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Séminaire de probabilités (Strasbourg), tome 6 (1972), p. 101-104
<http://www.numdam.org/item?id=SPS_1972__6__101_0>

Université de Strasbourg
Séminaire de Probabilités 1970/71

KRICKEBERG'S DECOMPOSITION FOR LOCAL MARTINGALES

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Any L^1- bounded martingale can be uniquely decomposed into two positive martingales possessing some additional property : this is the well known Krickeberg decomposition, which will be recalled below. In the present note we extend this fact to the local martingale case, following the same idea.

1. - Let Ω be a set, $\underline{\underline{F}}$ a Borel field of subsets of Ω, P a probability measure defined on $(\Omega,\underline{\underline{F}})$. We are given a family $(\underline{\underline{F}}_t)$ of Borel subfields of $\underline{\underline{F}}$, increasing and right continuous. We may, and do, assume that $\underline{\underline{F}}$ has been completed with respect to P, and that each $\underline{\underline{F}}_t$ contains all sets of measure zero. We assume that the reader knows the usual definitions, for example : stopping times, changes of times, martingales , etc (see [1]). We don't distinguish two processes X and Y such that for a.e. ω $X_t(\omega)=Y_t(\omega)$ $\forall t\geq 0$; this is important for the understanding of uniqueness statements below.

2. - All martingales considered below are assumed to be right continuous.

Proposition 1.- A martingale $X=(X_t,\underline{\underline{F}}_t)$ is L^1-bounded if and only if it can be written as the difference of two positive martingales. These martingales $X^{(1)}$ and $X^{(2)}$ then can be chosen so as to realize the equality

$$\sup_t E[|X_t|] = E[X_0^{(1)}] + E[X_0^{(2)}] \tag{1}$$

This decomposition then is unique.

Proof. The "if" part is clear. To prove the "only if" part, set

$$X_t^{(1)} = \lim_n E[X_n^+|\underline{\underline{F}}_t] \quad , \quad X_t^{(2)} = \lim_n E[X_n^-|\underline{\underline{F}}_t]$$

The monotone convergence theorem shows that if s<t, we have $E[X_t^{(i)}|\underline{\underline{F}}_s] = X_s^{(i)}$, i=1,2. This is the martingale equality, and since the family $(\underline{\underline{F}}_t)$ is right continuous we may assume that right continuous modifications of the above processes have been chosen. Then it is easy to

see that $X=X^{(1)}-X^{(2)}$, and that the equality (1) holds .

If we have another decomposition $X=Y-Z$ of X into two positive martingales, then $Y_t \geqq X_t^{(1)}$ and $Z_t \geqq X_t^{(2)}$. If this decomposition satisfies (1), we must have $E[Y_0]=E[X_0^{(1)}]$ and $E[Z_0]=E[X_t^{(2)}]$ and the uniqueness statement follows from it. It is interesting for the sequel to remark that the conclusion $Y_t \geqq X_t^{(1)}$, $Z_t \geqq X_t^{(2)}$ is true also if Y, Z are just assumed to be supermartingales $\geqq 0$.

3.- Definition 2. A process $X=(X_t, \underline{\underline{F}}_t)$ is said to be a local martingale if there exists an increasing sequence (T_n) of stopping times of $(\underline{\underline{F}}_t)$ such that $\lim_n T_n = \infty$ and for each n the process $(X_{t \wedge T_n} I_{\{T_n > 0\}}, \underline{\underline{F}}_t)$ is a martingale which belongs to the class (D).

To be short, we shall say that a stopping time T reduces the process X if $(X_{t \wedge T} I_{\{T>0\}})$ belongs to the class (D) - one may then show that it is a martingale - and we shall call a sequence T_n as above a fundamental sequence for the local martingale X.

Now we set $\|X\|_1 = \sup_T E[|X_T|]$, T ranging over the set of all a.s. finite stopping times. If $\|X\|_1 < \infty$, the local martingale is said to be bounded in L^1.

Theorem 3 . Let X be a local martingale. Then $\|X\|_1 = \sup_n E[|X_{T_n}|]$ for any fundamental sequence (T_n) consisting of a.s. finite stopping times. If X is L^1-bounded, then X can be written as the difference $X^{(1)}-X^{(2)}$ of two positive local martingales , which can be chosen so as to realize the equality

(2) $$\| X \|_1 = E[X_0^{(1)}]+E[X_0^{(2)}]$$

This decomposition then is unique.

Proof. We have $E[|X_{T_n}|] \leqq \| X \|_1$ for all n. Let T be any finite stopping time. A well known submartingale inequality gives us $E[|X_{T \wedge T_n} I_{\{T_n>0\}}|] \leqq E[|X_{T_n} I_{\{T_n>0\}}|]$, and $E[|X_T|] \leqq \sup_n E[|X_{T_n}|]$ now comes from Fatou's lemma. This proves the first statement.

Assume $\|X\|_1 < \infty$. Then X_0 is integrable. The process $(X_{t \wedge T_n})$ is a local martingale (stopping preserves the local martingale property) and belongs to the class (D), hence is a martingale of the class (D), and we have no need to insert $I_{\{T_n>0\}}$. For each n, denote by $X_t^{(1,n)}$ and $X_t^{(2,n)}$ the martingales appearing in the Krickeberg decomposition of $X_{t \wedge T_n}$.

The processes $X^{(1,n)}_{t\wedge T_{n-1}}$, $X^{(2,n)}_{t\wedge T_{n-1}}$ are positive martingales, and their difference is the martingale $X_{t\wedge T_{n-1}}$. Therefore we have

$$X^{(1,n)}_{t\wedge T_{n-1}} \geqq X^{(1,n-1)}_t \quad , \quad X^{(2,n)}_{t\wedge T_{n-1}} \geqq X^{(2,n-1)}_t$$

and $E[X^{(1,n)}_0 + X^{(2,n)}_0] = \sup_T E[|X_{T\wedge T_n}|]$. The processes $Y^{(i,n)}_t = X^{(i,n)}_{t\wedge T_n} I_{\{t \leqq T_n\}}$ (i=1,2) are supermartingales and increase with n, therefore their limit still is a right continuous process (see [1], chapter VI, theorem 16) . Denote this limit by $X^{(i)}_t$. We also have

$$X^{(i)}_t = \lim_n X^{(i,n)}_t$$

The processes $X^{(i)}_t$ are positive supermartingales, their difference is X_t , and we have $E[X^1_0 + X^2_0] \leqq \|X\|_1$ from Fatou's lemma - in fact, this must be an equality, since the reverse inequality is obvious. On the other hand, $X^{(i)}_{t\wedge T_k}$ is the limit of the increasing sequence of martingales $X^{(i,n+k)}_{t\wedge T_k}$ as $n\to\infty$. We now remark that $X^{(i,n+k)}_{t\wedge T_k} = E[X^{(i,n+k)}_{T_k} | \underline{\underline{F}}_t]$ and, using monotone convergence, that $X^{(i)}_{t\wedge T_k} = E[X^{(i)}_{T_k} | \underline{\underline{F}}_t]$. Hence this process is a class (D) martingale, T_k reduces $X^{(i)}$, which therefore is a local martingale. The existence part is proved.

To prove the uniqueness, consider another decomposition $X = Y^{(1)} - Y^{(2)}$ where $Y^{(1)}, Y^{(2)}$ are positive local martingales, and $E[Y^{(1)}_0 + Y^{(2)}_0] = \|X\|_1$. Note that $Y^{(i)}$ is a supermartingale , i=1,2. Stopping at time T_k, and using our remark at the end of the proof of proposition 1, we get that $Y^{(i)}_{t\wedge T_k} \geqq X^{(i,k)}_t$. Letting $k\to\infty$, we have $Y^{(i)} \geqq X^{(i)}$, and the condition on expectations implies $E[Y^{(i)}_0] = E[X^{(i)}_0]$. The positive supermartingale $Y^{(i)} - X^{(i)}$ being equal to 0 for t=0 must be identically 0, and the theorem is proved.

<u>Corollary</u> 1. <u>For any local martingale</u> X

$$(\forall\lambda>0) \ , \ \lambda P\{ \sup_t |X_t| > \lambda \} \leqq \|X\|_1$$

<u>Corollary</u> 2. <u>If</u> $\|X\|$ <u>is finite</u>, <u>then</u> X_t <u>converges a.s. to an integrable random variable as</u> $t\to\infty$.

<u>Proof</u>. X is the difference of two positive supermartingales.

<u>Remark</u> that for any normal change of time $\Theta = (\underline{\underline{F}}_t, \theta_t)$ we have $(\Theta X)^{(i)}_t = X^{(i)}_{\theta_t}$, i=1,2 .

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