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A LOCAL FIXED POINT INDEX FOR COMPACT MAPS WITH NON COMPACT DOMAINS

by Ronald J. KNILL

The local fixed point index of an isolated compact set of fixed points of a map f was developed by Leray [2] whenever the domain and range of f is a convexoid space. This is well known and without getting involved in the precise context it may be said that the local fixed point index is usually not directly obtainable from its definition. Leray gave a number of properties whose function was to aid in the computation of the index in specific situations. One of the most useful of these properties is that the Leray index coincides with the Lefschetz index whenever both are defined.

Recently Fuller [3] has defined an index of an isolated compact set of periodic orbits of a C^{∞} dynamical system [5] but he has observed certain obstacles to the existence of a Leftschetz formula for his index [4].

It is the purpose of this paper to describe an index which includes both the Leray and Fuller indices and for which there is a Lefschetz formula. The index to be described is a floating index, taking its values not in the integers, but in a group which depends contravariantly on the domain, covariantly on the range, of the map f being studied. An important problem is the identification of this group, which is simple in the case of dynamical systems. We will only briefly give here the properties of the floating index, with neither a specific definition nor proofs. These will appear in full in a longer version of this paper. The reader should however be able to determine for himself the validity of the Lefschetz formula given here, in the case of simplicial maps.

We generalize the usual data given for a dynamical system as follows. Suppose given

$$(P, X, O_1 \times O_2, W)$$

where P and X are regular Hausdorff spaces $O_1 \times O_2$ is an open subset of $P \times X$, and W is a compact subset of X . Let $C\ell(O_1 \times O_2)$ denote the closure of $O_1 \times O_2$ and suppose there is given a continuous function

$$f : C\ell(O_1 \times O_2) \to W .$$

Then let

$$F(f) = \{(\tau, x) \in C\ell(O_1 \times O_2) : f(\tau, x) = x\} .$$

We say that $F(f)$ is nondegenerate if it is a compact subset of $O_1 \times O_2$. Let H^* and H_* denote the Čech cohomology and homology functors respectively, with coefficients in a field K. Let $H^*_c(O_1) = H^*(P, P\backslash O_1)$.

THEOREM 1. <u>There exists a functor</u> J_* <u>which is contravariant in</u> $(P-O) \times X$ <u>and covariant in</u> W, $J_*(O_1 \times X;W)$ <u>is a graded</u> $H^*(P \times X)$ <u>module and there is a natural module homomorphism which preserves total degree</u> (<u>where total degree</u> $H^k \otimes H_\ell = \ell - k$),

$$\varepsilon_* : H^*_c(O_1) \otimes H_*(W) \to J_*(O_1 \times X, W) .$$

If $F(f)$ is nondegenerate there is a floating index of $F(f)$ in $J_o(O_1 \times X;W)$ which we denote by

$$i\{f(\tau,x) = x; O_1 \times O_2\} .$$

If this index is not zero then $F(f)$ is non empty. Furthermore it has the following properties.

<u>Homotopy independence</u>. If

$$f_t : Cl(O_1 \times O_2) \to W \qquad (0 \leq t \leq 1)$$

is a homotopy such that the union of all the sets $F(f_t)$ is a compact subset of $O_1 \times O_2$ then

$$i\{f_o(\tau,x) = x; O_1 \times O_2\} = i\{f_1(\tau,x) = x; O_1 \times O_2\} .$$

<u>Additivity</u>. If $O_1 \times O_2$ is the disjoint union of open sets $O_i \times O_j$, where i and j range over finite index sets, then

$$i\{f(\tau,x) = x; O_1 \times O_2\} = \sum_{i,j} i\{f(\tau,x) = x; O_i \times O_j\} .$$

<u>Commutativity</u>. Given the data $(P', X', O_1' \times O_2', W')$ and given continuous functions

$$h : P' \to P , \qquad r : X' \to X , \qquad j : W \to W'$$

subject to the conditions $h^{-1} O_1 = O_1'$, $r^{-1} O_2 = O_2'$ and $rj(w) = w$ for every $w \in W$, then let

$$f' = j \circ f \circ (h \times r) .$$

Then the homomorphism $J_*(h \times r;j)$ induced by $(h \times r;j)$ preserves indices :

$$J_*(h \times r;j) i\{f(\tau,x) = x; O_1 \times O_2\} = i\{f'(\tau',x') = x', O_1' \times O_2'\} .$$

<u>Agreement with the Lefschetz formula</u>. If P is compact and $P \times X = O_1 \times O_2$, let

$$L_m(f) \in H^m(P) \otimes H_m(W)$$

be the m^{th} Lefschetz index of f (as defined in the sequel). Then

$$\varepsilon_* \Sigma_m L_m(f) = i\{f(\tau,x) = x; O_1 \times O_2\} .$$

Definition of $L_m(f)$. This is defined only in the case when the induced cohomology homomorphism

$$H^*(f) : H^*(W) \to H^*(P \times X)$$

has finite rank. This is assumed in the statement of "Agreement with the Lefschetz formula". In this case $H^*(f)$ may be regarded in the obvious way as an element

$$H^*(f) = \sum_{i,j,m,n} Z^m_{ij} \otimes Z^{n-m}_j \otimes Z^i_n \in H^*(P) \otimes H^*(X) \otimes H_*(W) .$$

Let the cap product pairing be denoted by

$$\cap : H^*(X) \otimes H_*(W) \to H_*(W) .$$

Then define $L_m(f)$ by the formula

$$L_m(f) = \sum_{i,j,n} Z^m_{ij} \otimes (Z^{n-m}_j \cap Z^i_n) , \quad m \geqq 0 .$$

COROLLARY. If $P = 0$, is compact, $X = O_2$, and if $H^*(f)$ has finite rank, if ε_* is a monomorphism and it $L_m(f) \neq 0$ for some $m \geqq 0$, then $F(f)$ is non empty.

Comment. Suppose that P is a point. Then $L_o(f)$ is the classical Lefschetz index. Thus the previous corollary includes the classical Lefschetz fixed point theorem and in particular the latter holds whenever ε_* is a monomorphism. This turns out to be the case in significantly more cases than when ε_* is an isomorphism. On the other hand if X is one of the classical counter-examples such as Borsuk's [1], then $\varepsilon_* = 0$.

THEOREM 2. If G is a connected compact n-dimensional Lie Group and $f : G \times G \to G$ is the group multiplication, then

$$L_n(f) = Z^n \otimes Z_n$$

where Z^n and Z_n are dual generators of $H^n(G)$ and $H_n(G)$.

By taking $G = S^1$ = circle group one may derive the Fuller index using the commutativity property :

THEOREM 3. Let $f : R^1 \times M \to M$ be a (C^∞) dynamical system with an isolated periodic orbit through a point x_o of period $p > 0$. Let O_1 be an ε-neighborhood of p, let O_2 be a small tubular neighborhood of the orbit of f at x_o. We may take W to be of the homotopy type of S^1.

Then

$$\varepsilon_* : H^*_c(O_1) \otimes H_*(W) \to H_*(O_1 \times M;W)$$

is an isomorphism. The zero degree term on the left is $K = H^1_c(O_1) \otimes H_1(W)$, so identify K with $H_0(O_1 \times M;W)$ by means of ε_*. If T is the Poincaré map of a local section of f at x_0 [4], then the local fixed point index of T in a neighborhood of x_0 is

$$i\{f(\tau,x) = x; O_1 \times O_2\} .$$

THEOREM 4. _The natural transformation_

$$\varepsilon_* : H_c^*(O_1) \otimes H_*(W) \to J_*(O_1 \times X;W)$$

is an isomorphism if either

(i) $P \times X$ _is locally compact and_ X _is a finite dimensional cohomology manifold_ ;

(ii) P _and_ X _are possibly infinite dimensional topological manifolds modelled on locally convex topological vector spaces_ ;

(iii) $P \times X$ _is paracompact and_ P, X _are neighborhood retracts of spaces of types_ (i) _or_ (ii).

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