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A SIMPLE PROOF OF DOOB'S CONVERGENCE THEOREM

by K.L. CHUNG

Doob's version of the fundamental convergence theorem of potential theory asserts that if (f_n) is a decreasing sequence of excessive function and f is the supermedian function $\inf_n f_n$, then the set where f differs from $\hat{f}$ (its regularized function) is semi-polar. Many beautiful proofs of this result are available in the literature. Here is a trivial one.

By truncation we can assume f_1 is bounded. Set $A = \{f \geq \hat{f} + \varepsilon\}$, $\varepsilon > 0$. Since f is fine u.s.c., $\hat{f}$ fine continuous, A is finely closed. Calling P_A the usual réduite operator, we have

$$f \geq P_A f \geq P_A(\hat{f} + \varepsilon) = P_A \hat{f} + P_A \varepsilon$$

(the first $\geq$ follows from $f_n \geq P_A f_n$, the second $\geq$ from the fact that the measures $P_A(x, \cdot)$ are carried by A). Applying the semi-group

$$P_t f \geq P_t P_A \hat{f} + P_t P_A \varepsilon$$

As $t \rightarrow 0$, we get

$$\hat{f} \geq P_A \hat{f} + P_A \varepsilon$$

At a point x regular for A , this implies the absurd inequality $\hat{f}(x) \geq \hat{f}(x) + \varepsilon$. Thus A has no regular point, and $\{f > \hat{f}\} = \bigcup \{f \geq \hat{f} + \frac{1}{n}\}$ is semi-polar.

The same proof shows that if f is nearly Borel, positive and such that $f \geq P_K f$ for every compact set K (it is well known that f then is supermedian), then $\{f > \hat{f}\}$ is semi-polar. One must just take care to apply the above reasoning, not to A (which isn't known to be finely closed) but to $K \subset A$, compact. If x is regular for A one chooses an increasing sequence K_n of compact subsets of A such that $T_{K_n} \downarrow 0$ P^x -a.s. and gets the same contradiction as above.

As a matter of fact the latter version, given very recently by Gettoor and Rao, is the one that I stumbled into. On going over the above proof with Sieveking he pointed out the even shorter cut to the original Doob version.