Séminaire Janet.
Mécanique analytique
et mécanique céleste

F. A. E. Pirani
Gareth Williams

Rigid motion in a gravitational field

Séminaire Janet. Mécanique analytique et mécanique céleste, tome 5 (1961-1962), exp. n° 8-9, p. 1-16

<http://www.numdam.org/item?id=SJ_1961-1962__5__A8_0>

Séminaire JANET
(Mécanique analytique et Mécanique céleste)
5e année, 1961/62, n° 8-9

9 et 10 février 1962

RIGID MOTION IN A GRAVITATIONAL FIELD

by F. A. E. PIRANI and Gareth WILLIAMS.

Abstract. - Integrability conditions are constructed for Born's equations of rigid motion in a gravitational field. It is shown that the angular velocity of a rigid test body in vacuo must be of constant magnitude, and a new proof is given for the Herglotz-Noether theorem, which states that in the absence of a gravitational field every rotating rigid motion is isometric.

1. Introduction.

In Newtonian mechanics, a body is called rigid if the distance between every pair of particles remains constant. How can this concept of rigidity be extended to relativistic mechanics ? First of all, the idea of distance between particles must be generalized from space to space-time. It seems natural in space-time to define the distance between neighbouring particles, at least, as space-time interval measured orthogonal to the world-line of one of them. Distance so defined has a simple physical description in terms of idealized operations with light-signals or radar (cf. SYNGE [18]).

A definition of rigidity in Minkowskian space-time, based on this concept of distance, was proposed fifty years ago, first by BORN [1] for rectilinear rigid motion, then, independently, by HERGLOTZ [2] and by F. NOETHER [7] for general rigid motion. According to this definition :

A body is called rigid if the distance between every neighbouring pair of particles, measured orthogonal to the world line of either of them, remains constant along the world-line.

As SALZMAN and TAUB [15] and, independently, SYNGE [17] suggested, one may with exactly the same words define a rigid body in Riemannian space-time, that is to say, in a gravitational field. The equations of rigid motion, defined in this way, have been studied extensively by RAYNER [9], [14]. Here we work out the conditions of integrability of these equations, and deduce some simple consequences.

HERGLOTZ and NOETHER found that in the minkowskian case the definition restricts rigid motions much more than one might have expected from Newtonian preconceptions. In fact, a rigid body in Minkowskian space-time has only three degrees of freedom - apart from special cases, the motion of the body may be fixed completely by giving the motion of a single particle. As a result, only certain motions are possible ; for example, along any world-line the angular velocity must be constant.

It has in the past been inferred from this that the definition is not after all a reasonable one, and that a less restrictive definition should be found. In our view, on the contrary, these restrictions of the motion may be explained, at least in a general way, by the argument that the application of forces to a body to change its momentum or angular momentum must distort the body, so that a body which is required to be rigid cannot be subject to such forces. From this point of view, severe restrictions are to be expected, and in fact it is then necessary to ask why uniformly accelerated rectilinear rigid motion, which was discovered by BORN, is not also forbidden. We suspect that this is more-or-less an accident, connected with the principle of equivalence, but since the argument is anyhow only a heuristic one we shall not pursue it any further.

The motion of a rigid body in a gravitational field may be studied from two distinct points of view. While its kinematical behaviour is in all cases determined by the equations of rigid motion, it may be regarded dynamically either as a heavy body which contributes to the gravitational field or as a test body whose influence on the field is negligible.

If the body is regarded as a heavy body, then it is to be assumed that the kinematical velocity, which enters the equations of rigid motion, is the same as the dynamical velocity, which is the timelike eigenvector of the energy-momentum tensor. In making this identification we follow that point of view of SYNGE [18] about the physical interpretation of the energy-momentum tensor.

If the body is regarded as a test body, then there is no necessary connection between the kinematical velocity and the energy-momentum tensor, but nevertheless the energy-momentum tensor need not vanish, for one may consider the motion of a rigid body in a medium as well as in vacuo. Is rigid motion through an arbitrary medium likely to be interesting ? We consider that besides motion in vacuo, only motion through a medium which is spacewise homogeneous, and which may therefore have a cosmological interpretation, is likely to be of interest for test bodies.

The notation is established, and some formulae of Lie differentiation are introduced, in § 2. Rigid bodies are defined in § 3 and integrability conditions for the equations of rigid motion derived in § 4. Various simple applications are given in § 5 .

2. Notations and formulae.

Rather than mix notations or add to the multiplicity already existing, we employ as far as possible the Hamburg notation developed by JORDAN and his collaborators [3], [6].

Lower case Latin indices $a, b, c, \ldots$ range and sum over $1, 2, 3, 4$.

Lower case Greek indices $\alpha, \beta, \gamma, \ldots$ range and sum over $1, 2, 3$.

Round brackets denote symmetrization : $A_{(ab)} = \frac{1}{2}(A_{ab} + A_{ba})$, etc.

Square brackets denote antisymmetrization : $A_{[ab]} = \frac{1}{2}(A_{ab} - A_{ba})$, etc.

The metric tensor g_{ab} has signature $+ + + -$; its inverse is g^{ab}.

The Kronecker delta is $\delta_a{}^b \begin{cases} = 1 & \text{if } a = b \\ = 0 & \text{otherwise.} \end{cases}$

The alternating (oriented) tensor is $\eta_{abcd} = \eta_{[abcd]}$; $\eta_{1234} = \{-\det(g_{ab})\}^{-1/2}$.
Partial differentiation is denoted ∂_c or $_{/c}$: $A_{ab/c} = \partial_c A_{ab} = \partial A_{ab}/\partial x^c$.
Covariant differentiation is denoted ∇_c or $_{//c}$:

$$A_{ab//c} = \nabla_c A_{ab} \quad \text{and} \quad A_{ab//cd} = (A_{ab//c})_{//d} \qquad .$$

The Riemann tensor is defined by the Ricci identity

$$A_{a//[bc]} = -\frac{1}{2} R^k{}_{abc} A_k \tag{2.1}$$

for any vector A_a.
The Ricci tensor is $R_{ab} = -R^k{}_{abk}$ and the curvature scalar is $R = R^k{}_k$.
The Einstein tensor is $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$.
Units are chosen so that $c = 1$ and $K = 1$ (K is Einstein's gravitational constant).

A continuous medium, in particular a finite body, may be specified by giving the equations of its world-lines

$$x^a = x^a(y^\beta, s)$$

in terms of three parameters y^β, which distinguish the world-lines, and a fourth, s, which is conveniently chosen to be proper time along each world-line.

The 4-velocity is $u^a = \frac{\partial x^a}{\partial s}$, with $u^a u_a = -1$.

Absolute differentiation along a world-line is denoted by D/Ds or by a dot $^\circ$; thus

$$\overset{\circ}{A}_{ab\ldots} = \frac{D}{Ds} A_{ab\ldots} = A_{ab\ldots//k}\, u^k \qquad .$$

A central role in the theory of rigid bodies is played by the projection operator

$$h_a{}^b = \delta_a{}^b + u_a u^b \iff h_{ab} = g_{ab} + u_a u_b = h_{ba} \quad .$$

At any point, contraction of a tensor index with this operator effects projection into the local rest-frame (instantaneous space) at that point, for

$$h_a{}^b u_b = 0$$

and for any z_b orthogonal to u^b,

$$z_b u^b = 0 \iff h_a{}^b z_b = z_a \quad .$$

One new notation which we hope will reduce clutter and give some geometrical insight will be used : the symbol $\perp$ before a tensor expression denotes that after all other indicated contractions in that expression have been carried out, each remaining free index is to be projected with $h_a{}^b$. Thus

$$\perp R_{abcd} = h_a{}^p h_b{}^q h_c{}^r h_d{}^s R_{pqrs}$$

while

$$\perp R_{abck} u^k = h_a{}^p h_b{}^q h_c{}^r R_{pqrk} u^k \quad .$$

In the theory of rigid bodies, much simplification is achieved by the use of Lie derivatives. Roughly speaking, the apparatus of Lie derivatives enables us to answer the question : Given a correspondence between two regions of a manifold, how can we compare geometric objects of the same type, defined in the two regions ? In the present context it helps us to identify physical quantities which are conserved during the motion of a rigid body. The theory of Lie derivatives, and its interpretation in terms of displacements of geometrical figures, is explained very clearly in the books of SCHOUTEN [16] and YANO [19] ; here we collect only the relevant formulae.

The Lie derivative of an arbitrary tensor $T_{ab\ldots}{}^{cd\ldots}$ over any vector field ξ_k is given by

(2.2)
$$\underset{\xi}{\mathcal{L}} T_{ab\ldots}{}^{cd\ldots} = T_{ab\ldots}{}^{cd\ldots}{}_{//h}\, \xi^h + \xi^k{}_{//a} T_{kb\ldots}{}^{cd\ldots} + \xi^k{}_{//b} T_{ak\ldots}{}^{cd\ldots} + \ldots$$
$$- \xi^c{}_{//k} T_{ab\ldots}{}^{kd\ldots} - \xi^d{}_{//k} T_{ab\ldots}{}^{ck\ldots} - \ldots$$

In particular, if s is a scalar, then

$$\underset{\xi}{\mathcal{L}} s = s_k \xi^k \quad ,$$

if z_b is a covariant vector, then

$$\underset{\xi}{\mathcal{L}} z_b = z_{b//k} \xi^k + \xi^k{}_{//b} z_k \quad ,$$

and if z^b is a contravariant vector, then

$$\underset{\xi}{\pounds}\, z^b = z^b{}_{//k}\, \xi^k - \xi^b{}_{//k}\, z^k \quad .$$

It may be shown (for example, from (2.2)) that the Lie derivative of a tensor is a tensor of the same type, that the Lie derivative of a sum is the sum of the Lie derivatives, and that Lie differentiation of a product obeys Leibnitz's rule.

The Lie derivative of the Riemannian connection Γ^c_{ab} is given by

$$\underset{\xi}{\pounds}\, \Gamma^c_{ab} = \xi^c{}_{//ab} + R^c{}_{abk}\, \xi^k \tag{2.3}$$

which together with (2.2) yields the commutation rules

$$(\underset{\xi}{\pounds}\, \nabla_a - \nabla_a \underset{\xi}{\pounds})\, z^c = z^b \underset{\xi}{\pounds}\, \Gamma^c_{ab} \quad , \tag{2.4}$$

where z^c is any contravariant vector, and similarly

$$(\underset{\xi}{\pounds}\, \nabla_a - \nabla_a \underset{\xi}{\pounds})\, z_b = - z_c \underset{\xi}{\pounds}\, \Gamma^c_{ab} \quad , \tag{2.5}$$

where z_b is any covariant vector. We note that

$$\pounds\, g_{ab} = 2\xi_{(a//b)} \tag{2.6}$$

and may easily deduce that since $g_{ab//c} = 0$,

$$\underset{\xi}{\pounds}\, \Gamma^c_{ab} = \frac{1}{2}\, g^{ck}(\nabla_a \underset{\xi}{\pounds}\, g_{bk} + \nabla_b \underset{\xi}{\pounds}\, g_{ak} - \nabla_k \underset{\xi}{\pounds}\, g_{ab}) \quad . \tag{2.7}$$

It may be shown by differentiation of this equation that

$$2\nabla_{[d} \underset{\xi}{\pounds}\, \Gamma^c_{a]b} = \underset{\xi}{\pounds}\, R^c_{bad} \quad .$$

These results about Lie derivatives apply to differentiation over an arbitrary vector field ξ^k ; in the following, nearly all Lie derivatives are over the velocity field u^k , and to reduce the clutter we shall abbreviate by writing $\pounds$ for $\underset{u}{\pounds}$.

3. Equations of rigid motion.

We now give analytic form to the definition of rigid motion stated in § 1. The short argument is to remark that distances in the local rest-frame must be preserved along each world-line ; these distances are determined by h_{ab} , which must therefore be Lie-transferred along the world-lines :

$$\pounds\, h_{ab} = 0 \quad . \tag{3.1}$$

To justify this argument, we carry out a rather longer derivation.

Consider two neighbouring world-lines

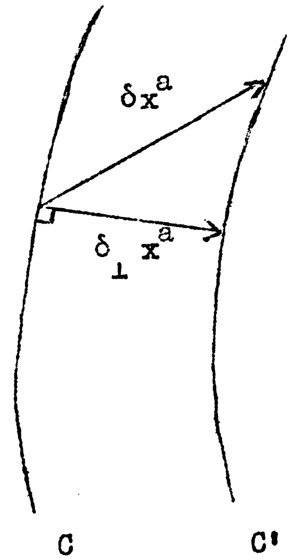
$$C \; : \; x^a = x^a(y^\beta\, , s) \text{ and } C' \; : \; x^a = x^a(y^\beta + \delta y^\beta\, , s)$$

in a rigid body. A displacement vector δx^a from C to C', joining points with corresponding values of s, is given by

$$\delta x^a = \frac{\partial x^a}{\partial y^\beta}\, \delta y^\beta \qquad ,$$

and the corresponding orthogonal displacement vector is

$$\delta_\perp x^a = h^a{}_b\, \delta x^b \qquad .$$



The orthogonal distance $\delta_\perp \ell$ from C to C' is given by

$$\delta_\perp \ell^2 = g_{ab}\, \delta_\perp x^a\, \delta_\perp x^b = h_{ab}\, \delta x^a\, \delta x^b \qquad ,$$

and the demand that it remain constant along C is simply

$$0 = \frac{d}{ds}(\delta_\perp \ell^2) = \frac{D}{Ds}(h_{ab}\, \delta x^a\, \delta x^b) = (\dot u_a u_b + u_a \dot u_b)\, \delta x^a\, \delta x^b + 2h_{ab}\, \delta x^a\, \frac{D}{Ds}\left(\frac{\partial x^b}{\partial y^\beta}\right) \delta y^\beta$$

which by virtue of

$$\frac{\partial}{\partial y^\alpha}\frac{\partial}{\partial s} = \frac{\partial}{\partial s}\frac{\partial}{\partial y^\alpha}$$

reduces to

$$0 = [u_{(a//b)} + \dot u_{(a}\, u_{b)}]\, \delta x^a\, \delta x^b$$

and since this must hold for every world-line C' near to C, it follows that

(3.2) $$0 = u_{(a//b)} + \dot u_{(a}\, u_{b)} = {}^\perp u_{(a//b)}$$

which is just a more explicit form of (3.1). We note that there are here six independent first order non-linear partial differential equations for the u_a.

In his discussion of the kinematics of continuous media, EHLERS [6] has given a decomposition of $u_{a//b}$ for general timelike unit vector field u_a;

(3.3) $$u_{a//b} = \omega_{ab} + \sigma_{ab} = \frac{1}{3}\,\theta h_{ab} - \alpha_a\, u_b$$

where

$\omega_{ab} = h_a{}^c\, h_b{}^d\, u_{[c//d]} = {}^\perp u_{[a//b]}$ is the angular velocity ,

$\sigma_{ab} = h_a{}^c\, h_b{}^d (u_{(c//d)} - \frac{1}{3}\,\theta h_{cd}) = {}^\perp u_{(a//b)} - \frac{1}{3}\,\theta h_{ab}$ is the shear velocity,

$\theta = u^a{}_{//a}$ is the <u>expansion velocity</u>, and

$\alpha_a = \dot{u}_a = u_{a//b}\, u^b$ is the <u>accelaration</u>.

These definitions imply that

(3.4) $$\omega_{(ab)} = \sigma_{[ab]} = \sigma^a{}_a = 0\ , \qquad \omega_{ab}\, u^b = \sigma_{ab}\, u^b = \alpha_b\, u^b = 0 \qquad .$$

Substitution of (3.3) into (3.2) shows immediately that necessary and sufficient conditions for rigidity are

$$\sigma_{ab} = \theta = 0 \qquad .$$

That is, <u>a continuous medium is kinematically rigid if and only if its velocity field is shear-free and expansion-free</u>.

Consequently, for a rigid body, $u_{a//b}$ is completely determined by the angular velocity and the acceleration :

(3.5) $$u_{a//b} = \omega_{ab} - \alpha_a\, u_b \qquad .$$

Occasionally it is convenient to make use of the <u>angular velocity vector</u>

(3.6) $$\omega^a = \frac{1}{2}\, \eta^{abcd}\, u_b\, \omega_{cd}$$

which points along the instantaneous axis of rotation. This definition implies that

$$\omega_{ab}\, \omega^b = u_b\, \omega^b = 0 \qquad .$$

The following easily derived formulae are useful in calculations : For an arbitrary medium (and in particular, for a rigid body) :

(3.7) $$\begin{aligned} \pounds u^a &= 0 \qquad , \\ \pounds u_a &= \alpha_a \qquad , \\ \pounds h_a{}^b &= \alpha_a\, u^b \qquad . \end{aligned}$$

For a rigid body

(3.8) $$\pounds h^{ab} = 2\alpha^{(a}\, u^{b)} \qquad ,$$

(3.9) $$\pounds a_b = \dot{\alpha}_b - \alpha_c\, \alpha^c\, u_b - \omega_{bc}\, \alpha^c = \perp \dot{\alpha}_b - \omega_{bc}\, \alpha^c \qquad ,$$

(3.10) $$\pounds \omega_{ab} = \dot{\omega}_{ab} + 2u_{[a}\, \omega_{b]c}\, \alpha^c = \perp \dot{\omega}_{ab} \qquad .$$

The simplification gained by the use of Lie derivatives may be understood, and exploited, in a different way, by adapting the coordinate system to the rigid body. This amounts simply to taking the parameters y^β and s as a new coordinate system (with s as the new x^4). Equations which (so far as is shown) hold only in this coordinate system will be written with $\stackrel{*}{=}$. Thus

(3.11) $u^a \stackrel{*}{=} \delta^a_4$, $u_a \stackrel{*}{=} g_{a4}$, $u_4 = g_{44} \stackrel{*}{=} -1$, $h_{a4} \stackrel{*}{=} 0$, $h^a_4 \stackrel{*}{=} 0$,

(3.12) $$\pounds T_{ab\ldots}{}^{cd\ldots} \stackrel{*}{=} \frac{\partial}{\partial x^4} T_{ab\ldots}{}^{cd\ldots} \quad .$$

From (3.11) and (3.12) it follows that the equations of rigid motion (3.1) may be written

(3.13) $$\frac{\partial}{\partial x^4} h_{\alpha\beta} \stackrel{*}{=} 0 \quad .$$

Thus in the adapted coordinate system, $h_{\alpha\beta}$ depends only on the three coordinates y^γ ; it is therefore possible to interpret (*) $h_{\alpha\beta}$ as the metric of a certain three-dimensional space $\hat{V}$, which is in fact the quotient space (space-time)/(world-lines). We write

(3.14) $$\hat{g}_{\alpha\beta} \stackrel{*}{=} h_{\alpha\beta}$$

for the metric tensor. Simple calculations show that the contravariant metric tensor and the connection of $\hat{V}$ are given by

(3.15) $$\hat{g}^{\alpha\beta} \stackrel{*}{=} g^{\alpha\beta}$$

and by

(3.16) $$\hat{\Gamma}^\gamma_{\alpha\beta} \stackrel{*}{=} \Gamma^\gamma_{\alpha\beta} - g^{\gamma\delta}[2u_{(\alpha}\,\omega_{\beta)\delta} + u_\alpha\, u_\beta\, \alpha_\delta]$$

respectively, where $\Gamma^\gamma_{\alpha\beta}$ are some of the components of the space-time connection Γ^c_{ab} .

The adapted coordinate system and the quotient space $\hat{V}$ may be used to derive covariant equations from covariant equations with rather little calculation. For example, let z_a be a vector field orthogonal to u^a whose Lie derivative along each world-line vanishes :

(3.17) $$z_a\, u^a = 0 \ , \qquad \pounds z_a = 0 \quad .$$

In the adapted coordinate system, these conditions become

(3.18) $$z_4 \stackrel{*}{=} 0 \ , \qquad \frac{\partial}{\partial x^4} z_\alpha \stackrel{*}{=} 0 \quad ,$$

so that z_α may be interpreted as a vector field of $\hat{V}$. Let $\hat{\nabla}_\beta$ denote the operator of covariant differentiation in V . A short calculation shows that

(3.19) $$\hat{\nabla}_\beta\, z_\alpha \stackrel{*}{=} \perp \nabla_\beta\, z_\alpha \quad .$$

However it follows immediately from (3.18) that

(*) We are indebted to C. B. RAYNER for suggesting this approach.

$$\frac{\partial}{\partial x^4} \hat{\nabla}_\beta z_\alpha \overset{*}{=} 0 \quad .$$

From (3.12) and (3.19) this may be written

(3.20) $$\pounds \perp \nabla_\beta z_\alpha \overset{*}{=} 0 \quad .$$

However $h^a_4 \overset{*}{=} 0$, so that any expression of the form $\perp T_{ab\ldots}$ vanishes identically in the adapted coordinate system whenever one or more indices take the value 4. Therefore (3.20) is equivalent to

$$\pounds \perp \nabla_b z_a \overset{*}{=} 0 \quad ,$$

but this is a tensor equation, and therefore valid in every coordinate system :

(3.21) $$\pounds \perp \nabla_b z_a = 0 \quad .$$

The deduction of (3.21) from (3.17) may also be carried out directly, but requires more work. We shall use this easier method again in the next section.

4. Integrability conditions.

We now suppose a particular Riemannian space-time to be given, so that g_{ab}, Γ^c_{ab} and R_{abcd} and its derivatives are known, and seek integrability conditions for the equations of rigid motion (3.1) or (3.2). From the point of view of the quotient space, just developed, this amounts to finding the conditions which must be satisfied in a general coordinate system in order for there to exist an adapted coordinate system in which $\frac{\partial}{\partial x^4} h_{\alpha\beta} = 0$.

We cannot solve equations (3.2) algebraically for the first (partial or covariant) derivatives of the u_a, because there are only six independent equations for twelve independent derivatives. However, if we adjoin additional variables, namely the α_a and the ω_{ab}, to the u_a, we can then derive a set of equations which may be solved to give the first derivatives of all these variables in terms of the variables themselves. Integrability conditions for this larger system may then be derived in the usual way. Let us call the variables u_a, α_{az} and ω_{ab} the "selected variables". We first observe that equation (3.5) :

(4.1) $$u_{a//b} = \omega_{ab} - \alpha_a u_b$$

already gives $u_{a//b}$ algebraically in terms of the selected variables. The next step is to calculate $\alpha_{b//c}$ and $\omega_{ab//c}$. Expansion of $\alpha_{b//c} = (u_{b//k} u^k)_{//c}$ and use of the Ricci identity (2.1) quickly yields

(4.2) $$\alpha_{b//c} = \dot{\omega}_{bc} - \dot{\alpha}_b u_c - \alpha_b \alpha_c + \omega_b{}^k \omega_{kc} - \omega_{bk} \alpha^k u_c - R_{bhck} u^h u^k$$

and in a similar way, with the help of the identity

$$\omega_{ab//c} = 3\omega_{[ab//c]} + 2\omega_{c[b//a]}$$

one finds

(4.3) $\omega_{ab//c} = -\overset{\circ}{\omega}_{ab}\, u_c + 2\alpha_{[a}\omega_{b]c} - \omega_{ab}\,\alpha_c - 2u_{[a}\omega_{b]}{}^{k}\,\omega_{kc} - \perp R_{abck}\, u^k$.

Equation (4.2) may also be derived by setting $\xi^a = u^a$, $z_b = u_b$ in equation (2.5).

The task would now be completed, but for the appearance of $\overset{\circ}{\alpha}_b$ on the right hand side of (4.2), and $\overset{\bullet}{\omega}_{bc}$ on the right hand sides of both (4.2) and (4.3).

It turns out that these unwanted derivatives may be eliminated with the help of the first and second integrability conditions of (4.3) itself. Calculation of $\alpha_{b//[cd]}$ shows that the integrability conditions for (4.2) are satisfied identically in virtue of (4.1), (4.2) and (4.3). Calculation of $\omega_{ab//[cd]}$, on the other hand, yields after some work

(4.4) $$\mathcal{L}(3\omega_{ab}\,\omega_{cd} + \perp R_{abcd}) = 0$$.

This equation may be derived comparatively easily in two other ways :

(i) by projecting the identity (2.8) with $h_a{}^b$ on all indices (and replacing ξ^a by u^a) ; or

(ii) by differentiation of (3.19), which yields $\hat{\nabla}_\gamma\,\hat{\nabla}_\beta\, z_\alpha \overset{*}{=} \perp\nabla_\gamma(\perp\nabla_\beta\, z_\alpha)$, and, on antisymmetrization in β , γ

(4.5) $$\hat{R}_{\alpha\beta\gamma\delta} \overset{*}{=} \perp(3\omega_{\alpha\beta}\,\omega_{\gamma\delta} + R_{\alpha\beta\gamma\delta})$$

(where $\hat{R}_{\alpha\beta\gamma\delta}$ is the Riemann tensor of $\hat{V}$), since z_α is arbitrary at each point of $\hat{V}$. Equation (4.5) now leads to (4.4) by the argument which led from (3.19) to (3.21). From this point of view we observe that (4.4) is equivalent to equation (2.5) of [12].

We may now obtain $\overset{\bullet}{\omega}_{ab}$ algebraically in terms of the selected variables from (4.4). Contraction of (4.4) with h_{bc} and h^{ad} yields in turn

(4.6) $$\mathcal{L}(3\omega_a{}^b\,\omega_{bd} + \perp R_{abcd}\, h^{bc}) = 0$$

and

(4.7) $$\mathcal{L}(3\omega^{ab}\,\omega_{ab} - R_{abcd}\, h^{ad}\, h^{bc}) = 0$$,

whence

(4.8) $$\omega^{ab}\,\mathcal{L}\omega_{ab} = \frac{1}{6}\,\mathcal{L}h^{ad}\, h^{bc}\, R_{abcd} = \frac{1}{3}\,\mathcal{L}G_{bd}\, u^b\, u^d$$.

On the other hand, contraction of (4.4) with ω^{ab} yields

(4.9) $$\omega^{ab}\,\omega_{ab}\,\pounds\omega_{cd} + \omega_{cd}\,\omega^{ab}\,\pounds\omega_{ab} = -\tfrac{1}{3}\,\omega^{ab}\,\pounds\perp R_{abcd} \quad .$$

In the rest of this section, we assume that

$$\omega_{ab} \not\equiv 0$$

and define the (positive) magnitude ω of the angular velocity by

(4.10) $$\omega_{ab}\,\omega^{ab} = 2\omega^2 \quad .$$

Then from (4.8) and (4.9) it follows that

(4.11) $$\pounds\omega_{cd} = (6\omega^2)^{-1}[\omega_{cd}\,\pounds G_{bd}\,u^b u^d - \omega^{ab}\,\pounds\perp R_{abcd}]$$

and from (3.10) we may express $\dot{\omega}_{ab}$ algebraically in terms of the selected variables, so that equation (4.3) may now be written

(4.12) $$\omega_{ab//c} = [-(6\omega^2)^{-1}\,\omega_{ab}\,\pounds G_{hk}\,u^h u^k + (6\omega^2)^{-1}\,\omega^{pq}\,\pounds\perp R_{abpq}]\,u_c$$
$$+ 2u_{[a}\omega_{b]k}\,\alpha^k u_c + 2\alpha_{[a}\omega_{b]c} - \omega_{ab}\,\alpha_c - 2u_{[a}\omega_{b]}{}^k\,\omega_{kc} - \perp R_{abck}\,u^k$$

giving $\omega_{ab//c}$ algebraically in terms of the selected variables. To express (4.2) in similar form, it is necessary to eliminate $\dot{\alpha}_b$. This may be done by differentiating (4.4). We have seen already ((3.17), (3.21)) that

$$z_a\,u^a = \pounds z_a = 0 \quad \text{implies} \quad \pounds\perp\nabla_b\,z_a = 0 \quad .$$

It follows from this, by application of Leibnitz's rule that (4.4) implies

(4.13) $$\pounds\perp(3\omega_{ab}\,\omega_{cd} + \perp R_{abcd})_{//e} = 0 \quad .$$

This equation, and its sequels,

$$\pounds\perp[\perp(3\omega_{ab}\,\omega_{cd} + \perp R_{abcd})_{//e}]_{//f} = 0$$

$$\pounds\perp[\perp\{\perp(3\omega_{ab}\,\omega_{cd} + \perp R_{abcd})_{//e}\}_{//f}]_{//g} = 0$$

and so on, seem to be hitherto unknown consequences of the equations of rigid motion (3.1). They are equivalent to the equations

$$\frac{\partial}{\partial x^4}\hat{\nabla}_\varepsilon R_{\alpha\beta\gamma\delta} \overset{*}{=} 0 , \qquad \frac{\partial}{\partial x^4}\hat{\nabla}_\zeta\hat{\nabla}_\varepsilon R_{\alpha\beta\gamma\delta} \overset{*}{=} 0 , \qquad \frac{\partial}{\partial x^4}\hat{\nabla}_\eta\hat{\nabla}_\zeta\hat{\nabla}_\varepsilon R_{\alpha\beta\gamma\delta} \overset{*}{=} 0 , \ldots$$

and so on.

We need

(4.14) $$\perp\omega_{ab//e} = 2\alpha_{[a}\omega_{b]e} - \omega_{ab}\,\alpha_e - \perp R_{abek}\,u^k \quad ,$$

which follows from (4.12) and

$$(4.15)\quad \perp(\perp R_{abcd})_{//e} = \perp(h_a{}^p\, h_b{}^q\, h_c{}^r\, h_d{}^s\, R_{pqrs})_{//e}$$
$$= \perp R_{abcd//e} + \omega_{ae} \perp R_{pbcd}\, u^p + \omega_{be} \perp R_{apcd}\, u^p$$
$$+ \omega_{ce} \perp R_{abpd}\, u^p + \omega_{de} \perp R_{abcp}\, u^p \quad .$$

Substitution from (4.14) and (4.15) into (4.13) yields

$$(4.16)\quad \pounds \perp [3(2\alpha_{[a}\omega_{b]e} - \omega_{ab}\,\alpha_e - R_{abek}\,u^k)\,\omega_{cd} + 3\omega_{ab}(2\alpha_{[c}\omega_{d]e} - \omega_{cd}\,\alpha_e - R_{cdek}\,u^k)$$
$$+ 2\omega_{e[a}R_{b]pcd}\,u^p + 2\omega_{e[c}R_{d]pab}\,u^p + R_{abcd//e}] = 0 .$$

If this equation is contracted with $h^{ac}\, h^{bd}$, it may, with the aid of (4.4) be solved for $\pounds\alpha_e$ in the form

$$(4.17)\quad \pounds\alpha_e = (12\omega^2)^{-1}[\delta_e^h + (2\omega^2)^{-1}\,\omega^h{}_m\,\omega^m{}_e][4\alpha^c\,\pounds \perp R_{bcdh}\,h^{bd} + 4\alpha_h\,\pounds G_{bd}\,u^b\,u^d$$
$$- 6\omega^{ab}\,\pounds \perp R_{abhk}\,u^k + 4\omega^a{}_h\,\pounds \perp R_{ka}\,u^k + 2\pounds \perp G_{bd//h}\,u^b\,u^d$$
$$- \perp R^{ab}{}_{hk}\,u^k\{(\omega^2)^{-1}\,\omega_{ab}\,\pounds G_{pq}\,u^p\,u^q - (\omega^2)^{-1}\,\omega^{rs}\,\pounds \perp R_{abrs}\}$$
$$+ \perp \tfrac{2}{3} R_k{}^a\,u^k\{(\omega^2)^{-1}\,\omega_{ah}\,\pounds G_{pq}\,u^p\,u^q - (\omega^2)^{-1}\,\omega^{rs}\,\pounds \perp R_{ahrs}\}] .$$

Substitution from (4.11) and (4.17) into (4.2) now yields $\alpha_{b//c}$ algebraically in terms of the selected variables :

$$(4.18)\quad \alpha_{b//c} = (6\omega^2)^{-1}[\omega_{bc}\,\pounds G_{hk}\,u^h\,u^k - \omega^{hk}\,\pounds \perp R_{bchk}] - u_c[\pounds\alpha_b + u_b\,\alpha_h\,\alpha^h + \omega_{bh}\,\alpha^h]$$
$$- \alpha_b\,\alpha_c + \omega_b{}^h\,\omega_{hc} - \omega_{bh}\,\alpha^h\,u_c - R_{bhck}\,u^h u^k .$$

We now have $u_{a//b}$, $\alpha_{b//c}$ and $\omega_{ab//c}$ expressed algebraically in terms of u_a, α_b and ω_{ab}, with coefficients which are functions of g_{ab} and its derivatives. Calculation of further integrability conditions will lead back to (4.4) and to its derivatives, starting with (4.13).

5. Consequences of the integrability conditions.

a. Angular velocity of a heavy body and of a test body. - RAYNER has shown [9] that the angular velocity of a heavy rigid body must be a constant. His result follows easily from (4.7), which may be written in the form

$$(5.1)\quad \pounds(3\omega^2 + G_{ab}\,u^a\,u^b) = 0 \quad .$$

The condition for a heavy body is simply

$$(5.2)\quad G_{ab}\,u^b = \rho u_a$$

where ρ is the density. From the contracted Bianchi identity $G^{ab}{}_{//b} = 0$ it follows that in the case of a rigid body

$$\pounds\rho = \dot\rho = \rho_{/a}\, u^a = (\rho u^a)_{//a} = (G^{ab}\, u_b)_{//a} = G^{ab}\, u_{b//a} = -\,2G^{ab}\,\alpha_{(b}u_{a)}$$
$$= -\,\rho u^b\,\alpha_b = 0 \quad ,$$

while from (5.1) and (5.2),

(5.3) $$\pounds(3\omega^2 - \rho) = 0$$

so that

$$\dot\omega = 0 \quad .$$

For a rigid test body the same result may be deduced immediately from (5.1).

b. <u>The Herglotz-Noether theorem.</u> - Every isometric motion of space-time with timelike generators is also a rigid motion, although the converse is not true. This is easily seen from Killing's equations for an isometric motion

(5.4) $$\underset{\xi}{\pounds}\, g_{ab} = 0 \quad .$$

If now $\xi_a = \xi u_a$, where $u_a\, u^a = -1$, these equations imply

(5.5) $$\xi_{/(a}u_{b)} + \xi u_{(a//b)} = 0$$

and contraction of this equation with u^b yields

$$\dot\xi u_a - \xi_{/a} + \xi\alpha_a = 0 \quad .$$

Contraction with u^a in turn yields

$$\dot\xi = 0$$

so that

(5.6) $$\alpha_a = \xi^{-1}\,\xi_{/a} = (\log\xi)_{/a}$$

and so (5.5) becomes $\xi(\alpha_{(a}u_{b)} + u_{(a//b)}) = 0$ which is precisely the set of equations of rigid motion, since $\xi \neq 0$.

However, it follows from (5.6) that, for an isometric motion,

(5.7) $$\alpha_{[a//b]} = 0 \quad ,$$

and from the manner of derivation it follows that this is evidently the sufficient as well as the necessary condition that a rigid motion be isometric. Equation (5.7) does not hold for all rigid motions, but HERGLOTZ [2] and independently NOETHER [9], proved that :

<u>In flat space-time, every rotating rigid motion is isometric.</u>

This may easily be proved as a consequence of the integrability conditions (4.4) and (4.13), as follows :

In flat space-time, the integrability conditions (4.4) reduce to

$$\pounds\omega_{ab}\,\omega_{cd} = 0$$

and contraction with ω^{ab}, then with ω^{cd}, quickly yields

(5.8) $$\pounds\omega_{cd} = 0 \quad .$$

The integrability conditions for this equation, which may also be found directly from (4.13), are

$$\pounds \perp (\omega_{cd//e}) = 0$$

or, from (4.3),

$$\pounds(\alpha_c\,\omega_{de} + 2\alpha_{(d}\omega_{e)c}) = 0 \quad .$$

Antisymmetrization on d and e yields, with the help of (5.8), the equation

(5.9) $$\omega_{de}\,\pounds\alpha_c = 0 \quad ,$$

but (4.2) implies that

$$\alpha_{[a//b]} = \pounds\omega_{ab} + u_{[a}\pounds\alpha_{b]}$$

and from (5.8) and (5.9) it now follows that

$$\omega_{de}\,\alpha_{[a//b]} = 0$$

which, in view of (5.7), completes the proof.

c. Transformations preserving the equations of rigid motion. - The question, whether the Herglotz-Noether theorem holds in an arbitrary space-time, may be answered in the negative, in the limited sense that there exist non-isometric rotating rigid congruences of (timelike) curves, representing rigid bodies moving through media. We have not yet been able to settle the more interesting question, whether there are non-isometric motions of rigid heavy bodies or of rigid test bodies in vacuo. The other question is settled by carrying out the following transformation, which preserves the rigid property while spoiling the isometry :

Let u^a be a (unit) vector field satisfying the equations of rigid motion, and let s_a be a unit vector field everywhere orthogonal to u^a (and therefore necessarily spacelike). Then the transformation

$$g_{ab} \to \bar{g}_{ab} = g_{ab} + xu_a\,u_b + 2yu_{(a}s_{b)} - (1-x)^{-1}\,y^2\,s_a\,s_b$$

$$u^a \to \bar{u}^a = (1-x)^{-1/2}\,u^a$$

$$s_a \to \bar{s}_a = s_a \quad ,$$

where x (< 1) and y are arbitrary functions of position, is easily seen to preserve the equations of rigid motion, but not the conditions for an isometric motion. Such a transformation may be carried out without annihilating the angular velocity. Unfortunately the transformation of the Ricci tensor is in general very complicated.

d. Non-rotation rigid bodies. - If $\omega_{ab} = 0$ then the world-lines of the body admit a family of orthogonal hypersurfaces, and further simplification may be achieved in the adaptation of the coordinate system to the congruence, by taking the hypersurfaces to be surfaces $x^4 = \text{Cte}$, as well as choosing the world-lines as the lines of x^4. The equations of rigid motion reduce to

$$g_{\alpha\beta 14} = 0 \qquad (\alpha , \beta = 1 , 2 , 3)$$

and the metric may be taken in the form

$$ds^2 = g_{44}(x^\alpha , t)\, dt^2 + g_{\alpha\beta}(x^\gamma)\, dx^\alpha\, dx^\beta \qquad .$$

The motion is isometric if and only if g_{44} is of the form

$$g_{44} = T(t)\, g'_{44}(x^\alpha) \qquad ,$$

in which case $T(t)$ may be absorbed and the metric is static. A calculation, or inspection of equation (4.3), shows that the conformal tensor must be of Petrov type I or D , so that space-times of conformal types II , III and N cannot admit non-rotating rigid bodies. That this is no longer true for rotating bodies follows from the existence of the anti-Mach metric of OSZVATH and SCHÜCKING [8], which is a type N vacuum metric admitting a timelike isometric motion.

Acknowledgements.

We are grateful to R. H. BOYER and C. B. RAYNER for valuable discussions and criticism of the manuscript. This work was carried out during the tenure by one of us (G. W.) of a D. S. I. R. Research Studentship.

BIBLIOGRAPHY

[1] BORN (Max). - Die Theorie des starren Elektrons in der Kinematik des Relativitätsprinzips, Annalen der Physik, t. 30, 1909, p. 1-56.

[2] HERGLOTZ (G.). - Über den vom Standpunkt des Relativitäts-prinzips aus als "starr" zu bezeichnenden Körper, Annalen der Physik, t. 31, 1910, p. 393-415.

[3] JORDAN (P.), EHLERS (J.) und KUNDT (W.). - Strenge Lösungen der Feldgleichungen der Allgemeinen Relativitätstheorie, I., Akad. Wiss. Lit., Mainz, Abh. math.-naturw. Kl., 1960, p. 21-105.

[4] JORDAN (P.), EHLERS (J.) und SACHS (R. K.). - Strenge Lösungen der Feldgleichungen der Allgemeinen Relativitätstheorie, II., Akad. Wiss. Lit., Mainz, Abh. math.-naturw. Kl., 1961, p. 1-62.

[5] JORDAN (P.) und KUNDT (W.). - Strenge Lösungen der Feldgleichungen der Allgemeinen Relativitätstheorie, III., Akad. Wiss. Lit., Mainz, Abh. math.-naturw. Kl., 1961, p. 85-97.

[6] JORDAN (P.), EHLERS (J.), KUNDT (W.) und SACHS (R. K.). - Strenge Lösungen der Feldgleichungen der Allgemeinen Relativitätstheorie, IV., Akad. Wiss. Lit., Mainz, Abh. math.-naturw. Kl., 1961, p. 791-837.

[7] NOETHER (Fritz). - Zur Kinematik des starren Körpers in der Relativtheorie, Annalen der Physik, t. 31, 1910, p. 919-944.

[8] OSZVATH (M.) and SCHÜCKING (E.). - Recent developments in general relativity.

[9] RAYNER (Charles B.). - Mouvement rigide en relativité générale, C. R. Acad. Sc. Paris, t. 248, 1959, p. 929-932.

[10] RAYNER (Charles B.). - Sur une solution générale des équations intérieures d'Einstein pour un mouvement de groupe, C. R. Acad. Sc. Paris, t. 248, 1959, p. 2725-2727.

[11] RAYNER (Charles B.). - Trois remarques concernant un théorème récent, C. R. Acad. Sc. Paris, t. 249, 1959, p. 1327-1328.

[12] RAYNER (Charles B.). - Une forme simple pour le tenseur de Ricci sous conditions de rigidité, C. R. Acad. Sc. Paris, t. 249, 1959, p. 1461-1463.

[13] RAYNER (Charles B.). - Sur une simplification des équations extérieures d'Einstein pour un mouvement de groupe, C. R. Acad. Sc. Paris, t. 249, 1959, p. 1614-1616.

[14] RAYNER (Charles B.). - Le concept d'un mouvement rapide, Atti Accad. naz. Lincei, Rendiconti, Cl. Sc. fis. mat. nat., Série 8, 1959, t. 26, 1959, p. 478-483.

[15] SALZMAN (G.) and TAUB (A. H.). - Born-type rigid motion in relativity, Phys. Rev., t. 95, 1954, p. 1659-1669.

[16] SCHOUTEN (J. A.). - Ricci-calculus, 2nd edition. - Berlin, Springer-Verlag, 1954 (Grundlehren der mathematischen Wissenschaften, 10).

[17] SYNGE (J. L.). - Seminar at the Institute for advanced Studies of Dublin (unpublished).

[18] SYNGE (J. L.). - Relativity, the special theory. - Amsterdam, North-Holland publishing Company, 1960.

[19] YANO (Kentaro). - The theory of Lie derivatives and its applications. - Amsterdam, North-Holland publishing Company, 1955 (Bibliotheca Mathematica, 3).