomposition to give completely elementary proofs of (1) and (2) thus answering a question raised by Luis Baez-Duarte [3]. Let us add in passing that our method also gives inequalities for class B mappings of Gundy. For terminology not defined here we refer to [4].

For random variables $f_0, \ldots, f_N$, $f_N^*$ will denote $\max_{0 \leq i \leq N} |f_i|$.

We shall show that if the martingale $X_0, \ldots, X_N$ is non-negative (1) and (2) can be replaced by

\begin{align*}
(3) & \quad \mathbb{P}[\max_{n} |g_n| > a] \leq 13 \mathbb{E}[|X_0|] \\
(4) & \quad \mathbb{P}[\max_{n} S_N > a] \leq 13 \mathbb{E}[|X_0|].
\end{align*}

* Prof. Neveu pointed out to P.A.Meyer that he has given in his Cours de 3e Cycle on martingale theory, Paris 1969/70, a proof of the Burkholder maximal lemma which is very closely related to that of Prof. Rao.
Lemma. Let $Z_0, \ldots, Z_N$ be a square integrable super martingale and $Z_i = M_i - A_i$ be its Doob-decomposition. Then

$$E[M_N^2] \leq E[Z_N^2] + 2 \sum_{i=0}^{N-1} E[Z_i (A_{i+1} - A_i)].$$

Proof. Noting $E[(Z_{i+1}-Z_i) | F_i] = A_{i+1} - A_i$,

$$E[M_{i+1}^2 - M_i^2] = E[(M_{i+1} - M_i)^2]$$

$$= E[(Z_{i+1} - Z_i)^2 + 2 (A_{i+1} - A_i)(Z_{i+1} - Z_i) + (A_{i+1} - A_i)^2]$$

$$= E[(Z_{i+1} - Z_i)^2] - E[(A_{i+1} - A_i)^2]$$

$$\leq E |(Z_{i+1} - Z_i)^2|$$

$$= E[Z_{i+1}^2 - Z_i^2] + 2 E[Z_i (Z_{i+1} - Z_i)]$$

$$= E[Z_{i+1}^2 - Z_i^2] + 2 E[Z_i (A_{i+1} - A_i)].$$

And since $M_0 = Z_0$ we get

$$E[M_N^2] = E[M_0^2] + \sum_{i=0}^{N-1} E[M_{i+1}^2 - M_i^2]$$

$$\leq E[Z_0^2] + \sum_{i=0}^{N-1} E[Z_{i+1}^2 - Z_i^2] + 2 \sum_{i=0}^{N-1} E[Z_i (A_{i+1} - A_i)]$$

$$= E[Z_N^2] + 2 \sum_{i=0}^{N-1} E[Z_i (A_{i+1} - A_i)].$$

That proves the Lemma.

If $Z_1 \geq 0$, $E[A_N] \leq E[M_N] = E[Z_0]$ and we have

Corollary. If $0 \leq Z_1 \leq a$ is a super martingale and $Z_1 = M_1 - A_1$ its Doob decomposition then

$$E[M_N^2] \leq 3a E[Z_0].$$
Now let us prove (3). Let $a > 0$ and $Z_1 = X_1 \wedge a$. Let $Z_1 = M_1 - A_1$ be its Doob decomposition and

$$\begin{align*}
U_n &= Z_0 V_0 + \sum_{1}^{n} (Z_{i-1} - Z_{i-1}) V_i \\
V_n &= V_0 M_0 + \sum_{1}^{n} (M_{i-1} - M_{i-1}) V_i.
\end{align*}$$

By martingale inequality $\mathbb{P}(X_N > a) \leq \mathbb{E}(X_0)$. On the set $(X_N \leq a)$, $g_n = U_n$ for all $n$. Thus

$$\mathbb{P}(g_n > a) \leq \mathbb{P}[g_n > a] + \mathbb{P}[g_n > a, X_N \leq a]$$

$$\leq \mathbb{E}[X_0] + \mathbb{P}[U_N > a].$$

Clearly $|U_n| \leq |V_n| + A_n$ (note that $|V_i| \leq 1$ and $A_n \geq 0$) and $|V_n| + A_n$ is a submartingale. Submartingale inequality gives

$$\mathbb{P}[U_N > a] \leq \mathbb{P}[(|V| + A)_N > a]$$

$$\leq \frac{1}{a^2} \mathbb{E}[(|V_N| + A_N)^2]$$

$$\leq \frac{2}{a^2} \mathbb{E}[V_N^2 + A_N^2]$$

$$\leq \frac{4}{a^2} \mathbb{E}[M_N^2],$$

since $\mathbb{E}[V_N^2] \leq \mathbb{E}[M_N^2]$ and $A_N \leq M_N$. Using (6) and that $Z_0 \leq X_0$

$$\mathbb{P}[U_N > a] \leq \frac{12}{a} \mathbb{E}[X_0].$$

Together with (7) this gives (3).

As another example of application of the Lemma let us derive (4).

Put again $Z_1 = X_1 \wedge a$, and let $Z_1 = M_1 - A_1$ be its Doob decomposition.

$$\mathbb{P}[S_N(X) > a] \leq \mathbb{P}[X_N > a] + \mathbb{P}[S_N(X) > a, X_N \leq a] \leq \frac{1}{a} \mathbb{E}[X_0] + \mathbb{P}[S_N(Z)^2 > a^2].$$
Clearly \( S_N(Z)^2 \leq 2S_N(M)^2 + 2S_N(A)^2 \leq 2S_N(M)^2 + 2A_N^2 \).

\[
P[S_N(Z)^2 > a^2] \leq P[S_N(M)^2 + A_N^2 > \frac{a^2}{2}]
\]

\[
\leq \frac{2}{a^2} E[S_N(M)^2 + A_N^2]
\]

\[
= \frac{2}{a^2} E[M_N^2 + A_N^2]
\]

\[
\leq \frac{\frac{4}{a}}{2} E[M_N^2] \leq \frac{13}{a} E[X_0^2].
\]

This together with (8) gives (4). Similar argument applies to any class \( B \) mapping. We remark that (3) implies (4) but with 5\( \frac{1}{2} \) instead of 13.

References


