Séminaire Jean Leray. Sur les équations aux dérivées partielles

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Holomorphic semi-groups

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§ 1. The Theorem. Let $X$ be a locally convex, sequentially complete, linear topological space, and let $L(X,X)$ be the set of all linear continuous mappings defined on $X$ with values in $X$.

A system $\{T_t; t \geq 0\}$ of mappings $T_t \in L(X,X)$ is called an equi-continuous semi-group of class $(C_0)$ if:

1. $T_t T_s = T_{t+s}$, $T_0 = I$ (the identity),
2. $\lim_{t \to t_0} T_{t} x = T_{t_0} x$ for every $t_0 \geq 0$ and $x \in X$,
3. for every continuous semi-norm $p$ in $X$, there exists a continuous semi-norm $q$ in $X$ such that $p(T_t x) \leq q(x)$ for all $t \geq 0$ and $x \in X$.

We shall prove the following

Theorem. The three propositions (I), (II) and (III) given below are mutually equivalent:

I. For every $t > 0$ and $x \in X$, $T_t' x = \lim_{h \to 0} \frac{T_{t+h} - T_t}{h} x$

exists, and, for a suitable positive constant $C$, the system of mappings

$\{(C t T_t')^n; 1 \geq t > 0, n = 0, 1, 2, \ldots\}$

is equi-continuous.
(II) The Taylor expansion
\[ T_\lambda x = \sum_{n=0}^{\infty} \frac{|\lambda - t|^n}{n!} T^{(n)}_t x \]
converges for every \( x \in X \) and every complex number \( \lambda \) with \( |\arg \lambda| < \tan^{-1}(C\varepsilon^{-1}) \), in such a way that the system of mappings \( \{ e^{-\lambda T_t} ; |\arg \lambda| < \tan^{-1}(C\varepsilon^{-1}) \} \) is, for a certain \( k > 0 \), equi-continuous.

(III) The infinitesimal generator \( A \) of \( T_t \) defined by
\[ A x = \lim_{t \downarrow 0} \frac{T_t x - I}{t} \]
satisfies the condition that the resolvent \( (\lambda I - A)^{-1} \) exists as a mapping \( \in L(X, X) \) for \( \Re(\lambda) > 0 \) and the system of mappings \( \{ [C_1 \lambda (\lambda I - A)^{-1}]^n ; \Re(\lambda) \geq 1, \ n = 0, 1, 2, \ldots \} \) is equi-continuous for a certain positive constant \( C_1 \).

§ 2. The Sketch of the Proof of the Theorem. We first recall known facts concerning equi-continuous semi-groups \( T_t \) of class \( (C_0) \):

The domain \( D(A) \) of \( A \) is dense in \( X \); for every \( \lambda \) with \( \Re(\lambda) > 0 \), the resolvent \( (\lambda I - A)^{-1} \) exists and \( \in L(X, X) \); \( (\lambda I - A)^{-1} x = \int_0^\infty e^{-\lambda t} T_t x dt \) for every \( x \in X \) and \( \lambda \) with \( \Re(\lambda) > 0 \); the system of operators \( \{ [\sigma (\sigma I - A)^{-1}]^m ; \sigma > 0, \ m = 0, 1, 2, \ldots \} \) is equi-continuous; for every \( x \in D(A) \), we have
\[ \frac{dT_t x}{dt} = A T_t x = T_t A x, \quad t \geq 0, \]
concerning \( T_t^{(n)} = (T_t^{(n-1)})_t \), we have the Lemma. Let \( T_t x \in D(A) \) for every \( t > 0 \) and \( x \in X \). Then \( T_t x \) is infinitely differentiable in \( t \) and
Proof. For any $t > t_0$ with $t > t_0$, we have, by (1) and (4),
\[ T_t x = AT_t x = T_{t-t_0} AT_{t_0} x. \]
Hence
\[ T_t^n x = (T_{t-t_0})^n x = (AT_{t_0})^n x = AT_{t_0} x = A T_{t_0} x = A T_{t_0} x = (T_{t_0}^2)^n x. \]
We have only to repeat the same reasoning to obtain (5).

**Proof of the Theorem**

(1) implies (II). By the equi-continuity of $\left\{ [(CT_t)^n]_{t > 0}, 1 > t > 0 \right\}$, we obtain, remembering (5),
\[ p_{(\lambda-t)^n} T_t x < \left| \frac{\lambda-t}{\lambda} \right|^n \frac{1}{t^n} \sum_{k=0}^{n-1} \frac{1}{\lambda^n} p(\frac{t}{n}) CT_t^n x < \left| \frac{\lambda-t}{\lambda} \right| c^{-1} e^n q(x). \]
Hence the first part of (II) is proved.

Next consider the semi-group $\{ S_t \}$ defined by
\[ S_t = e^{-t} T_t. \]
Then
\[ tS_t = te^{-t} T_t - te^{-t} T_t. \]
Thus remembering that $0 < te^{-t} < 1$ for $t > 0$, we easily see that
\[ \left\{ (2^{-k} tS_t)^n ; t > 0, n = 0,1,2,... \right\} \]
is with a certain $k > 0$, equi-continuous. Hence, as above, we see that $e^{-\lambda T_\lambda}$, which is an holomorphic extension of $S_t$, satisfies the condition that
\[ \left\{ e^{-\lambda T_\lambda} ; |\arg \lambda | < \tan^{-1} \left( \frac{c}{\lambda} \right) \right\} \]
is equi-continuous.

(II) implies (III). Differentiating \( (\lambda I - A)^{-1} = \int_0^\infty e^{-\lambda t} T_t dt \) with respect to $\lambda$, we obtain, for $\lambda = \sigma + i \tau$ with $\sigma > 0$,
\[ [(\sigma + 1 + i\tau)(\sigma + 1 + i\tau) - A]^{-1} x \]
\[ = \frac{(\sigma + 1 + i\tau)^{n+1}}{ln} \int_0^\infty e^{-(\sigma + 1 + i\tau)} T^n S_t x dt, x \in X. \]
Let \( \tau < 0 \). Then, by Cauchy's integral theorem, we obtain, for
\[
0 < \Theta < \tan^{-1}\left(\frac{\cos^{-1}\Theta}{2}\right),
\]

\[
\left[\left((\sigma + 1 + i \tau)(\sigma + 1 + i \tau) I - A\right)^{-1}\right]^{n+1} = 
\frac{(\sigma + 1 + i \tau)^{n+1}}{\ln} \int_{0}^{\infty} e^{-(\sigma + i \tau) r} e^{i \Theta} r^n e^{i \Theta} dr.
\]

Hence, by the equi-continuity of \( \{S_{i \Theta} : \tau \gg 0\} \), we have
\[
p \left(\left[(\sigma + 1 + i \tau)(\sigma + 1 + i \tau) I - A\right)^{-1}\right]^{n+1}\right) 
\leq q(x) \cdot \frac{|\sigma + 1 + i \tau|^{n+1}}{\ln} \int_{0}^{\infty} e^{-(\sigma \cos \Theta - \tau \sin \Theta) r} r^n dr
\]
\[
= q(x) \cdot \frac{|\sigma + 1 + i \tau|^{n+1}}{|\tau \sin \Theta - \sigma \cos \Theta|^{n+1}}
\]

since \( 0 < \Theta < \frac{\pi}{2} \), \( \tau < 0 \) and \( \sigma \gg 0 \), we easily see that the second factor on the right is \( \leq C_1^n \).

We also obtain similar estimate for the case \( \tau > 0 \).

Thus (II) implies (III).

(III) implies (I). We have, from
\[
p \left(\left[C_1 \lambda_0 (\lambda_0 I - A)^{-1}\right]^{n+1} \right) \leq q(x) \text{ with } \operatorname{Re}(\lambda_0) \gg 1,
\]

the inequality
\[
p \left(\left[\lambda - \lambda_0 (\lambda_0 I - A)^{-1}\right]^{n+1} \right) \leq \frac{||\lambda - \lambda_0||^n}{C_1^{n} |\lambda_0|^n} q(x)
\]

Thus, if \( \operatorname{Re}(\lambda_0) \gg 1 \) and \( \frac{|\lambda - \lambda_0|}{C_1 |\lambda_0|} < 1 \), the series
converges and represents the resolvent \((\lambda_0 - \lambda)^{-1} I - A)^{-1}\) in such a way that

\[
\left( 1 - \frac{|\lambda - \lambda_0|}{C_1 |\lambda_0|} \right)^{-1}
\]

when \(|\lambda| \rightarrow \infty\) in the domain of the complex \(\lambda\)-plane which lies on the right of a oriented path (see the figure).

\[C_2(s) = \sigma(s) + i \tau(s) \quad (-\infty < s < \infty)\]

such that

\[
\lim_{s \to \infty} \tau(s) = \infty, \quad \lim_{s \to -\infty} \tau(s) = -\infty,
\]

\[
\lim_{s \to \infty} \frac{\sigma(s)}{\tau(s)} < 0, \quad \lim_{s \to -\infty} \frac{\sigma(s)}{\tau(s)} > 0
\]

We thus can define, for \(t > 0\),

\[
(7) \quad \hat{T}_t x = \frac{1}{2\pi i} \int_{C_2(s)} e^{\lambda t} (\lambda - A)^{-1} x \, d\lambda
\]

If we are able to show that

\[
(8) \quad \hat{T}_t^n = T_t^n
\]

then, by

\[
\left( T_t^n \right)_x = \frac{1}{2\pi i} \int_{C_2(s)} e^{\lambda t} (\lambda - A)^{-1} x \, d\lambda = (T^n)_{t/n} x,
\]

we obtain

\[
(tT^n)_t x = \frac{1}{2\pi i} \int_{C_2(s)} e^{\lambda t} (t\lambda)^n (\lambda - A)^{-1} x \, d\lambda
\]

which implies (I).
We shall prove (8). To this purpose, we first prove (9), (10) and (11):

1. For every $x_0 \in D(A)$,

2. For every $x \in X$ and $t > 0$,

3. $\hat{T}_t x$ is of exponential growth when $t \to \infty$.

(11) is clear from (7). (10) is proved from

$$
\hat{T}_t x - A\hat{T}_t x = \hat{T}_t x = \frac{1}{2\pi i} \int_{C_2(s)} e^{\lambda t} \{ (\lambda I - A)^{-1} x - A(\lambda I - A)^{-1} x \} d\lambda
$$

by shifting the path of integration $C_2(s)$ to the left.

To prove (9), we take a $\lambda_0$ with $\Re(\lambda_0) > 0$ on the right of the path $C_2(s)$ and take $y_0 \in X$ such that $x_0 = (\lambda_0 I - A)^{-1} y_0$.

Then

$$
\hat{T}_t x_0 = \hat{T}_t (\lambda_0 I - A)^{-1} y_0 = \frac{1}{2\pi i} \int_{C_2(s)} e^{\lambda t} (\lambda I - A)^{-1} (\lambda_0 I - A)^{-1} y_0 d\lambda
$$

$$
= \frac{1}{2\pi i} \int_{C_2(s)} e^{\lambda t} \frac{1}{\lambda_0 - \lambda} (\lambda I - A)^{-1} y_0 d\lambda
$$

$$
= \frac{1}{2\pi i} \int_{C_2(s)} e^{\lambda t} \frac{1}{\lambda_0 - \lambda} (\lambda_0 I - A)^{-1} y_0 d\lambda
$$

The second integral on the right is $0$ as may be seen by shifting the path of integration $C_2(s)$ to the left. Hence, by (6),

$$
\lim_{t \to \infty} \hat{T}_t x_0 = \frac{1}{2\pi i} \int_{C_2(s)} \frac{1}{\lambda_0 - \lambda} (\lambda I - A)^{-1} y_0 d\lambda
$$

$$
= (\lambda_0 I - A)^{-1} y_0 \quad \text{(the residue at } \lambda = \lambda_0) .
$$
We are now ready to prove (8). Put

\[ y_t = \hat{T}_t x_0 - T_t x_0. \]

Then \( \lim_{t \to 0^+} y_t = 0 \), \( y'_t = Ay_t \quad (t > 0) \) and \( y_t \) is of exponential growth as \( t \to \infty \). Hence, for sufficiently large \( \Re(\lambda) \), we obtain

\[
\lambda \int_0^\infty e^{-\lambda t} y_t \, dt = \int_0^\infty e^{-\lambda t} A y_t \, dt = \int_0^\infty e^{-\lambda t} y'_t \, dt
\]

by partial integration. But, since every \( \lambda \) with \( \Re(\lambda) > 0 \) is in the resolvent set of \( A \), we must have

\[
\int_0^\infty e^{-\lambda t} y_t \, dt = 0 \quad \text{for all } \lambda \text{ if } \Re(\lambda) \text{ is sufficiently large.}
\]

This proves that \( y_t = 0 \), i.e., \( \hat{T}_t x_0 = T_t x_0 \) for every \( x_0 \in D(A) \). As \( D(A) \) is dense in \( X \), we obtain \( \hat{T}_t = T_t \).

Remark. In the case when \( X \) is a Banach space, the equivalence of (II) and (III) is proved by E. Hille and R. S. Phillips [1]. The condition (I) was observed by K. Yosida [1], [2]. The theorem given in the present note is adapted from K. Yosida [3].

In the case when \( X \) is a Banach space, we can construct, from any equicontinuous semi-group \( T_t \) of class \( (C_0) \), a holomorphic semi-group \( \tilde{T}_{t,\alpha} = \tilde{T}_t \) as follows K. Yosida [4], V. Balakrisnan [5] and T. Kato [6]; Consider

\[
\int_{t,\alpha} (\lambda) = \frac{1}{2\pi i} \int_{\sigma - i \infty}^{\sigma + i \infty} e^{Z\lambda - Z\alpha t} \, dZ
\]
Where \( \sigma > 0, t > 0, \lambda \geq 0 \) and \( 0 < \alpha < 1 \); we here take the branch of the function \( z^\alpha \) in such a way that

\[
\text{Re}(z^\alpha) > 0 \text{ for } \text{Re}(z) > 0. \]

Then

\[
(13) \quad \tilde{T}_{t, \alpha} x = \tilde{T}_t x = \int_0^\infty \tilde{T}_{t, \alpha} (s) T_s x \, ds.
\]

REFERENCES


