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Randomness and determinism : a mathematical approach

by

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If one accepts the hypothesis that the mechanisms of natural phenomena are deterministic, as all scientists do practically vhen working at a microscopic level, then it is not immediately seen that probability and statistics enter the picture at all. In fact, given enough information about the system to be studied and enough time and ability in calculation, all of the desired results can be determined. The idea of randomness and the techniques of probability and statistics, however, are very useful in such situations, and this usefulness arises in particular when the observer of such a system can describe the deterministic mechanism involved, but does not have enough knowledge concerning the initial data or the parameters of the system, or enough calculatory power, to describe the future with precision. In other words, and this is a point wich we shall make more precise below, whether a phenomenon "appears" random or not ot an observer, depends to a great extent on "how much" information he has collected concerning the phenomenon.

In these lectures, I shall endeavor to describe an ongoing mathematical effort to classify and recognize different types of randomness in deterministic situations of a special type. First of all, let me state clearly that I am assuming a "classical" situation (as opposed for instance in physics to a "quantum" situation whose description inherently contains probabilistic matter at present), in which the observer has complete knowledge of the deterministic mechanism involved, but lacks information on the initial state and power to observe the system completely at any given time. For simplicity, I shall also assume that time is discrete. After describing the basic ideas of the general theory, I would like to present a few examples from my personal research which I deem interesting for further study.

§ 1. DYNAMICAL SYSTEMS

In the following definition, what I have in mind is a deterministic physical system which can assume a certain collection of possible states, and which obeys a rigid rule in moving from state at successive time units. The total knowledge which is possessed concerning the initial state is contained in a probability distribution on the set of possible states, and I consider the simple case in which this probability distribution is stationary w.r.t. the movement of the system. (Hence we should be able to estimate the probability distribution from previous manipulation of the system).

Définition :

A (discrete time stationary) dynamical system is a triple (Ω,P,T) where

1) Ω is a compact metric space (the set of possible states of the system)

2) P is a probability distribution on (the Borel sets of) Ω

3) T is a homeomorphism from Ω_1 to Ω_1, where Ω_1 is a residual subset of Ω whith $P(\Omega_1) = 1$, and T preserves the probability distribution P. (the state $\omega \in \Omega_1$ changes to the state $T(\omega) \in \Omega_1$ after one time unit).

To be precise, one should call the above concept an "almost topological dynamical system", to distinguish it from a mesurable dynamical system (in which (Ω,P) is simply an abstract probability space and T a (measurable) automorphism of (Ω,P) and from a topological dynamical system, in which T is e measure preserving homeomorphism of Ω.

From a conceptual point of view, there are (at least) two "different" types of dynamical systems. One the one hand, there are systems airising from deterministic situations in the physical sciences, such as a gas in a box, billiards on different types of "tables", population models in biology, etc. On the other hand, dynamical systems such as two-sided infinite coin tossing, Markov chains (finite state space) with stationary distributions, etc., are well-know to probabilists and exhibit clearly a probabilistic nature. It is clearly of interest, both philosophically and practically, to be able to "see" the probabilistic structure in the second type of examples inside of the first type of examples. This is the basic goal in the following definition.

Définition :

Let (Ω,P,T) and (Ω',P',T') be two dynamical systems. A finitary homomorphisme from (Ω,P,T) to (Ω',P',T') is a continuous map $\phi : \Omega_2 \to \Omega_2'$,

where

1) $\Omega_2 \subset \Omega_1$ and $\Omega_2' \subset \Omega_1'$ are residual subsets of full probability in Ω and Ω'

2) $\phi P = P'$ and $\phi \circ T = T' \circ \phi$.

If ϕ is a homeomorphism (i.e. $\phi^{-1} : \Omega_2' \to \Omega_1'$ exists and is continuous), then ϕ is a finitary isomorphism.

For a complete discussion of the technical details involved in the definitions above, we refer to [D-K]. An immortant distinction between the framework above and that of classical ergodic theory is that the former gives us at least some hope of physically recognizing the probabilistic nature of a mechanical system, in view of the continuity assumptions.

§ 2. MEASUREMENTS

When observing a dynamical system, I shall thake the point of view that the observation yelds only a finite number of possible results, and that continuity of measurement is to be expected.

Définition :

Let (Ω,P,T) be a dynamical system. A partition ξ (i.e. a means of observation, or a measuring instrument) is an ordered finite collection $\xi = (\xi_o , \xi_1 ,..., \xi_{n-1})$ of open disjoint subsets of Ω such that

$$P\left(\bigcup_{j=o}^{n-1} \xi_j \right) = 1$$

If ξ is a partition of (Ω,P,T) and if ω is a state of Ω, then we can associate to ω the total sequence of past, present and future measurements of it by ξ . Let

$$\phi(\omega) = x = (..., x_{-1}, x_o, x_1, ...)$$

defined by

$$x_t = k \quad \text{iff} \quad T^t(\omega) \in \xi_k .$$

Then $\phi(\omega)$ is well-defined for P-almost all points $\omega \in \Omega$, and if we let X denote the closure of $\{\phi(\omega) : \omega \in \Omega, \phi(\omega) \text{ defined}\}$ in the infinite product space $\{0, 1, ..., n-1\}^{\mathbb{Z}}$, S the shift transformation on X, and Q the image of the probability distribution P under ϕ , then (X,Q,S) is a dynamical system and ϕ is a finitary homomorphism from (Ω,P,T) to (X,Q,S) .

Définition :

The dynamical system (X,Q,S) is called the finitary factor associated to ξ . The partition ξ is called a finitary generator if ϕ is a finitary isomorphism.

Obviously, the measurement ξ will (eventually) extract the totality of information contained in the dynamical system (Ω,P,T) and it is interesting to know when this is possible and how to obtain a finitary generator (with prescribed properties).

§ 3. ENTROPY

The average of information contained in an ongoing measurement is called its entropy. The idea of applying information theory to dynamical systems is due to Kolmogorov.

Let $\xi = (\xi_0, \xi_1, \ldots, \xi_{n-1})$ be a partition of (Ω, P, T). The entropy of ξ is defined as

$$H(\xi) = - \sum_{j=o}^{n-1} P(\xi_j) \log P(\xi_j) .$$

The partition

$$\xi_s^t = \bigvee_{r=s}^{t} T^{-r}\xi ,$$

where " V " denotes common refinement, corresponds to measuring the system between times s and t . It is not hard to show that the limit

$$h(\xi) = \lim_{t \to \infty} \frac{1}{t} H(\xi_o^{t-1})$$

exists, using subadditivity properties of the function H ; the quantity $h(\xi)$ is called <u>mean entropy</u> of ξ . Obviously $0 \leq h(\xi) \leq \log n$. Finally, the <u>entropy $h(T)$ of the dynamical system</u> (Ω, P, T) is defined as

$$h(T) = \sup_{\xi} h(\xi)$$

the sup being taken over all partitions.

If ϕ is a finitary homomorphism from (Ω, P, T) to (Ω', P', T') and if ξ' is a partition of (Ω', P', T'), then $\phi^{-1} \xi'$ is a partition (modulo P) of (Ω, P, T) and $h(\xi') = h(\phi^{-1}\xi')$. Therefore entropy of dynamical systems does not increase under finitary homomorphisms, and is invariant under finitary isomorphism.

One of the basic theorems of entropy theory, due to Sinai, is that if ξ is a generator, then $h(T) = h(\xi)$.

§ 4. THE GENERATOR THEOREM

If a dynamical system (Ω,P,T) possesses a finitary generator, then its entropy $h(T)$ is obviously finite. There is one other obstruction to the existence of a finite generator, and that is the existence of a set of periodic points with positive probability. The dynamical system (Ω,P,T) is said to be *aperiodic* if the set of periodic states has probability zero.

Theorem (see [D-K] , [D]) :

If (Ω,P,T) is aperiodic and if $h(T) < \infty$, then (Ω,P,T) has a finitary generator $\xi = (\xi_0,\xi_1,\ldots,\xi_{n-1})$ with

$$n = [e^{h(T)}] + 1$$

The reader should be cautioned that this theorem is an existence theorem, and that in general it is not easy to find a "nice" generator. If, however T is *expansive* (i.e. there is a positive number δ such that for any ω, $\omega' \in \Omega$ with $\omega \neq \omega'$, there existe t with $d(T^t\omega , T^t\omega') \geq \delta$), then any partition whose sets all have diameter less than δ will be a generator. This case often arises in practice. Finally, we remark that the hypothesis $h(T) < \infty$ is satisfied for a large class of dynamical systems ([Ku]).

§ 5. ERGODICITY

For measurable dynamical systems, one generally defines ergodicity as the absence of a measurable set of states invariant under the motion T and with probability strictly between zero and one. A more natural and equivalent definition in the situation described above is to say that (Ω,P,T) is *ergodic* if the weak of large numbers holds for every measurement (partition) of the dynamical system. It then follows that the strong law of large numbers holds for any $\mathbb{L}^1$ - function on (Ω,P) (individual ergodic theorem). Thus ergodicity is the property which allows us to use the law of large numbers for measurements on the system or, in other words, which insures that almost all sample paths of the system have the same *average* behavior.

Thus it is important to determine whether a given dynamical system is ergodic. In general, this problem is difficult, even for simple systems, and I shall give some examples in the second part.

A homomorphic image of an ergodic system is ergodic, so that ergodicity is an isomorphism invariant. A particular type of ergodicity occurs in a dynamical system when the strong law of large numbers holds for each measurement and _every initial_ state $\omega \in \Omega$. A dynamical system which this property is called _strictly ergodic_. An interesting open problem in this direction is given in the second part.

§ 6. ASYMPTOTIC INDEPENDENCE

It is never true for a dynamical system that every measurement on it (in the above sense) satisfies the central limit theorem, or the law of the iterated logarithm. To obtain such results, it is necessary to impose some asymptotic independence contidion (in general, exponential decay) on the partition itself. I shall not describe in detail such results (see e.g. [I,L],[G]). Recent work ([D-K]) has shown that if a smooth dynamical system (Ω,P,T) is finitarily isomorphic to a Bernouilli scheme (Ω',P',T') (= infinite coin toss), and if the isomorphism ϕ satisfies a "finite expected coding time" property, then "smooth" measurements on (Ω,P,T) do satisfy the central limit theorem and the law of the iterated logarithm. This shows that certain types of isomorphisms between "deterministic" systems and independent processes will imply probabilistric behavior of a large class of measurements on the deterministic system.

One of the long-outstanding open problems in classical ergodic theory concerne asymptotic independence.

A dynamical system is k-miwing if for any measurable sets $A_1,\dots.A_k$,

$$\lim_{\substack{t_j \to \infty \\ 1<j<k}} P(A_1 \cap T^{t_1} A_2 \cap T^{t_1+t_2} A_3 \cap \dots \cap T^{t_1+\dots+t_{k-1}} A_k) = \prod_{j=1}^{k} P(A_j)$$

It is unknown whether there exists a dynamical system which is 2-mixing but not k-mixing for some $k > 2$.

§ 7. BERNOULLI SCHEMES

A Bernoulli scheme is a dynamical system (Ω,P,T) where Ω is the set of all two-sided infinite sequences whose terms can take finitely many values, P is a product measure on Ω, and T is the (left) shift transformation on Ω. If $\{0,1,\dots,n-1\}$ is the set of values, then the Bernoulli scheme is uniquely determinde by a probability vector $p = (p_o, p_1, \dots p_{n-1})$, and is a probabilistic model for infinite independent repetition of an experiment whose outcomes are $0,1,\dots,n-1$ with respective probabilities $p_o, p_1, \dots, p_{n-1}$. The partition ξ of Ω into points of Ω having the same coordinate at zero is a finitary generator, and

$$h(\xi) = -\sum_{i=o}^{n-1} p_i \log p_i$$ because of independence.

Thus Sinai's theorem (§3) yelds

$$h(T) = -\sum_{i=o}^{n-1} p_i \text{ long } p_i$$

for the Bernoulli scheme (Ω,P,T) determined by the probability vector p. It is not hard to see that Bernoulli schemes are ergodic, and even k-mixing for all $k \geq 2$.

Now suppose that (Ω,P,T) and (Ω',P',T') are two dynamical systems and ϕ is a finitary homomorphism from (Ω,P,T) to (Ω',P',T'), and suppose that (Ω',P',T') is a Bernoulli scheme with probability vector $p = (p_o, \dots, p_{n-1})$ and zero coordinate partition $\xi' = \xi'_o, \dots, \xi'_{n-1})$. Then the partition $\xi = \phi^{-1}(\xi')$ of (Ω,P,T) obviously "looks" like ξ'. That is, the sequence of measurements on (Ω,P,T) using the partition ξ is essentially an independent repetition of an experiment with probabilities $p = (p_o, \dots, p_{n-1})$. That, if we measure (Ω,P,T) with the partition ξ and if our only knowledge of the initial position is contained in the probability distribution P, we shall see a probabilistic (independent) experiment, even though (Ω,P,T) is deterministic. One of the basic goals of the study of dynamical systems is to determine, for a given (Ω,P,T), which Bernoulli schemes it has a homomorphic images and what the corresponding partitions $\phi^{-1}(\xi)$ are like. This study is in its beginning stages, and very few results are known, especially in the case of physical systems.

I would like to finish this paragraph with a discussion of a problem which arises naturally in the above setting. This is the question of when two different Bernoulli schemes are finitarily isomorphic, or more generally, when is one a finitary image of the other. In more probabilistic language, how many kinds of "complete" randomness exist ?

In the case of dynamical systems on sequence spaces with the shift transformation (i.e. shift dynamical systemsp), finitary homomorphisms and isomorphisms have a natural interpretation in terms of sequential coding. If $\omega = (\ldots, \omega_{-1}, \omega_o, \omega_1, \ldots) \in \Omega$ and $\phi(\omega) = \omega'$ $(\ldots, \omega'_{-1}, \omega'_o, \omega'_1, \ldots) \in \Omega'$ where ϕ is finitary, then ϕ can be described as follows. We begin by examining the zero coordinate ω_o of ω, then the coordinates ω_{-1}, ω_o, ω_1 of ω, etc. and we suppose that we have a stopping rule $\tau : \Omega \to N$ (finite a.e.) which will tell us to stop at the examination of $(\omega_{-\tau(\omega)}, \ldots, \omega_o, \ldots, \omega_{\tau(\omega)})$.

For each value $m \in \mathbb{N}$, we also suppose that we have a "dictionary" δ_m given, which tells us for each stopped sequence $(\omega_{-m}, \ldots, \omega_m)$ of length $2m + 1$ what the coordinate $\omega'_o = \phi(\omega)_o$ should be if $\tau(\omega) = m$, i.e. if ω is stopped at time m :

$$\delta_m(\omega_{-m}, \ldots, \omega_m) = \omega'_o$$

It is not hard to see ([D-K]) that any finitary homomorphism ϕ between shift dynamical systems can be described with the aid of a stopping rule τ and a set of dictionaries δ_m. (To obtain the k^{th} coordinate ω'_k of the image $\omega' = \phi(\omega)$, simply shift the original point ω by k and apply the above procedure). Thus a finitary homomorphism is a sequential code, in the above sense, and our question concerning different kinds of randomness can also be formulated as : when can a stationary sequence be transformed into another type of stationary sequence by means of a sequential code ? (The words "stationary", are important here).

This problem was solved recently in two articles ([K-S] 1977 and 1978) .

Theorem :

If (Ω,P,T) and (Ω',P',T') are two Bernoulli schemes with corresponding probability vectors p and p' , and entropies

$$h = h(T) = - \sum_i p_i \log p_i$$

$$h' = h(t') = - \sum_i p_i' \log p_i' \quad ,$$

then,

a) There is a finitary homomorphism ϕ from (Ω,P,T) to (Ω',P',T') iff $h \geq h'$,

and

b) there is a finitary isomorphism ϕ between (Ω,P,T) and (Ω',P',T') iff $h = h'$.

The proof is constructive, but the homomorphisms (isomorphisms) obtained are far from being unique, and it would be interesting to try to reduce the arbitrary nature of ϕ by imposing additional conditions.

Several extensions of these results have recently appeared [A-DJR] , [DJ] , [K-S] 1979), and in particular it has been shown that ergodic automorphisms of the two-torus possess independent open generators. An interesting open problem is whether this result remains valid for higher-dimensional tori or other compact groups.

§ 8. MEASURABLE DYNAMICAL SYSTEMS

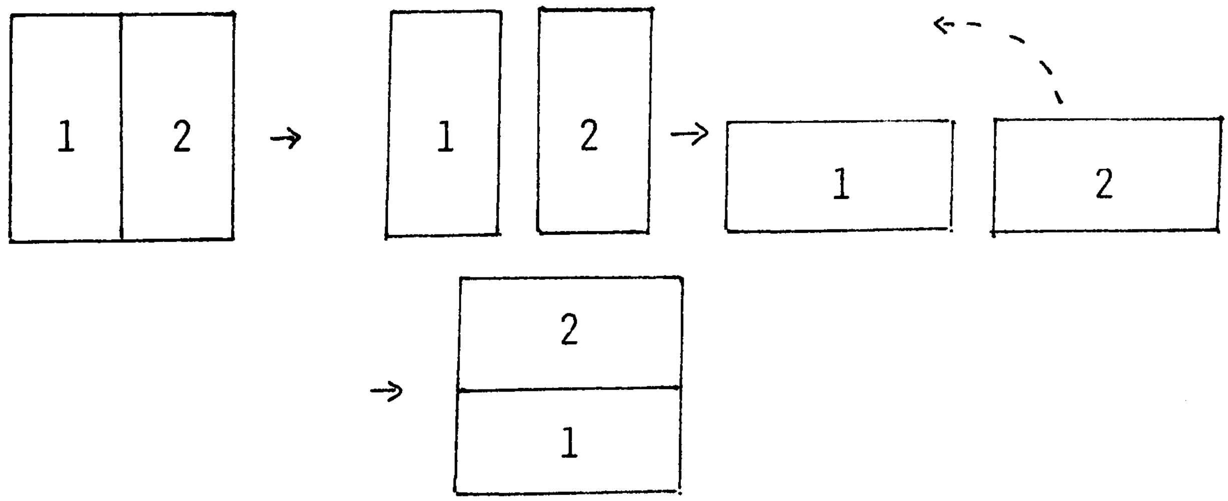
In the preceding presentation, I have violated the historical sequence of events in presenting concepts and results in the framework of almost topological dynamical systems, a relatively recent concept. My reason for doing this is to be able to describe the potential usefulness of this theory for applications. It would not be fair to do so without giving essential credit to the development of the basic ideas of dynamical systems where it was first done, namely in measurable dynamical systems. In particular, §3 comes from measurable ergodic theory and is due to Kolmogorov and Sinai, the measurable counterpart of the theorem of §4 is due to W. Krieger (here W. Parry should also mentioned, and an essential idea in the proof of Krieger comes from K. Meshalkin's construction), and the measurable counterpart of §7 has been developed by Ja. Sinai and D. Ornstein. The results in the almost topological theory should, in my opinion, be viewed as (perhaps interesting and hopefully useful) improvements and in some cases simplifications of the basic results in the measurable case, developed by the above and many other authors.

§ 9. EXAMPLES AND PROBLEMS

Instead of giving a set of examples which have been completely studied, I shall try to indicate the major directions of research, at the risk of omitting important items. This list should therefore not be considered as complete and certainly reflects the tastes of the author. The first example is the standard motivation for the study of dynamical systems.

1. The baker's transformation

Let Ω the unit squarein $\mathbb{R}^2$, with uniform distribution P. The transformation T consists of first cutting Ω into two equal pieces along the line $x = \frac{1}{2}$, then contracting each piece by a factor $\frac{1}{2}$ in the y direction and expanding each piece by a factor 2 in the x direction, and finally putting the right piece above the left one:



It is not hard to see, using the binary expansions of the coordinates x and y of $\omega = (x,y) \in \Omega$ that the above partition into sets 1 and 2 is a finitary generator and that the corresponding finitary isomorphism ϕ (see §2) has as image the Bernoulli scheme with probability vector $(\frac{1}{2}, \frac{1}{2})$.

2. Gas in a box

Consider the dynamical system of N molecules which are situated in a box and undergo elastic collisions with each other and with the walls of the box. At the outset, the system is assumed to have a fixed energy, and Ω is the set of all vectors in $\mathbb{R}^{6N}$ (three position coordinates and three velocity coordinates) having the given energy. Liouville's theorem yelds a probability distribution P on Ω invariant under motion, and $T : \Omega \to \Omega$ denotes the passage of a time unit (which we may fix arbitrarily).

In the case $N = 1$ it is easy to see what happens (here (Ω,P,T) is not ergodic). If $N = 2$, it was shown by Sinai that (Ω,P,T) is ergodic and work of Gallavotti and Ornstein shows that (Ω,P,T) is measure theoretically isomorphic to a Bernoulli scheme. To my knowledge, nothing further is known.

3. Smooth dynamical systems

A smooth dynamical system is a dynamical system (Ω,P,T) where Ω is a compace C^∞-manifold, P is a probability distribution having a smooth density function on Ω, and T is a C^∞-diffeomorphism. The basic question here is : what types of dynamical systems can arise from smooth dynamical systems ? Outside of finite entropy, no obstructions are known either in the measurable or almost topological framework. Pecin has shown that if $\dim \Omega = 2$ ans if $h(tT) > 0$, (Ω,P,T) ergodic, then (Ω,P,T) breaks up into a finite number of components which are measure theoretically isomorphic to Bernoulli schemes, and his results have been used to construct smooth dynamical systems on any compact manifold which are measurably isomorphic to Bernoulli schemes.

Another open question is whether a compact manifold can admit a strictly ergodic diffeomorphism of positive entropy, preserving the natural measure.

4. *Billiards in polygons*

Consider one billiard ball moving at a fixed velocity in a fixed polygon. If the angles of the polygon are not all rational, no example has been shown either to be ergodic or not ergodic. This problem (for right triangles) is connected to the motion of two masses m_1 and m_2 on the unit interval with elastic reflections from each other and from the end points of the interval. If $\frac{1}{\pi}$ arc tg$\sqrt{\frac{m_1}{m_2}}$ is irrational, is this system ergodic ? The entropy of these systems is zero.

5. *Interval exchange transformations*

An interval exchange transformation is a map $T : \Omega \to \Omega$ with $\Omega = [0,1]$, wich consists in cutting up Ω into a finite number of pieces and permuting these pieces in a given order. Thus T depends on a vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ and a permutation $\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\}$ where α gives the length of the pieces. Something is known concerning interval exchange transformations, but the following interesting question remains open : if σ is irreductible (in the sense that $\sigma(\{1,\ldots,k\}) = \{1,\ldots,k\}$ implies $k = n$), then is T strictly ergodic for almost all vectors α ?

6. *Classification of dynamical systems with infinite measure*

A recurrent random walk on $\mathbb{Z}$ (or $\mathbb{Z}^2$) gives rise to a dynamical system (Ω,P,T) in a natural manner, where P is not a probability measure but an infinite measure (corresponding to the infinite invariant initial "distribution" which is counting measure on $\mathbb{Z}$). These systems seem to be natural generalisation of Bernoulli schemes in the infinite measure case. Their entropies can be defined in a reasonable way and are all infinite. Il we restrict ourselves to the class of recurrent walks with finite variance, then it is not known whether any two such walks are isomorphic or not. (Here the isomorphism may multiply the measure by a constant factor, since the measures are infinite).

I have tried in the above to give a short sample, without references, of some interesting problems concerning dynamical systems, and I shall gladly supply references on request.

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