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Holomorphic semi-groups

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§ 1. The Theorem. Let $X$ be a locally convex, sequentially complete, linear topological space, and let $L(X,X)$ be the set of all linear continuous mappings defined on $X$ with values in $X$.

A system $\{T_t; t \geq 0\}$ of mappings $T_t \in L(X,X)$ is called an equi-continuous semi-group of class $(C_0)$ if:

1. $T_t T_s = T_{t+s}$, $T_0 = I$ (the identity),

2. $\lim_{t \to t_0} T_t x = T_{t_0} x$ for every $t_0 \geq 0$ and $x \in X$,

3. for every continuous semi-norm $p$ in $X$, there exists a continuous semi-norm $q$ in $X$ such that

$$p(T_t x) \leq q(x)$$

for all $t \geq 0$ and $x \in X$.

We shall prove the following

Theorem. The three propositions (I), (II) and (III) given below are mutually equivalent:

1. For every $t > 0$ and $x \in X$, $T_t x = \lim_{h \to 0} \frac{T_{t+h} - T_t}{h} x$

exists, and, for a suitable positive constant $C$, the system of mappings

$$\{(C t T_t)^n; 1 \geq t > 0, n = 0, 1, 2, \ldots\}$$

is equi-continuous.
(II) The Taylor expansion

\[ T_\lambda x = \sum_{n=0}^{\infty} \frac{\lambda - t^n}{n!} T^{(n)}_t x \]

converges for every \( x \in X \) and every complex number \( \lambda \) with \( |\arg \lambda| < \tan^{-1}(Ce^{-1}) \), in such a way that the system of mappings \( \{ e^{-\lambda} T_\lambda ; |\arg \lambda| < \tan^{-1}(Ce^{-1}) \} \) is, for a certain \( k > 0 \), equi-continuous.

(III) The infinitesimal generator \( A \) of \( T_t \) defined by

\[ Ax = \lim_{t \downarrow 0} \frac{T_t - I}{t} x \]

satisfies the condition that the resolvent \( (\lambda I - A)^{-1} \) exists as a mapping \( \in L(X, X) \) for \( \text{Re}(\lambda) > 0 \) and the system of mappings

\[ \left\{ [C_1 \lambda (\lambda I - A)^{-1}]^n ; \text{Re}(\lambda) \geq 1, \ n = 0, 1, 2, \ldots \right\} \]

is equi-continuous for a certain positive constant \( C_1 \).

§ 2. The Sketch of the Proof of the Theorem. We first recall known facts concerning equi-continuous semi-groups \( T_t \) of class \((C_0)\):

The domain \( D(A) \) of \( A \) is dense in \( X \); for every \( \lambda \) with \( \text{Re}(\lambda) > 0 \), the resolvent \( (\lambda I - A)^{-1} \) exists and \( \in L(X, X) \); \( (\lambda I - A)^{-1} x = \int_0^{\infty} e^{-\lambda t} T_t x dt \) for every \( x \in X \) and \( \lambda \) with \( \text{Re}(\lambda) > 0 \); the system of operators

\[ \left\{ [\sigma (\sigma I - A)^{-1}]^m \ ; \ \sigma > 0, \ m = 0, 1, 2, \ldots \right\} \]

is equi-continuous; for every \( x \in D(A) \), we have

\[ \frac{dT_t x}{dt} = AT_t x = T_t Ax , \quad t > 0 , \quad (4) \]

concerning \( T_t^{(n)}_t = (T_t^{(n-1)})_t \), we have the

Lemma. Let \( T_t x \in D(A) \) for every \( t > 0 \) and \( x \in X \). Then \( T_t x \) is infinitely differentiable in \( t \) and
Proof. For any $t_o > 0$ with $t > t_o$, we have, by (1) and (4),

$$T_t^x = AT_t x = T_{t-t_o} AT_{t_o} x .$$

Hence

$$T_t^n x = T_{t-t_o} AT_{t_o} x = AT_{t-t_o} AT_{t_o} x = AT_{t/2} AT_{t/2} x = (T_{t/2}^2)^n x .$$

We have only to repeat the same reasoning to obtain (5).

**Proof of the Theorem**

(1) implies (II). By the equi-continuity of $\left\{ \left( CT_t^t \right)^n ; n \geq 0, 1 \geq t > 0 \right\}$, we obtain, remembering (5),

$$p\left( \frac{\lambda-t}{\ln t} T_t^n x \right) \leq \frac{\lambda-t}{t^n} \frac{n}{\ln t} \frac{1}{c^n} p\left( \frac{\lambda-t}{\ln t} CT_t^n x \right) \leq \left( \frac{\lambda-t}{t} \right)^n q(x) .$$

Hence the first part of (II) is proved.

Next consider the semi-group $\{ S_t \}$ defined by

$$S_t = e^{-t} T_t .$$

Then

$$tS_t^n = te^{-t} T_t^n - te^{-t} T_t^n .$$

Thus remembering that $0 \leq te^{-t} \leq 1$ for $t > 0$, we easily see that

$$\left\{ \left( 2^{-k} tS_t^n \right)^n ; t > 0, n = 0, 1, 2, \ldots \right\}$$

is with a certain $k > 0$, equi-continuous. Hence, as above, we see that $e^{-\lambda} T_{\lambda}$, which is an holomorphic extension of $S_t$, satisfies the condition that

$$\left\{ e^{-\lambda} T_{\lambda} ; \left| \arg \lambda \right| < \tan^{-1} \left( \frac{c_t^{-1}}{2k} \right) \right\}$$

is equi-continuous.

(II) implies (III). Differentiating $(\lambda I - A)^{-1} = \int_0^\infty e^{-\lambda t} T_t dt$ with respect to $\lambda$, we obtain, for $\lambda = \sigma + 1 + i\tau$ with $\sigma > 0$,

$$[(\sigma + 1 + i\tau)/(\sigma + 1 + i\tau)I - A]^{-1} x$$

$$= \frac{1}{\ln n} \int_0^\infty e^{-\left( \sigma + 1 + i\tau \right) t} t^n S_t x dt, x \in X .$$
Let $\tau < 0$. Then, by Cauchy's integral theorem, we obtain, for

$$0 < \Theta < \tan^{-1}\left(\frac{\pi}{2}\right),$$

$$\left[(\sigma + 1+i \tau)\left[(\sigma + 1+i \tau)(-\lambda - A)^{-1}\right]^{n+1}\right]_{x} = \frac{(\sigma + 1+i \tau)^{n+1} \int_{0}^{\infty} e^{-\sigma - i \tau} r e^{i \theta} r^n S r e^{i \theta} x e^{i \theta} dr}{\ln}.$$

Hence, by the equi-continuity of $\{S_{i \theta} : r \geq 0\}$, we have

$$p \left(\left[(\sigma + 1+i \tau)(\sigma + 1+i \tau)(-\lambda - A)^{-1}\right]^{n+1}_{x}\right)$$

$$\leq q(x) \cdot \frac{|\sigma + 1+i \tau|^{n+1}}{\ln} \int_{0}^{\infty} e^{(-\sigma \cos \theta + \tau \sin \theta)^{r} r^n dr}$$

$$= q(x) \cdot \frac{|\sigma + 1+i \tau|^{n+1}}{|\tau \sin \theta - \sigma \cos \theta|^{n+1}}$$

since $0 < \Theta < \frac{\pi}{2}$, $\tau < 0$ and $\sigma > 0$, we easily see that the second factor on the right is $\leq C_1^n$.

We also obtain similar estimate for the case $\tau > 0$.

Thus (II) implies (III).

(III) implies (I). We have, from

$$p \left(\left[\lambda_{\alpha}(-\lambda_{\alpha} - A)^{-1}\right]^{n+1}_{x}\right) \leq q(x) \text{ with Re} \left(\lambda_{\alpha}\right) \geq 1,$$

the inequality

$$p \left(\left[\lambda - \lambda_{\alpha}(-\lambda_{\alpha} - A)^{-1}\right]^{n+1}_{x}\right) \leq \frac{|\lambda - \lambda_{\alpha}|^{n}}{C_1 n \left|\lambda_{\alpha}\right|^n} q(x)$$

Thus, if Re $\left(\lambda_{\alpha}\right) \geq 1$ and $\frac{|\lambda - \lambda_{\alpha}|}{C_1} \left|\lambda_{\alpha}\right| < 1$, the series
\[ \sum_{n=0}^{\infty} (\lambda - \lambda)^n (\lambda I - A)^{-n+1} x \]

converges and represents the resolvent \((\lambda I - A)^{-1}\) in such a way that

\[ p((\lambda I - A)^{-1} x) \leq (1 - \frac{|\lambda - \lambda_0|}{|\lambda_0|})^{-1} q(x) = 0 \quad \frac{1}{|\lambda|} \]

when \(|\lambda| \to \infty\) in the domain of the complex \(\lambda\)-plane which lies on the right of a oriented path (see the figure)

\[ C_2(s) = \sigma(s) + i \tau(s) \quad (-\infty < s < \infty) \]

such that

\[ \lim_{s \to \infty} \tau(s) = \infty, \quad \lim_{s \to -\infty} \tau(s) = -\infty, \]

\[ \lim_{s \uparrow \infty} \frac{\sigma(s)}{\tau(s)} < 0, \quad \lim_{s \downarrow -\infty} \frac{\sigma(s)}{\tau(s)} > 0 \]

We thus can define, for \(t > 0\),

\[ T_t^x = \frac{1}{2 \pi i} \int_{C_2(s)} e^{\lambda t} (\lambda I - A)^{-1} x \, d\lambda \]

If we are able to show that

\[ (T_t^n)^x = \frac{1}{2 \pi i} \int_{C_2(s)} e^{\lambda t} \lambda^n (\lambda I - A)^{-1} x \, d\lambda = (T_{t/n})^n x, \]

we obtain

\[ (tT_t^n)^x = \frac{1}{2 \pi i} \int_{C_2(s)} e^{\lambda t} (t \lambda)^n (\lambda I - A)^{-1} x \, d\lambda \]

which implies (I).
We shall prove (8). To this purpose, we first prove (9), (10) and (11):

(9) \( \lim_{t \to 0} \hat{T}_t x_0 = x_0 \) for every \( x_0 \in D(A) \),

(10) \( \hat{T}_t x = \Delta t x \) for every \( x \in X \) and \( t > 0 \),

(11) \( \hat{T}_t x \) is of exponential growth when \( t \to \infty \).

(11) is clear from (7). (10) is proved from

\[
\hat{T}_t x - \Delta t x = \frac{1}{2\pi i} \int_{C_2(s)} e^{\lambda t} \left\{ (\lambda I - A)^{-1} x - A(\lambda I - A)^{-1} x \right\} d\lambda
\]

by shifting the path of integration \( C_2(s) \) to the left.

To prove (9), we take a \( \lambda_0 \) with \( \text{Re}(\lambda_0) > 0 \) on the right of the path \( C_2(s) \) and take \( y_0 \in X \) such that \( x_0 = (\lambda_0 I - A)^{-1} y_0 \).

Then

\[
\hat{T}_t x_0 = \hat{T}_t (\lambda_0 I - A)^{-1} y_0 = \frac{1}{2\pi i} \int_{C_2(s)} e^{\lambda t} (\lambda I - A)^{-1} (\lambda_0 I - A)^{-1} y_0 d\lambda
\]

\[
= \frac{1}{2\pi i} \int_{C_2(s)} e^{\lambda t} \frac{1}{\lambda_0 - \lambda} (\lambda I - A)^{-1} y_0 d\lambda
\]

\[
= \frac{1}{2\pi i} \int_{C_2(s)} e^{\lambda t} \frac{1}{\lambda - \lambda_0} (\lambda_0 I - A)^{-1} y_0 d\lambda
\]

The second integral on the right is 0 as may be seen by shifting the path of integration \( C_2(s) \) to the left. Hence, by (6),

\[
\lim_{t \to 0} \hat{T}_t x_0 = \frac{1}{2\pi i} \int_{C_2(s)} \frac{1}{\lambda_0 - \lambda} (\lambda I - A)^{-1} y_0 d\lambda
\]

\[
= (\lambda_0 I - A)^{-1} y_0 \quad \text{(the residue at } \lambda = \lambda_0)\]
We are now ready to prove (8). Put

\[ y_t = \hat{T}_t x_0 - T_t x_0. \]

Then \( \lim_{t \to 0} y_t = 0 \), \( y'_t = Ay_t \) (\( t > 0 \)) and \( y_t \) is of exponential growth as \( t \to \infty \). Hence, for sufficiently large Re(\( \lambda \)), we obtain

\[
\lambda \int_0^\infty e^{-\lambda t} y_t dt = \int_0^\infty e^{-\lambda t} Ay_t dt = \int_0^\infty e^{-\lambda t} y'_t dt
\]

by partial integration. But, since every \( \lambda \) with Re(\( \lambda \)) > 0 is in the resolvent set of \( A \), we must have

\[
\int_0^\infty e^{-\lambda t} y_t dt = 0 \quad \text{for all \( \lambda \) if Re(\( \lambda \)) is sufficiently large.}
\]

This proves that \( y_t = 0 \), i.e., \( \hat{T}_t x_0 = T_t x_0 \) for every \( x_0 \in D(A) \). As \( D(A) \) is dense in \( X \), we obtain \( \hat{T}_t = T_t \).

Remark. In the case when \( X \) is a Banach space, the equivalence of (II) and (III) is proved by E. Hille and R. S. Phillips [1]. The condition (I) was observed by K. Yosida [1], [2]. The theorem given in the present note is adapted from K. Yosida [3].

In the case when \( X \) is a Banach space, we can construct, from any equicontinuous semi-group \( T_t \) of class \( (C_\varepsilon) \), a holomorphic semi-group \( \widetilde{T}_{t,\varphi} = \tilde{T}_t \) as follows K. Yosida [4], V. Balakrisnan [5] and T. Kato [6]. Consider

\[
\int_{t,\varphi} (\lambda) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{\lambda z - 2\alpha z} t \, dz
\]
Where $\sigma > 0$, $t > 0$, $A > 0$ and $0 < \alpha < 1$; we here take the branch of the function $Z^\alpha$ in such a way that

$$\Re(z^\alpha) > 0 \text{ for } \Re(z) > 0.$$  Then

$$\tilde{t}^\alpha x = \tilde{t} x = \int_0^\infty \tilde{t}_s^\alpha (s) T_s x \, ds.$$  

REFERENCES