ISTVAN FARY

On a topological characterization of plane geometries

Topologie et géométrie différentielle. Cahiers du Séminaire dirigé par Charles Ehresmann, tome 7 (1965), exp. nº 1, p. 1-33

<http://www.numdam.org/item?id=SE_1965__7__A1_0>

CAHIERS
de
TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE
Juin 1964

Vol. VII

ON A TOPOLOGICAL CHARACTERIZATION OF PLANE GEOMETRIES

by Istvan FARY

Table of contents.

I. Introduction.

1. Let R be the Cartesian number plane, and H the group of all homeomorphismes $h : R \to R$ of R onto itself; we denote by H_+ the sub-group of those h's which conserve an orientation of R. Let G_E denote the group of all sense preserving isometries of the natural Euclidean metric of R (see (4) below).

If $f : R \to B$ is a homeomorphism of R onto the plane B of the Bolyai geometry, or the geometry of constant curvature < 0, and L_B is the group of all sense preserving isometries of B, then $G_B = f^{-1} L_B f$ is a sub-group of H_+; it is determined up to conjugacy in H. (Given another homeomorphism $g : R \to B$, we set $h = g^{-1} f$. Then we have $h^{-1}(g^{-1} L_B g) h = f^{-1} g g^{-1} L_B g g^{-1} f = G_B$). The groups G_E and G_B can be distinguished by a purely group theoretical property : G_E contains an invariant abelian sub-group of dimension 2, isomorphic to the additive group of R^2 (two-dimensional vector space over the field R^1 of reals), G_B, however, does not contain such a sub-group.

The standard unit two-sphere S^2 will be considered as the one-point compactification of the plane $S^2 = R \cup \omega$; let us denote σ the «spherical metric» of R, i.e.

The work of the author has been supported in part by National Science Foundation contract GP - 1610.

A summary of the present paper will probably appear in the Research Announcements of the American Mathematical Society.

the metric of S^2 cut down to R. Then both G_E and G_B are equi-continuous in the metric σ (see sections II.5, II.7, II.9); evidently, any sub-group of these groups shares this property. More generally, if $G \subset G_E$ or $G \subset G_B$, then any conjugate $h^{-1}Gh$ $(h \in H)$ is also equi-continuous with respect to σ (see section II.9, Theorem 9); this is not necessarily true, if we use another metric.

2. Our principal aim is to show that this is the only way to obtain equi-continuous transformation groups.

MAIN THEOREM. *Let G be an equi-continuous group of sense preserving homeomorphisms of R onto itself. Then G is conjugate in H to a sub-group of G_E or G_B.*

That is to say, supposing that

$$G \subset H_+, \ G \text{ is equi-continuous in } \sigma, \tag{1}$$

then there exists an $h \in H$ such that

$$\text{either } h^{-1}Gh \subset G_E \text{ or } h^{-1}Gh \subset G_B \tag{2}$$

holds true.

In a slightly different form our main result can be stated as follows.

THEOREM 1. *Every equi-continuous group of sense preserving homeomorphisms of R onto itself is contained in a group maximal with respect to these properties. A maximal group satisfying (1) is either conjugate to G_E or to G_B.*

The equivalence of the two wordings of the main result is plain. We come back to other formulations later.

3. Conjugacy in H is a strict equivalence relation : it is «equality up to coordinate transformation». To be more specific, let us suppose that

$$g = h^{-1}fh \tag{3}$$

holds true for the homeomorphisms f, g, h in H. Then g is described, in an appropriate coordinate system determined by h, by the same functions as f in the Cartesian coordinate system. For example, if $f \in G_E$, hence

$$\begin{cases} \xi' = \ \ \xi \cos\theta + \eta \sin\theta + \alpha \\ \eta' = -\xi \sin\theta + \eta\cos\theta + \beta \end{cases} \tag{4}$$

is its expression in Cartesian coordinate system, then g is given by the same functions in a suitable coordinate system. As a point of fact, the function φ of two real variables giving the first coordinate of fx, $x \in R$, is defined by $\xi(fx) = \varphi(\xi(x), \eta(x))$. If we introduce $\overline{\xi}(x) = \xi(hx)$, $\overline{\eta}(x) = \eta(hx)$, as new coordinates, we have $\overline{\xi}(gx) = \xi(hgx) = \xi(fhx) = \varphi(\xi(hx), \eta(hx)) = \varphi(\overline{\xi}(x), \overline{\eta}(x))$ defining the same function

φ. In particular, if in *(2)* the first alternative holds true, then all transformations of G are described by *(4)* in the coordinate system $\xi(hx), \eta(hx)$. Then G appears as a group of isometries of a «topological model» of the Euclidean geometry (see [14]).

4. Conjugacy in H conserves all properties of maps which may be termed «topological properties», hence it is a proper concept of «homeomorphism of maps» and «homeomorphism of families of maps». To illustrate this statement, let us suppose that

$$G' = h^{-1} G h \tag{5}$$

where G, G' are sub-sets of H, and $h \in H$. As H is a topological group (see section II.5), and $f \to h^{-1} f h$ is an endomorphism of the whole structure, G and G' correspond to one another under this map. Every $g \in G$ is a sub-set of $R \times R$, with a particular structure. The relation *(5)* means that the homeomorphism

$$h^{-1} \times h^{-1} : R \times R \to R \times R$$

maps each sub-set $\{(x, gx)\}$ onto $\{(h^{-1}x, h^{-1}gx)\} = \{(x, h^{-1}ghx)\} = \{(x, g'x)\}$, where $g \in G$ and $g' \in G'$. Thus a sub-set of $R \times R$ representing a $g \in G$ is mapped homeomorphically onto a sub-set representing a $g' \in G'$ by a homeomorphism of the whole space $R \times R$.

To sum up, using *(5)* as a definition of homeomorphism of sets of maps, we may state the following theorem.

THEOREM 2. *The Main Theorem solves the homeomorphism problem for equi-continuous transformation groups acting on surfaces.*

Of course, the wording of the theorem above is rather vague. However, we will not come back to a more precise formulation and a proof of Theorem 2 in the present paper. Nevertheless, we want to mention that, properly formulated, the theorem above generalizes the classical result on classification of surfaces, which corresponds to Theorem 2 applied to the identity transformation of a surface M. (See Theorem 27, Section VI.2).

5. Let us formulate an equivalent form of Theorem 1, which is useful in some steps of the proof. In what follows, we call surface a two dimensional, metric, C^0 manifold without boundary, which may or may not be compact.

THEOREM 3. *Let G be a group of sense preserving homeomorphisms of an orientable surface M onto itself, which is equi-continuous in a metric of M which extends to the one point compactification of M. Then M carries a complex structure preserved by all elements $g \in G$.*

The conclusion of the theorem can be worded differently : under the hypotheses of our theorem, there is a conformal structure invariant under all homeomorphisms $g \in G$.

6. We formulated above equivalent forms of our Main Theorem. Let us state presently important special cases, and easy corollaries.

In the course of the proof of the Main Theorem, we will use a deep result of Hilbert, which in turn is a weak special case of our theorem. In order to formulate this theorem of Hilbert (see [9]), we introduce the following notations :

$$(6) \qquad G_x = \{ g \in G, gx = x \}$$

$$(7) \qquad G(y) = \{ z \in R, z = gy \text{ for some } g \in G \}. \qquad (x \in R,\ G \subset H)$$

The first is, of course, the isotropy set (isotropy sub-group, in case G is a group) of the point $x \in R$; $G(y)$ is the trajectory of y under G. We refer to section III.4 for the definition of the concept of «three-rigidity» appearing in the formulation of Hilbert's theorem; we give here a loose, intuitive description of this concept. A transformation group $G \subset H$ is termed «three rigid» if three points from arbitrary neighborhoods of x, y, z can be transformed arbitrarily near to x', y', z' only if there is a transformation carrying x, y, z precisely into x', y', z' respectively (here x, y, z, x', y', z' are points of R, not necessarily distinct, and «transformation» refers to elements of the given G).

THEOREM (HILBERT). *If $G \subset H_+$ is a three-rigid transformation group of R, such that $G_x(y)$ is infinite for all x and y ($y \neq x$), then G is conjugate in H to G_E or G_B.*

This theorem of Hilbert gives topological characterization of the Euclidean and the Bolyai geometry. It could be paraphrased as follows. If a transformation group of the plane does not change distances too badly, and if its isotropy groups behave as they should in order to be the group of all isometries of a geometry of constant curvature ≤ 0, then it is the group of all isometries of a geometry of constant curvature ≤ 0.

In section III.4 we analyse three rigidity and will show that Hilbert's theorem, which, by the way, inspired his V^{th} problem (see [19]), is a special case of our Main Theorem.

Another known special case of our Main Theorem, very important in its proof, is the following (see [12], [13]).

THEOREM (KERÉKJÁRTÓ). *Let $g \in H_+$ be a homeomorphism, whose powers $\{ g^p; p = 0, \pm 1, \pm 2, . \}$ are equi-continuous in the metric σ. Then there exists an $h \in H$, such that*

$$h^{-1} g h \in G_E$$

holds true, i.e. $h^{-1} g h$ is given by (4) in Cartesian coordinates.

This special case of the Main Theorem serves, in particular, to «adjust» the hypotheses of Hilbert's theorem to the condition *(1)*. It is to be noted, however, that it does not imply trivially our main result. Given G satisfying *(1)*, as in the Main Theorem, there is an $h_g \in H$ for every $g \in G$, such that $h_g^{-1} g h_g \in G_E$ holds true, in virtue of Kerékjártó's theorem. By our theorem, h can be chosen for all elements of G simultaneously, so that *(2)* holds true. Let us also notice that the disjonction in *(2)* is necessary, although each element of G is conjugate to some element of G_E (in particular each element of G_B is conjugate to some element of G_E).

Among special cases of the Main Theorem let us also mention the following classical result (see [11]).

THEOREM (BROUWER). *If G is a finite group of sense preserving homeomorphisms of the 2-sphere S^2, then G is cyclic and conjugate, in the group of all homeomorphisms of S^2, to a group of rotations.*

Other similar classical results can also be deduced from our Main Theorem.

7. Finally, using the Main Theorem it is easy to formulate results analogous to the known theorems formulated above, and to analyse the hypotheses in them. For example, the Main Theorem can be used to give various topological characterizations of the Euclidean and Bolyai plane geometries.

THEOREM 4. *If (1) holds true, and there are two points a, b $(b \neq a)$ such that $G_a(x)$ and $G_b(y)$ are both infinite for some x, y then G is conjugate in H to G_E or to G_B, thus is the full group of sense preserving isometries of a geometry of constant curvature ≤ 0.*

THEOREM 5. *If (1) holds true and $\dim G \geq 3$, then G is the full group of sense preserving isometries of a geometry of constant curvature ≤ 0.*

The first of these theorems can be paraphrased as Hilbert's theorem. It is even easier to do this for Theorem 5 : if a group G does not change distances too badly, and is as large as can be, then it is the group of all isometries of a geometry of constant curvature ≤ 0.

Both theorems above are special cases of the theorem formulated below.

THEOREM 6. *Let P be a proposition which is true or false, if formulated for an equicontinuous transformation group acting on the plane, and has the same truth value for G as for $h^{-1}Gh$. If P is true for G_E and for G_B but false for proper sub-groups of these groups, then any group G satisfying (1) and having property P is conjugate to G_E or to G_B.*

Clearly, in Theorem 4, P is the proposition involving the trajectories of isotropy sub-groups, in Theorem 5 it is the statement «... is of dimension ≥ 3», which is meaningful as G inherits a topology from H (see Section II.5). Theorem 1 can also be formulated in such a manner. The property P is then «... is not properly contained in a group satisfying (1)».

As we mentioned earlier, there is a close connection between equi-continuity and three-rigidity (see Section III.4). Using this connection, we can formulate the results above, concerning three-rigid, or even two-rigid transformation groups. If we formulate Theorem 4 this way, we have a direct generalization of Hilbert's theorem.

8. I reached the results presented in this paper by a thorough study of Hilbert's paper [9] when I realized that the property concerning the trajectories is not essential in Hilbert's characterization of the geometries; the property serves only to exclude proper sub-groups of maximal equi-continuous groups. This conjecture was furthered when I learned Kerékjártó's theorem (see Section I.6). My former partial results are [6], [7]; these results are proved anew in the present paper.

II. Some General Properties of Equi-Continuous Transformation Groups.

1. Our ultimate aim is to study transformation groups acting on the plane, but we formulate some results more generally. We do not strive, however, for complete generality, whatever this may be, because some general statements are cumbersome in the present application. For a general theory see [5].

2. In what follows, we consider separable, metric spaces only, i.e. metric spaces satisfying the second countability axiom. Let R be such a space. If G is a group, and we are given a map

$$(1) \qquad \begin{cases} G \times R \to R \\ (g, x) \to gx \end{cases}$$

such that

$$ex = x \text{ for all } x \in R \qquad (e : \text{neutral element of } G)$$

$$(g_2 g_1)x = g_2(g_1 x)$$

then we say, somewhat loosely, that G is a transformation group acting on R.

$$N = \{ g \in G, gx = x \text{ for all } x \in R \}$$

is an invariant sub-group of G, and $G' = G/N$ is also a transformation group acting on R, if we agree that $(gN)x = gx$. Hence replacing G by G', if necessary, we may suppose

$$\text{if } gx = x \text{ for all } x \in R, \text{ then } g = e$$

holds true; such a transformation group is called effective. In what follows, we always consider effective transformation groups, without stating this necessarily.

3. For fixed $g \in G$, $x \to gx$ is a one-to-one map of R onto itself, its inverse being $x \to g^{-1}x$. If the map (1) is continuous in the discrete topology of G, $x \to gx$ is a homeomorphism.

4. Let H denote the set of all homeomorphisms of R onto itself. H is a group under composition of maps, and it is a transformation group of R, for which (1) is continuous in the discrete topology of H. Furthermore, for every transformation group G acting on R, for which (1) is continuous in the discrete topology of G, there is a natural homomorphism $G \to H$ mapping g into the homeomorphism $x \to gx$. If G is effective, this homeomorphism is an inclusion, hence G can be naturally identified to a subgroup of H. In what follows, we always make this identification, hence consider subgroups of H only.

5. Let σ be a bounded metric of R. Let us prove that

$$\Delta(f, g) = \sup \{ \sigma(fx, gx), x \in R \} \tag{2}$$

is a metric on H. Firstly, $\Delta(f, g) = 0$ implies $gx = fx$ for all x, thus $g = f$. Secondly, $\Delta(f, g) = \Delta(g, f)$ by symmetry on the right hand side of (2). Thirdly,

$$\begin{aligned} \Delta(f, g) - \varepsilon < \sigma(fx_o, gx_o) &\leq \sigma(fx_o, hx_o) + \sigma(hx_o, gx_o) \\ &\leq \Delta(f, h) + \Delta(h, g) \end{aligned}$$

for every $\varepsilon > 0$. Thus Δ is a metric for H. Also, it is easy to check that (1) is a continuous map. A sequence $\{f_n\}$ of elements of H converges to $g \in H$, if the maps converge uniformly in the uniform topology of R defined by σ. The metric Δ may change, if σ is replaced by another bounded metric inducing the same topology.

We always use the metric (2) when dealing with topological properties of the space G of a sub-group of H. Then the metric σ *on* R must be specified; of course, R will not be, in general, complete in this metric.

6. A sub-group G of H is called group of isometries of σ, if

$$\sigma(gx, gy) = \sigma(x, y) \tag{3}$$

for every pair of points x, y in R and every $g \in G$. The set of all elements of H satisfying (3) is the group of isometries of (R, σ); of course, this can be reduced to the neutral element e. We will use the following result of [4].

THEOREM 7. *If R is locally compact and has but a finite number of connected components, the group of isometries of any given bounded metric σ on R is a locally compact sub-group of H.*

7. Given a sub-group G of H, it is possible to introduce a new metric $\bar{\sigma}$ on the set R, in such a way that G be a group of isometries of $\bar{\sigma}$. If we also require that the topology of $\bar{\sigma}$ be the same as the given topology of R, the concept of equi-continuity comes up naturally. Let us introduce and discuss this concept.

DEFINITION 1. *A set K of elements of H is called equi-continuous, with respect to a given metric σ of R, if there is a function $\varphi(x, \varepsilon) > 0$ defined for all $x \in R$ and $\varepsilon > 0$, and such that*

$$\sigma(x, y) < \varphi(x, \varepsilon) \text{ implies } \sigma(kx, ky) < \varepsilon \tag{4}$$

for all $k \in K$. φ is called equi-continuity modulus of K.

REMARK. If there is a φ which is constant with respect to x, we may say that the equi-continuity is «uniform»; we will not be concerned with this special case. Let us also remark that some authors use the term «pointwise equi-continuous» instead of equi-continuous.

Let K be a given equi-continuous sub-set of H containing e, and $\varphi(x, \varepsilon)$ its modulus with respect to a given bounded metric σ. We introduce a new metric $\bar{\sigma}$ on the set R, by

$$\bar{\sigma}(x, y) = \sup \{ \sigma(kx, ky), k \in K \} \qquad (x, y \in R). \tag{5}$$

We have supposed $e \in K$, thus

$$\bar{\sigma}(x, y) \geq \sigma(x, y) \tag{6}$$

holds true. Naturally, $\bar{\sigma}(x, y) = 0$ implies $\sigma(x, y) = 0$, thus $y = x$; also $\bar{\sigma}(x, y) \geq 0$ by (6). Symmetry $\bar{\sigma}(x, y) = \bar{\sigma}(y, x)$ is evident. As to the triangle inequality :

$$\begin{aligned}\bar{\sigma}(x, y) - \varepsilon &\leq \sigma(k_o x, k_o y) \leq \sigma(k_o x, k_o z) + \sigma(k_o z, k_o y) \\ &\leq \bar{\sigma}(x, z) + \bar{\sigma}(z, y)\end{aligned}$$

if x, y, z are given, and $\varepsilon > 0$ is any number. This completes the proof to the effect that (5) is a metric of R.

The inequality (6) shows, furthermore, that :

$$\text{if } \lim \bar{\sigma}(x, x_n) = 0, \text{ then } \lim \sigma(x, x_n) = 0. \tag{7}$$

Until now, we have not used the equi-continuity of the family K. Using the modulus φ of the family, we will prove now the converse implication

$$\text{if } \lim \sigma(x, x_n) = 0, \text{ then } \lim \bar{\sigma}(x, x_n) = 0. \tag{8}$$

The hypothesis in (8) means that for $n \geq \bar{N}(\varepsilon)$, $\sigma(x, x_n) < \varphi(x, \varepsilon)$, hence $\sigma(kx, kx_n) < \varepsilon$ for all $k \in K$, thus $\bar{\sigma}(x, x_n) < \varepsilon$, implying the conclusion in (8). We proved then the following lemma.

LEMMA 1. *For any set* $K \subset H$, *(5) is a metric on the set* R; *in case* $e \in K$, *(6) holds true. If* K *is an equi-continuous family, the metric (5) induces the topology of* R, *in particular, both (7) and (8) hold true.*

In the future we will be interested in properties of sub-groups of H which are hereditary with respect to inclusion. The lemma below will show that, when dealing with hereditary properties, we may suppose G closed in H.

LEMMA 2. *Let* $K \subset H$ *be an equi-continuous set. Then its closure* $\overline{K}$ *in the space* (H, Δ) *(see (2)) is also equi-continuous.*

PROOF. Let $\varphi(x, \varepsilon)$ be the modulus of K. We claim that $\varphi(x, \varepsilon/2)$ is a modulus for $\overline{K}$. In order to prove this, let be given a $g \in \overline{K}$, and a sequence $k_n \in K$ such that $\lim \Delta(g, k_n) = 0$. Supposing then $\sigma(x, y) < \varphi(x, \varepsilon/2)$, we have

$$\sigma(gx, gy) \leq \sigma(gx, k_n x) + \sigma(k_n x, k_n y) + \sigma(k_n y, gy)$$
$$\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4}$$

if n is appropriately chosen.

THEOREM 8. *If* G *is an equi-continuous sub-group of* H, *then it is a group of isometries of some metric* $\overline{\sigma}$ *of* R. *The closure* $\overline{G}$ *of* G *in* H *is a locally compact group.*

PROOF. In virtue of Lemma 2, we may suppose $\overline{G} = G$. We consider a bounded metric σ of R, and we introduce $\overline{\sigma}$ by (5) with $K = G$. By Lemma 1, $\overline{\sigma}$ is a metric of the space R. Now for every $g \in G$,

$$\overline{\sigma}(gx, gy) = \sup\{\sigma(kgx, kgy) : k \in G\} = \sup\{\sigma(hx, hy) : h \in G\}$$
$$= \overline{\sigma}(x, y)$$

thus g is an isometry of $\overline{\sigma}$. By Theorem 7 the group of all isometries of $\overline{\sigma}$ is locally compact, thus G is a closed sub-space of a locally compact space. This completes the proof.

LEMMA 3. *Let* $G \subset H$ *be an equi-continuous family such that* R *be the union of compact sub-sets* A_i, $i \in I$, *for which*

$$\cup\{gA_i,\ g \in G,\ \text{or}\ g^{-1} \in G\}$$

is contained in a compact B_i. *Under these conditions* G *is precompact, i.e. its closure in* H *is compact.*

PROOF. Given a sequence $g_n \in G$, we can replace it by a sub-sequence which is convergent on a countable, everywhere dense set; let us denote now this sub-sequence by $\{g_n\}$. The restrictions $g_n | A_i$, i fixed, satisfy the conditions of the well known Ascoli theorem, hence $g^{(i)} = \lim (g_n | A_i)$ exists and is a continuous map, for every $i \in I$.

Now $g^{(i)}$ and $g^{(j)}$ coincide on an everywhere dense sub-set of $A_i \cap A_j$, thus there exists a map g such that $g \mid A_i = g^{(i)}$. Clearly, $g = lim\ g_n$. Hence g is a continuous map. We can apply the argument to the sequence $\{g_n^{-1}\}$; this proves that g is a homeomorphism.

DEFINITION 2. *Let $K \subset H$ be a given set of homeomorphisms. We say that K is bounded at $x \in R$, if $Kx = \{y = kx, k \in K\}$ is a precompact set.*

LEMMA 4. *Let R be locally compact, and K an equi-continuous sub-set of H. The set E of points of R where K is bounded is an open set. (It may be empty).*

PROOF. Let φ be the modulus of K. Given $x_o \in E$, the local compactness of R implies that there is an $\varepsilon > 0$ such that the ε-neighborhood U of Kx_o be precompact. By equi-continuity, $Kx \subset U$, if $\sigma(x_o, x) < \varphi(x_o, \varepsilon)$ thus K is bounded at each point of a $\varphi(x_o, \varepsilon)$-neighborhood of x_o. This completes the proof of the lemma.

9. The specific metric σ chosen on R had no importance until now. We will prove presently a theorem in which we have to narrow down the choice of σ.

Let R be a locally compact space and $\hat{R} = R \cup \omega$ its one point compactification. If $\hat{R}$ is metric, and $\hat{\sigma}$ is one of its distance functions, then $\sigma = \hat{\sigma} \mid R \times R$ is a bounded metric. Vice versa, given σ, metric of R, it may or may not be possible to extend it to a metric of $\hat{R}$. In the first case we will say that «σ has an extension to the one point compactification of R». Every $h \in H$ can be extended to $\hat{R}$, hence is uniformly continuous in the metric σ.

THEOREM 9. *Let R be locally compact, σ a metric having extension to the one point compactification of R, and $K \subset H$ an equi-continuous family in this metric. Then every conjugate set $h^{-1}Kh$ is equi-continuous in the same metric.*

PROOF. Let φ be the modulus of K and $\delta(\varepsilon)$ the modulus of continuity for both h and h^{-1}. Direct computation shows that

$$\psi(x, \varepsilon) = \delta(\varphi(hx, \delta(\varepsilon)))$$

is a modulus of equi-continuity of the set of maps $h^{-1}Kh$.

III. Equi-Continuous Transformation Groups Acting on the Plane.

1. In what follows, we will be concerned with special and specific questions, hence we specialize our notations and our hypotheses. From now on R always denotes the number plane, and σ is the metric on R obtained by stereographic projection from the sphere (see Section I.1); in particular, σ can be extended to the one-point compactification of R, which is S^2. H stands for the group of all homeomorphisms of R; H_+ is the sub-group of orientation preserving homeomorphisms. We consider sub-groups

$G \subset H_+$ satisfying two conditions :

(1) G is equi-continuous in σ $(G \subset H_+)$;

(2) G is closed in the space H,

where H has the metric Δ defined in (2) of Section II.5. Clearly, (1) is the hypothesis of the Main Theorem (Section I.2); in view of the Remarks in Section II.8, we may suppose (2).

2. If $\theta \equiv 0 \ (mod\ 2\pi)$ in (4) of Section I.3, the map defined is called translation t; if $\alpha = \beta = 0$, the formula gives a rotation r with angle $\theta (mod\ 2\pi)$.

TERMINOLOGY. A map $h^{-1}th$ is called topological translation, and $h^{-1}rh$ topological rotation with angle $\theta(mod\ 2\pi)$.

This terminology is justified by the following lemma.

LEMMA 5. *If $g \in H$ is conjugate to two rotations with angles θ_1 and θ_2, then $\theta_2 \equiv \theta_1$ $(mod\ 2\pi)$.*

PROOF. Let us suppose that

$$g = h_i^{-1} r_i h_i \qquad (i = 1, 2)$$

holds true, where r_1, r_2 are rotations and $h_i \in H$. Then $r_2 = k^{-1} r_1 k$, hence invariant sets under r_2 correspond to invariant sets under r_1. We consider an x_o, which is not the center of rotation of r_i, and the cyclic order of the points

$$(3) \qquad r_1^n x_o \qquad (n \text{ integer})$$

on the circle invariant under r_1. This set is mapped by k^{-1} into a similar set with respect to r_2. As the cyclic order in the set (3) in the same as in

$$r_2^n k^{-1} x_o$$

we have $\theta_2 \equiv \theta_1 \ (mod\ 2\pi)$.

In the case of a topological rotation, we can thus speak without ambiguity : (a) about its center, which is its only fixed point, unless it is the identity e; (b) about the angle $\theta(mod\ 2\pi)$ of the rotation. If $\theta/2\pi$ is irrational, the closure of (3) is an invariant circle, hence, for a topological rotation, which is not periodic, invariant Jordan curves are uniquely determined.

Any two translations are conjugate, thus any two topological translations are also conjugate in H.

3. Given a group $G \subset H_+$, satisfying conditions (1), (2), Kerékjártó's theorem (see Section I.6) shows that every $g \in G$ is either a topological translation or a topological rotation (see Terminology in the previous Section). This fact has important implications for sequences of transformations. The most striking corollary is this : if a sequence $g_n \in G$ converges in two distinct points it converges everywhere. Less surprising, but technically more important is the fact that a sequence $\{g_n\}$ bounded in one point is pre-compact. These facts follow from our special hypotheses (the group acts on the plane); the existence of a general invariant metric (see Theorem 8 in Section II.8) does not imply them.

THEOREM 10. *Let G be a group satisfying conditions $(1')$,(2), and $g_n \in G$ a given sequence of transformations of G, which is bounded in a point, thus such that $\{g_n x_o\}$ is precompact for some $x_o \in R$. Then the sequence $\{g_n\}$, sub-set of H, is pre-compact. If there are two-points x_o, x_1 $(x_1 \neq x_o)$, such that*

$$\lim g_n x_i = y_i \qquad (i = 0, 1) \tag{4}$$

exist, then

$$\lim g_n = g \tag{5}$$

exists and belongs to G. If $y_o = x_o$ and $y_1 = x_1$, then $g = e$.

PROOF. Set $S^2 = R \cup \omega$; $\hat{g} : R \to S^2$ denotes the map g with enlarged, compact range. By replacing the range of each map by a compact space, we can apply the Ascoli theorem to the equi-continuous sequence $\{\hat{g}_n\}$. This shows that the sequence can be thinned out and a convergent sub-sequence $\{\hat{g}_n : n \in I\}$ results :

$$k = \lim_{n \in I} \hat{g}_n \qquad (k : R \to R \cup \omega) \tag{6}$$

We define

$$\Omega = k^{-1} \omega \tag{7}$$

Our aim will now be to show

$$\Omega = \emptyset . \tag{8}$$

Before arriving to this conclusion, we have to discuss the structure of k, and apply Kerékjártó's theorem in two different ways.

As the complement of Ω contains $\{g_n x_o\}$, we have $\Omega \neq R$. Let us prove the the following statements :

$$k \mid R - \Omega \text{ is a one-to-one map} \tag{9}$$

$$k^{-1} \mid R - \Omega \text{ is continuous at each point of definition.} \tag{10}$$

PROOF OF (9). Let us suppose that, contrary to *(9)*, $k(x_1) = k(x_2) = y$, $x_2 \neq x_1$ x_1, x_2 in $R - \Omega$. We will show that this leads to a contradiction. As a point of fact, if $\varphi(y, \varepsilon)$ is the modulus of G, and $3\,\varepsilon < \sigma(x_1, x_2)$, the sequence $\{g_n^{-1}, n \in I\}$

takes points from the disc centered to y and of radius $\varphi(y, \varepsilon)$ into points at distance $> \varepsilon$, and this is a contradiction.

PROOF OF (10). On every compact set contained in R, k^{-1} is uniform limit of the equi-continuous set of functions g_n^{-1}, $n \in I$, by *(9)* above. Hence k^{-1} is continuous. Let us add here that there may be points x, where $k^{-1}x$ is not defined, or the set $g_n^{-1}x$ is not precompact.

From here on, we divide the proof in parts (a),(b) depending whether Kerékjártó's theorem applies to a map with fixed point or to a map without fixed point. In both cases we want to prove *(8)*.

(a) We suppose that there is a point $z_o \in R$ such that

$$(11) \qquad k z_o = z_o \qquad (z_o \in R).$$

Let us define

$$(12) \qquad \Omega_\infty = \{ x \in R, k^p x = \omega \text{ for some } p \geq 0 \};$$

$$(13) \qquad U = R - \Omega_\infty .$$

Then $z_o \in U$, and a disc centered to z_o also belongs to U by Section II.8, Lemma 4; in particular, U is not empty.

Let V be the interior of the component of z_o in U. By definition, and by the results above V is not empty, connected open set in R. Let us prove :

$$(14) \qquad V \text{ is simply connected.}$$

PROOF OF (14). Let us suppose that, contrary to *(14)*, there is a Jordan curve C in V whose bounded complement W contains a point $y \in \Omega_\infty$. From this hypothesis we arrive at a contradiction as follows. $\hat{g}_i^p(W)$, for some fixed p, contains points from arbitrary neighborhoods of ω, hence its boundary curve $g_i^p(C)$ has the same property as g_i^p is a homeomorphism. Let $y_i \in g_i^p(C)$ be such a point; $x_i = g_i^{-p} y_i$ is on C, and contains a convergent sub-sequence with limit point x. By equi-continuity, the sequence $g_i^p(x)$ converges to ω, contrary to the construction $C \subset V$ and the definition of V.

$$(15) \qquad k_o = k \mid V \text{ is a homeomorphism of } V \text{ onto } V;$$

$$(16) \qquad k_o \text{ conserves the orientation of } V;$$

$$(17) \qquad \{ k_o^p \} \text{ is an equi-continuous family.}$$

PROOF OF (15). No point of V belongs to the closure of Ω_∞, thus the sequences $\{ g_i^p \}$, $p = 0, \pm 1, \ldots$ are all bounded at every $x \in V$. Hence $k^{-1}x$ exists; it is continuous. Let P be a path from z_o (see *(11)*) to $x \in V$. Then $k^{-1}P$ is a path from z_o to $k^{-1}x$, thus $k^{-1}x \in V$. This proves *(15)*.

PROOF OF (16). On compact sub-sets of V, k_o is the uniform limit of sense preserving homeomorphisms. This remark, used in connection with some standard cohomology theory, or direct reasoning, enables one to prove *(16)*.

PROOF OF (17). By the proof of *(15)*, k_o^p is the limit of an equi-continuous sequence of transformations. (See Section II.7, Lemma 2).

From Kerékjártó's theorem we deduce immediately

$$(18) \qquad k_o \text{ is a topological rotation.}$$

In particular, we have

(19) There is a family of Jordan curves C_t, t real, $t > 0$, filling $V - z_o$ and invariant under k_o.

After this preparation, we will be able to prove

$$(20) \qquad V = R\,.$$

PROOF OF (20). If *(20)* is not true, there is a point y_o on the boundary $\overline{V} - V$ of V. Let us suppose now first that the rotation number θ of k_o is such that $\theta/2\pi$ is rational, hence there is an integer m such that k_o^m is the identity in V; we may suppose that $k^m y_o = \omega$. Then k^m, defined by the sequence $\{g_i^m : i \in I\}$ is not continuous at y_o, because $k^m \mid V$ is the identity, and $k^m y_o = \omega$. This is a contradiction. We consider now the case when $\theta/2\pi$ is irrational, and we fix a sequence m_k such that $m_k\theta \to 0$ $(mod\, 2\pi)$, thus $k_o^m k$ tend to the identity map in V. We consider now the table $\{g_i^m k\}$. For every fixed point $y_1 \neq y_o$, we can find a sub-sequence $i \in J$, $k \in K$, having the following properties :

$$(21) \qquad g_i^m k(y_o) \to \omega,\ g_i^m k(y_1) \to y_1 \quad i \in I(y_1), k \in K(y_1).$$

Hence, we can satisfy these conditions for a set Y everywhere dense in V, and a suitable sub-sequence $\{i \in J', k \in K'\}$. As above this leads to a contradiction.

This completes the proof of *(3)*, or *(20)*, in case *(11)* holds true.

(b) We suppose that *(11)* cannot be satisfied, hence

$$(22) \qquad kx \neq x \qquad (\text{for all } x \in R)$$

holds true. In this case we will use properties of «translation arcs» in the sense of Brouwer (see [1], [11], [16]).

Let us prove

$$(23) \qquad \Omega \cap k^{-1}\Omega = \emptyset\,.$$

In fact, if a point x is in $k^{-1}\Omega$, then $kx \in \Omega$. But $\Omega \subset R$, thus $kx \neq \omega$, hence $x \notin \Omega$. Thus the closed set Ω, and

$$\Omega_1 = k^{-1}\Omega$$

are disjoint. We set :

$$\Omega_p = k^{-1}\Omega_{p-1} \qquad (p \geq 2; \Omega_o = \Omega);$$

we want to show that the set

$$\Omega_* = \cup \Omega_p$$

is closed.

Let λ be a «translation arc» connecting $x_o \in \Omega_1$ and $kx_o \in \Omega$, not intersecting Ω except kx_o. Then

$$\Lambda = \cup k^{-p}\lambda$$

is a curve in the plane, which is a Jordan arc on S^2 connecting kx_o to ω. This follows from Brouwer's theory of translation arcs. Supposing that Ω_* is not closed, an appropriate sub-arc of Λ closed by a small segment is a Jordan curve whose bounded complement contains a fixed point (a z_o such that *(11)* holds true). This argument shows that Ω is closed.

Now $k\Omega_* = \Omega_*$, thus, if we set

$$U = R - \Omega_* \tag{25}$$

then we have $kU = U$. As R is not the union of a countable family of pairwise disjoint closed sets, $U \neq \emptyset$.

As in part (a), it is easy to see that

$$k_o = k \mid U \tag{26}$$

satisfies the conditions of Kerékjártó's theorem. By hypothesis *(22)* holds true, hence k_o is a topological translation. Hence there is an

E : translation domain for k_o limited by L, k_oL.

The set Ω_1 is closed, connected, and, if it is not empty, it will contain two points y_1, y_2 belonging to the closure E' of a $k_o^m E$, m integer. Then E' and $k_o^{-1}E'$ intersect in a point of the closure of $L' = k_o^m L$, and the same holds true for Ω, Ω_1. This contradiction shows that *(8)* holds true. We have thus established the first statement of the theorem.

Let us suppose that *(4)* holds true. Let h_1, h_2 be two cluster points of the sequence $\{g_n\}$. Then $h_2^{-1}h_1$ has two different fixed points, hence, by Kerékjártó's theorem, it is the identity. This shows that *(5)* holds true, and concludes the proof of the theorem.

4. Equi-continuity is closely related to the concept of n-rigidity introduced by Hilbert and used in his topological characterization of G_E and G_B (see Section I.6). A slight generalization of Hilbert's definition follows :

DEFINITION 3. *Let G be a given set of homeomorphisms of a space R onto itself. G is called n-rigid, if the following conditions are satisfied. Given the points x_{ir}, x_i, y_{is}, y_i, $i = 1, \ldots, n$, $r, s = 1, 2, \ldots$, such that*

$$x_i = \lim_{r \to \infty} x_{ir} \qquad (27)$$

$$y_i = \lim_{s \to \infty} y_{is} \qquad (28) \qquad (i = 1, \ldots, n)$$

and homeomorphisms $g_r \in G$ such that

$$g_r(x_{ir}) = y_{ir} \qquad (i = 1, \ldots, n; r = 1, 2, \ldots) \qquad (29)$$

then, by the condition of n-rigidity, there is at least one $g \in G$, such that

$$g(x_i) = y_i \qquad (i = 1, \ldots, n) \qquad (30)$$

holds true.

REMARKS. 1. A compact family of transformations is n-rigid, for every n. PROOF. Replace $\{g_r\}$ by a convergent sub-sequence, and take as g the limit of this sub-sequence 2. If $m \leq n$, then n-rigidity implies m-rigidity. PROOF. In the definition, we do not suppose $x_i \neq x_j$ for $i \neq j$. 3. We can use the concept of 1-rigidity in the definition of n-rigidity, as follows. If $R \times \ldots \times R$ is the product space of n equal factor, every $g \in G$ defines a map

$$\hat{g} : R \times \ldots \times R \to R \times \ldots \times R .$$

G is n-rigid, if and only if the set $\hat{G} = \{\hat{g} : g \in G\}$ is one-rigid.

THEOREM 11. *Let G be an equi-continuous group of orientation preserving homeomorphisms of the plane R onto itself, which is closed in H. Then G is n-rigid for all integers $n \geq 1$.*

PROOF. We use the notations of Definition 3 above. We take *(27)* for $i = 1$. Let us prove that the sequence g_r is bounded at the point x_1. Let D be a compact disc centered to y_1, and $\varepsilon > 0$ so small that the union of discs of radius ε intersecting D be compact (recall that we use the spherical metric of R). For $r \geq r_0$, $\sigma(x_1, x_{1r}) < \varphi(x_1, \varepsilon)$ and $x_{1r} \in D$, thus the disc of radius ε and centered to $g_r(x_1)$ intersects D. Hence, by Theorem 10, the closure of $\{g_r\}$ in G is compact, and our Remark 1 above applies.

THEOREM 12. *Let $G \subset H_+$ be a two rigid group of homeomorphisms. Then G is equicontinuous and closed in H.*

PROOF. Let us suppose that, contrary to the first claim of Theorem 12, there is a point $x_o \in R$, a sequence of points x_{or} and a sequence of $g_r \in G$, such that

$$(31) \qquad \lim_{r \to \infty} x_{or} = x_o$$

$$(32) \qquad \sigma(g_r(x_o), g_r(x_{or})) \geq \varepsilon_o$$

holds true. Passing to a sub-sequence g_r, $r \in I$, if necessary, we may suppose that

$$\lim_{r \in I} g_r(x_o) = y_o$$

$$\lim_{r \in I} g_r(x_{or}) = z_o$$

exist on the sphere $S^2 = R \cup \omega$. One of the points y_o, z_o may be the point ω, but, by (32), not both of them.

If $y_o \neq \omega$ and $z_o \neq \omega$ the two rigidity condition implies that there be a g mapping x_o into y_o and x_o into z_o. This is a contradiction, hence we have $y_o = \omega$ or $z_o = \omega$.

Let us suppose that $y_o = \omega$, and let C be the disc on S^2 centered to ω and of radius $\varepsilon_o/2$. The segment $[x_o, x_{or}]$ intersects the boundary of C in y_r, if r is large enough. Set $y_{or} = g_r^{-1} y_r$. Then y_{or} is on the segment $[x_o, x_{or}]$, hence $\lim y_{or} = x_o$. By thinning out the sequence I, we may suppose, we get a convergent sequence

$$\lim_{i \in J} y_r = y \qquad (J \subset I).$$

Again, we have a contradiction with the rigidity condition. The case, when $z_o = \omega$, is similar, thus we proved that G is an equi-continuous family.

Let us prove that G is closed in H. Given $g = \lim g_n, g_n \in G$. Then the conditions of Definition 3 are satisfied with $x_{ir} = x_i, y_{ir} = g_r(x_i)$, $i = 1, 2$. Thus there is an $h \in G$, such that $h(x_i) = g(x_i)$, $i = 1, 2$. As the closure $\bar{G}$ of G in H is an equicontinuous family by the first statement of Theorem 12, which has been proved already, and by Lemma 2 in Section II.6, Kerékjártó's theorem (Section I.6) applies to $h^{-1}g$ and shows that it is the identity, as it has two fixed points. Thus $g = h$, hence G is closed in H.

COROLLARY 1. *G is two rigid if and only if both conditions (1), (2) are satisfied.*

COROLLARY 2. *A two rigid group $\subset H_+$ is n-rigid for all integers $n \geq 1$.*

In view of these results, we can use the following definition.

DEFINITION 4. *If the group G acts on the plane, $G \subset H_+$, and is n-rigid for some $n \geq 2$, then it is termed rigid.*

Equivalently, G is termed rigid if it satisfies both conditions $(1), (2)$.

Let us emphasize again that this definition applies only to groups acting on the plane. For other transformation groups n-rigidity clearly implies m-rigidity, $m \leq n$, but I do not know other non trivial cases of converse implications.

By Theorem 8, hence, essentially by a simple application of the main result of [4], a rigid group is locally compact. The group of homoteties acting on R is locally compact but not rigid. It would be desirable to have a theory of groups acting rigidly on spaces.

IV. Proof of the Main-Theorem for Zero-Dimensional Groups.

1. We begin presently a deeper study of rigid groups G acting on the plane R. We will prove the Main Theorem (Section I.2) in case

$$(1) \qquad G \subset H_+ \text{ is rigid}$$

$$(2) \qquad \dim G = 0$$

(see Section III.4, Definition 4). We will use methods suitable to the case at hand. We consider the quotient space

$$(3) \qquad R^* = R/G$$

and the canonical map

$$(4) \qquad f : R \to R/G .$$

Our aim is to introduce a Euclidean or Bolyai metric in R, in which G is a group of isometries. We will find this via an appropriate conformal structure.

In view of Section II.7, Theorem 8, we can apply in the future results concerning locally compact groups in our study of rigid groups. The extensive and deep theory of locally compact groups is not always helpful, however, because we have to deal with «highly discontinuous groups». An exception is the Theorem 13 formulated below, where a non trivial result of the general theory has important implications in the present special case.

2. Unless the contrary is stated, we suppose $(1), (2)$ above. A point x^* of the quotient space (3) is sometimes thought of as a set

$$(4) \qquad x^* = G(x) = \{ y = gx, g \in G \}$$

in the plane R, sometimes a point of a quotient space. We set :

(5) $$E = \{ x \in R, G_x \neq \{ e \} \}$$

(see (6), (7) in Section II.6); in other words, E is the set of fixed points of topological rotations ($\neq e$) contained in G. We will prove that both E and

(6) $$E^* = f(E),$$

where f is the map (4), are discrete sub-spaces of R, R^* respectively.

The following result from the general theory of locally compact groups will be useful (see [15]).

THEOREME 13. *In a separable, metric, locally compact topological group G, which is zero-dimensional, the open sub-groups form a complete system of neighborhoods of the neutral element e.*

Thus, if U is any neighborhood of e in G, then there exists a sub-group K of G such that $K \subset U$, and K is open in G, in particular a neighborhood of e. Since K is an open set the quotient space G/K is discrete.

We will also need the following theorem from [15].

THEOREM 14. *There is a function $\eta(m)$, m integer, $\eta(m) > 0$, such that every periodic map $r : S^2 \to S^2$ of period m moves at least one point at a distance $\geq \eta(m)$.*

From these theorems, and the Theorem of Brouwer (Section I.6), we will deduce now the following result.

THEOREM 15. *Let $G \subset H_+$ be a zero-dimensional, rigid group acting on the plane. Then every $x \in R$ is the center of a topological disc $D(x)$ (i.e. a closed domain, limited by a Jordan curve), such that*

(7) $$\text{if } D(x) \cap gD(x) = \emptyset, \text{ then } gx = x, \text{ and } gD(x) = D(x).$$

Hence for $x \notin E$ (see (5)), the topological discs

(8) $$gD(x) \qquad (g \in G)$$

are pairwise disjoint, and

(9) $$f \mid D(x) \text{ is a homeomorphism} \qquad (x \notin E).$$

If, however, $x \in E$, $f \mid D(x)$ identifies points as a cyclic group of rotations of a disc; in particular, the quotient space is homeomorphic to a disc.

PROOF. Given $x_0 \in R$, let V be an open disc centered to x_0. Set

(10) $$U = \{ g \in G, gx_0 \in V \}.$$

By the definition of the metric Δ on H (see (2) in Section II.5), U is an open neighborhood of the identity e of G, and by Theorem 10, Section III.3, it is precompact. By

Theorem 13 above, U contains a sub-group K° which is open; its closure K is then a compact neighborhood of e in G, and a group.

If t is a topological translation, its powers are all distinct, and have no cluster points, hence such a t cannot belong to K. In other words all elements of K are topological rotations.

We extend each $g \in K$ into a map $S^2 \to S^2$. We have then a group with the following properties. K is compact, zero-dimensional, rigid, and every $g \in K$ is a topological rotation of S^2. We want to prove the following statement :

(11) there is an $\varepsilon_o > 0$, such that every $r \in G$ $(r \neq e)$ moves at least one point at a distance $\geq \varepsilon_o$.

PROOF OF (11). Supposing that (11) is not true, for every n, we have an $r_n \in G$, $r_n \neq e$, such that $\Delta(e, r_n) < 1/n$. By Theorem 14, the order of r_n tends to infinity with n, thus the rotation numbers θ_n tend to 0 with n. We may suppose that the centers of rotations converge to a point z. Given a number $\theta (mod\ 2\pi)$, we can take appropriate powers m_n of r_n in such a way that

$$r_n^{m_n}$$

will converge to an element of K which has $\theta (mod\ 2\pi)$ as rotation number. This means, however, that a full rotation group of dimension 1 is contained in K, which is a contradiction with the hypothesis $dim\ K = 0$.

Let W be a neighborhood of e in K defined by $W = \{ g \in K, \Delta(e, g) < \varepsilon_o/2 \}$, where ε_o is the number appearing in (11). By (11) this neighborhood is reduced to the neutral element e, thus K is discrete. As it is compact, it follows immediately that it is a finite set.

In particular, $K(x_o)$ is a finite set, as well as $G(x_o) \cap U$ in (10) is finite, and using the equicontinuity of the group, it is easy to find a topological disc $D(x)$ having the properties formulated in the theorem, whose proof is thus complete.

THEOREM 16. *If G is a zero-dimensional rigid group acting on the plane ($G \subset H_+$), then $R^* = R/G$ is a surface, i.e. a connected, separable metric space, which is a C° two-manifold without boundary. The set of points E of centers of topological rotations of G (see (5)) is discrete in R, and its image E^* is a discrete sub-space of R/G.*

PROOF. Let $x^* \in E^*$ be given, and be $x \in x^*$ (see (4)); let $D(x)$ be the disc whose existence is established in Theorem 15. Two points x_1, x_2 of $D(x)$ are equivalent under G, if and only if $x_2 = g^p x_1$, where g is the generator of G_x. By Theorem 15, $D(x)^*$ is homeomorphic to a disc(although, $f \mid D(x)$ is not a homeomorphism). In particular, any two points of $D(x)^*$ have disjoint neighborhoods.

Let $x^* \notin E^*$ be given. Then by Theorem 15, the disc $D(x)$ is mapped by $f \mid D(x)$ homeomorphically onto $D(x)^*$, and $f^{-1}D(x)^*$ is a family of pairwise disjoint topological discs. This proves that R^* is a Hausdorff space, a metric manifold, and that E^* is discrete in R^*. The proof of Theorem 6 is thus complete.

3. As E is a discrete sub-space of R, it is countable. We choose a countable family $\{U_i\}$ such that U_i $(i = 1, 2, \dots)$ be a topological disc of the type $D(x)$ of Theorem 15. Then there is a homeomorphism

$$h_i : U_i \to B \qquad (B = \{(\xi_1, \xi_2), \xi_1^2 + \xi_2^2 \leq 1) \tag{12}$$

such that

$$\rho = h_i g h_i^{-1} \tag{13}$$

is a Euclidean rotation of angle $2\pi / m_i$, where g is the generator of G_x, $x \in E$. The map h_i defines a Euclidean metric in U_i, in which g is a rotation. Let us call ds_1^2 the Riemannian metric in

$$U = \cup U_i$$

whose restriction to U_i is the Euclidean metric mentioned above (see (12), (13); $g \in G_x$ is a rotation in this metric).

By the choice of U (see Theorem 15), $fU = U^*$ is the union of disjoint discs. Let V^* be an open set containing the closure of $R^* - U^*$, and such that, if we set

$$V = f^{-1} V^*$$

then $U_i \cap V$ be the set $\frac{1}{2} < \xi_1^2 + \xi_2^2 \leq 1$ in the coordinate system used in (12). We choose a C^∞ Riemannian metric in V^*, and we denote by ds_2^2 the arc element obtained in V by $f \mid V$.

Let u, v be a C^∞ partition of the unity, belonging to the covering U, V of R:

$$u(x) \geq 0,\ v(x) \geq 0,\ \ u, v\ \ C^\infty \text{ in } R \tag{14}$$

$$u(x) + v(x) = 1 \tag{15}$$

$$u \mid \overline{V} = 0, \qquad v \mid \overline{U} = 0. \tag{16}$$

It is easy to see that u can be chosen such that

$$u(gx) = u(x) \qquad \text{(for every } g \in G). \tag{17}$$

It is enough to average u in some U_i by a finite group G_{x_0}, and define it coherently in the other discs $U_j = g\,U_i$. By (15), we also have

$$v(gx) = v(x).$$

We define now a Riemannian metric ds^2 in R by

$$ds^2 = u\,ds_1^2 + v\,ds_2^2 \,. \tag{18}$$

Then ds^2 is invariant under all transformations $g \in G$. Of course, we do not claim, that the curvature of this metric is constant.

4. The plane R, given the metric *(18)*, can be mapped conformally either onto the whole complex plane C by a

$$\varphi : R \to C$$

or onto the open unit disc $D = \{ z \in C, |z| < 1 \}$ by

$$\psi : R \to D \,.$$

In the first case

$$G' = \varphi G \varphi^{-1}$$

is a group of conformal maps of C onto itself, in the second case

$$G'' = \psi G \psi^{-1}$$

is a group of conformal transformations of D onto itself (see [2]).

In the first case the Hermitian metric of C is invariant under G', in the second case the Boyai metric of D is invariant under G'' (Poincaré model, see [2]). Transporting the appropriate Riemannian metric onto R, we have represented G as a group of isometries of a Riemannian metric of constant curvature.

This completes the proof of the Main Theorem, and of Theorem 1, in the special case of zero-dimensional groups G. Also, Theorem 3 is proved, when $M = R$ and $\dim G = 0$.

V. Proof of the Main Theorem for Positive-Dimensional Groups.

1. In the case of positive dimensional groups acting on the plane, we have to use a method which is very different from the one used in Section 4 for zero-dimensional groups. We will use presently the continuous map

$$F : G \to R,\ Fg = gx_o \qquad (x_o \text{ fixed in } R) \tag{1}$$

and its properties. The main fact is again the result of Theorem 10, Section III.3, which implies presently that F is a «proper» map, that is to say

(2) *if C is compact in R, then $F^{-1}C$ is compact.*

PROOF OF (2). $F^{-1}C = \{ g \in G, gx_o \in C \}$, hence by the first statement of Theorem 10, Section III.3, this set is precompact thus, by continuity of F, compact. The quotient space R / G will not be so important as in the previous Section. Also, in Section 4, the deep results used were Kerékjártó's and Brouwer's theorem on cyclic group. In the present case

$$(3) \qquad dim\ G \geq 1$$

Hilbert's theorem (Section I.6) will be the most important. Not only the final form of Hilbert's result as stated in Section I.6 will be used, but some other results of [9] will be formulated and used.

2. Let us formulate first a theorem of [9], using hypotheses as stated in the original form. We use here the following notations: P stands for the full rotation group, i.e. the group of transformations *(4)* in Section I.3 with $\alpha = \beta = 0$ and unrestricted θ.

THEOREM 17. *Let $G \subset H_+$ be a group acting on the plane R, and such that*

$$(4) \qquad G \text{ is three-rigid}$$

$$(5) \qquad \text{for every } x(\neq a),\ G_a(x) \text{ is infinite}.$$

Then the sub-group G_a of G is conjugate in H to the full group of rotations of the Euclidean metric, i.e.

$$(6) \qquad h^{-1} G_a h = P$$

holds true for an appropriate $h \in H$.

The essential part of Hilbert's proof of this theorem is, of course, the analysis of the homogeneous space $G_a(x)$, and its «position» in the plane. Then the family of all the curves $G_a(x)$ must be analysed.

F. Riesz showed in [17] how Hilbert's method of proving that $G_a(x)$ is a Jordan curve can be used to prove Schoenfliesz' theorem stated below.

THEOREM 18. *Let J be a compact set in R, whose complement contains two components U, V with the following properties. Let $a \in U$, $b \in V$ be chosen. For every $x \in J$, there is a Jordan arc from a to b intersecting J in the given point x only. Under these conditions, J is a Jordan curve, i.e. homeomorphic to a circle, its complement being $U \cup V$.*

The so called Jordan curve theorem, complemented with the statement on the «accessibility of points of a Jordan curve», states that every Jordan curve has the properties formulated as hypotheses of Theorem 18.

Although Theorem 17 does not imply Theorem 16 immediately, its use simplifies Hilbert's proof. (This does not contradicts to the fact, that Hilbert's method gives the simplest known proof of Theorem 17). Using these classical results, let us prove a stronger form of Theorem 7.

THEOREM 19. *Let G be a rigid group acting on the plane ($G \subset H_+$), and such that $G_a(x)$ be infinite for some x. Then (6) holds true.*

PROOF. Let us show that a G_a satisfying the conditions of Theorem 19 has property *(5)*; as to condition *(4)*, we have the results of Section III.4.

Let us suppose that, contrary to our claim, there is a point $x_o \neq a$, such that $G_a(x_o) = \{g_o x_o, \ldots, g_k x_o\}$ is a finite set. Then for every g in G_a there is a g_i, such that $gx_o = g_i x_o$, hence $g^{-1}g_i$ has two fixed points a and x_o. By Kerékjártó's theorem this implies $g^{-1}g_i = e$, thus $G_a = \{g_o, \ldots, g_k\}$. Then $G_a(x)$ is also finite contrary to hypothesis. Hence, Theorem 17 implies Theorem 19, whose proof is thus complete.

THEOREM 20. *Let G be a rigid group, $G \subset H_+$, such that*

$$\dim G_a \geq 1 \tag{7}$$

for a sub-group of isotropy. Then (6) holds true.

PROOF. Let us take $x_o \neq a$ in the definition of F in *(1)*. By Kerékjártó's theorem F is a homeomorphism (F is continuous and *(2)* holds true). Thus $\dim G_a(x_o) \geq 1$, hence $G_a(x_o)$ is not finite. Hence the conditions of Theorem 19 are satisfied, and the conclusion follows.

3. The results above concern the action of the compact one-dimensional sub-groups of G. We need similar results concerning the non compact one-dimensional sub-groups. We have first a classical result due to Brouwer.

THEOREM 21. *Let the additive group R^1 of real numbers act effectively on R. Then this transformation group is conjugate in H to the one parameter sub-group of G_E given by*

$$\xi' = \xi + \alpha, \quad \eta' = \eta \qquad (\alpha \in R^1) \tag{8}$$

in Cartesian coordinates.

With this statement at hand, we can easily prove the following result.

THEOREM 22. *Suppose that the rigid group $G \subset H_+$, has the following properties :*

$$G_a = \{e\} \text{ for every } a \in R \tag{9}$$

$$\dim G = 1 \tag{10}$$

Under these conditions.

$$G^e \text{ component of the neutral element } e \in G \tag{11}$$

is conjugate in H to the transformation group (8). For G itself, we have either $G = G^e$, or G is conjugate in H to the transformation group given by

$$\xi' = \xi + \alpha, \quad \eta' = \eta + m \qquad (\alpha \in R^1,\ m \text{ integer}) \tag{12}$$

in Cartesian coordinates.

PROOF. We suppose first that G is connected, thus $G = G^e$. We choose arbitrarily $x_o \in R$, fix it, and consider the map F in *(1)*. We will prove :

(13) $$F : G \to G(x_o) \text{ is a homeomorphism.}$$

By condition *(9)*, F is one-to-one. If we restrict F to a compact neighborhood of a $g \in G$, the image will be compact, hence the conclusion follows by *(2)*, *(9)* (see [10]).

We want to show now that G is isomorphic to the additive group R^1 of reals, and apply Theorem 21. The first, and most difficult step is to show, proving the conditions of Theorem 18, that

(14) $$F(G) \cup \omega$$

is a Jordan curve on the sphere $S^2 = R \cup \omega$; the structure of G is then easy to determine.

By *(13)* and hypothesis $G^e = G$, $G(x_o)$ is a connected sub-space of R. Because of *(10)*, every sufficiently small open neighborhood of e in G has a non-empty boundary, thus there is a sequence $\{g_n\}$ of maps in G, $g_k \neq g_l$, if $k \neq l$, converging to e. In particular

(15) $$\lim_{n \to \infty} g_n(x_o) = x_o .$$

For n fixed, we have

(16) $$\lim_{p \to \infty} g_n^p(x_o) = \omega$$

on the sphere S^2, as every $g \in G$ is a topological translation by hypothesis *(9)*. Equi-continuity of G, and *(15)*, *(16)* show that for every $\varepsilon > 0$, there is a sequence of points

(17) $$\ldots, g_n^{-2} x_o, g_n^{-1} x_o, x_o, g_n x_o, g_n^2 x_o, \ldots$$

clustering at ω only on the sphere.

Let us prove the following statement. Given $x \in G(x_o)$ and $y \notin G(x_o)$, there is a Jordan arc J with endpoints x, y, and such that

$$J \cap G(x_o) = \{x\};$$

under these conditions we say that «the point $x \in G(x_o)$ is accessible from y». The proof is as follows. Clearly,

$$G(x) \cap G(y) = \emptyset \qquad (y \neq x).$$

Let $[a, b]$ be a segment connecting the points $a \in G(x)$, $b \in G(y)$, and such that $(G(x) \cup G(y)) \cap [a, b]$ be the pair of points a, b. (Connect any point of $G(x)$ to any point of $G(y)$, and take appropriate sub-segment of this segment. Such a sub-segment will exist as both $G(x)$ and $G(y)$ are closed and disjoint). Now there is a map $g \in G$

such that $ga = x_o$; gb is on $G(y)$ which is connected and does not intersect $G(x)$; hence $g([a, b])$ completed with an appropriate arc gives a J having the stated properties.

We consider the set *(14)*, and will prove that ω is accessible from every point $y \in R - G(x_o)$.

If in *(17)* we choose an n large enough, $g_n y$ is so near to y that the circle passing through $g_n y$ and centered to y does not intersect $G(x_o)$. Then, by a construction of Terasaka (see [18]), x_o is on a translation arc λ such that

$$\Lambda_+ = \cup \{ g_n^p \lambda : p = 0, 1, \dots \} \tag{18}$$

does not intersect $G(x_o)$. As g_n is a topological translation, the set $\Lambda \cup \omega$ is a Jordan arc. This proves accessibility of ω on $G(x_o) \cup \omega$.

We want to prove now that the complement of *(14)* on S^2 has two components at least. Let us consider a translation arc for g_n in *(17)* connecting x_o and $g_n x_o$. Then

$$\Lambda = \cup \{ g_n^p \lambda : p = 0, \pm 1, \pm 2, \dots \} \tag{19}$$

is such that $\Lambda \cup \omega$ is a Jordan curve on the sphere S^2; for n large the diameter of $g_n^p \lambda$ is $< \varepsilon$. As both $G(x)$ and Λ are invariant under g_n, it is easy to construct a pair of points a, b separated by $G(x_o)$.

We have thus established all hypotheses of Theorem 18; the conclusion is that $G(x_o) \cup \omega$ is a Jordan curve on the sphere S^2. $G(x_o)$, thus by *(13)*, the space of G itself is homeomorphic to R^1. As the real line has but one topological group structure, the conditions of Theorem 21 are satisfied and the conclusion is : G is conjugate in H to the group *(8)*.

In the case $G \neq G^e$, we have these conclusions for G^e. Then there is an $h \in G$, $h \notin G^e$. For all such $h's$, we construct the set

$$h\, G^e(x_o) \qquad (h \in G^e). \tag{20}$$

These sets are pairwise disjoint; completed by ω, each is a Jordan curve on S^2. Supposing that there is a sequence $h_n \in G$, $h_n \notin G^e$, such that $\lim h_n(x_o) = x_o$, the equi-continuity of G would imply that the union of the sets *(20)* is everywhere dense in the plane. As *(1)* is a homeomorphism (by the hypothesis *(10)*), this set is locally compact, thus identical to the plane. This is, however, in contradiction with *(10)*. This shows the impossibility of our hypothesis, hence we have

(21) there is an h_1, such that $h_1^{-1} G(x_o)$, $h_1 G(x_o)$ separate $G(x_o)$ from all sets *(20)*.

This implies then

$$(22) \qquad \{ h G^e(x_o) : h \in G^e \} = \{ h_1^p G(x_o) : p = 0, \pm 1, \dots \}.$$

Let us map now the closed strip $-1 \leq \eta \leq 1$ by k_1 onto the strip $h_1^{-1}G(x_o), h_1 G(x_o)$ in such a way that $\eta = 0$ be mapped onto $G(x_o)$ and that $k_1 G^e k_1^{-1}$ be the restriction of the group (8). Using maps $l^p k_1^{-1}$, $l(\xi, \eta) = (\xi, \eta + p)$, p integer, this map can be extended to the whole plane, and this proves the last conclusion of Theorem 22, whose proof is complete.

4. In the previous sections we proved under various conditions that the group G is locally Euclidean, i.e. has a neighborhood of e which is homeomorphic to a Euclidean space. The theorem below describes the structure of two-dimensional, locally Euclidean, simply connected groups. The theorem is due to Brouwer, and was the first result toward the complete solution by Montgomery, Zippin and Gleason of Hilbert's Vth Problem.

THEOREM 23. *Let G be a separable, metric group, which is connected, simply connected, and contains an open set homeomorphic to R^2, i.e. the two-dimensional vector space over the field R^1 of reals. If G is abelian, it is isomorphic, as a topological group, to the additive group of R^2. If it is non abelian, it is isomorphic to the group*

$$(23) \qquad T(x) = ax + b \qquad (a, b \text{ reals}; a > 0 : x \text{ variable}).$$

In both cases G carries a left invariant Riemannian metric of constant curvature ≤ 0, hence an underlying conformal structure.

In (23) it is understood that the group operation is $(c, d)(a, b) = (ca, cb + d)$, and the variable is used to express the group conveniently. This group acts on the upper half-plane $C_+ = \{ z = x + iy, y > 0 \}$ as follows : $T(z) = az + b$. In particular, T maps i into $ai + b$. It is thus clear that (23) is simply transitive on C_+, whose structure can be pulled back onto G.

THEOREM 24. *Let us suppose that for a rigid group $G \subset H_+$, we have*

$$(24) \qquad \dim G = 2 .$$

Then G is locally Euclidean. G^e does not contain rotations, is simply transitive on R, and it is conjugate in H to a sub-group of G_E or to a sub-group of G_B. The plane R carries a Riemannian metric of constant curvature ≤ 0, preserved by all transformations $g \in G$.

PROOF. Let us suppose $\dim G(x_o) \leq 1$. By a known result in dimension theory (see [8], p. 102) $\dim G_a \geq 1$ for some $a \in R$. By Theorem 21, G_a is conjugate to the full rotation group. $\dim G \neq \dim G_a$ implies that there is a $g \in G$ such that $b = ga \neq a$.

$G_b = g G_a g^{-1}$, hence isomorphic to the full rotation group. We have proved : for every $x \in G(a)$, G_x is conjugate to the full rotation group. We will prove that these conditions imply :

$$(25) \qquad dim\ G \geq 3$$

contrary to our hypotheses.

For every $v \in G_z(y)$, we have $g^{-1} G_y g = G_z$, where $g \in G_z(y)$ and $gz = y$. Now, if w is in the component of $R - G_z(y)$ containing z, then $G_y(w)$ intersects $G_z(y)$, hence G_w is the full rotation group. This proves in particular : FG (see *(1)*) contains an open set such that, if v is in this set, G_v is the full group of rotations.

By this property, FG is an open sub-space of R, as it is homogeneous. At the same time, it is a closed sub-set by Theorem 10, Section III.3. Thus $F(G) = R$. Hence the Theorem of Hilbert, Section I.6 applies, implying *(25)* in particular, which is in contradiction with *(24)*.

We have thus the following result : under the conditions of Theorem 24

$$(26) \qquad dim\ G(x_o) = 2 .$$

$$(27) \qquad G_a \text{ is finite for every } a \in R.$$

The result *(26)* implies, by a theorem on dimension theory (see [8], p. 135) that $G(x_o)$ contains an open sub-set of R; by homogeneity it is thus R (see previous paragraph). Hence *(26)* can be replaced by

$$(28) \qquad G(x_o) = R .$$

Furthermore, *(27)* can be improved as follows :

$$(29) \qquad G_b = g^{-1} G_a g \qquad (gb = a)$$

for an appropriate $g \in G$, hence all groups G_a are cyclic of the same order.

Let us consider now the sub-group G^e. We will show that there is a neighborhood V of e in G^e such that $F \mid V$ is a homeomorphism of V onto an open set of R (see *(1)*). This will prove then that G is locally Euclidean.

Let a, b $(b \neq a)$, be two fixed points in R. Let us suppose that for every integer n, there is a topological rotation r_n such that

$$\sigma(a, r_n a) < \frac{1}{n}, \text{ and } \sigma(b, r_n b) < \frac{1}{n} \qquad (r_n \neq e).$$

We will arrive at a contradiction. By *(29)* the order of every r_n divides the same integer m. Hence, for n large enough we can construct two disjoint Jordan domains, each invariant under r_n. Each of these domains should contain a fixed point of r_n which is in contradiction with the fact that r_n is a topological rotation $\neq e$. We have thus the following result : there is an $\varepsilon_o > 0$ such that, if $g \in G$, $\sigma(a, ga) < \varepsilon_o$ and $\sigma(b, gb) < \varepsilon_o$, then g is a topological translation.

Set $V = \{g \in G, \sigma(a, ga) < \varepsilon_o/3, \sigma(b, gb) < \varepsilon_o/3\}$. If $g_1(x_o) = g_2(x_o)$ with g_1, g_2 in V, then $g_2^{-1}g_1$ satisfies the previous conditions, and has a fixed point x_o. Thus $g_2 = g_1$. We see thus; $F \mid V$ is one-to-one. By *(2)*, $F \mid V$ is then a homeomorphism. Now $F(V)$ contains inner points, hence G^e is locally Euclidean. Thus Theorem 23 applies, and the conclusion of Theorem 24 follows easily.

5. The last partial result needed before the complete proof of our Main Theorem is the following.

THEOREM 25. *Let G be a rigid group in H_+, such that*

$$\dim G \geq 3 . \tag{29}$$

Then G is conjugate in H to G_E or G_B, hence R carries a Riemannian metric of constant curvature ≤ 0, in which G is a group of isometries.

PROOF. For every map F in *(1)*, $\dim FG \leq 2$, thus $\dim G_y \geq 1$ for some y. As in a previous proof this implies the hypotheses of Hilbert's theorem (Section I.6), thus the statement of the theorem.

6. PROOF OF THE MAIN THEOREM. We consider a G satisfying the conditions of the Main Theorem. By Section III.4, we may suppose that G is rigid. We have the following possibilities :

$$\dim G = 0 \tag{30}$$

$$\dim G = 1 \tag{31}$$

$$\dim G = 2 \tag{32}$$

$$\dim G \geq 3. \tag{33}$$

In case of *(30)*, G has the properties stated in the Main Theorem in virtue of Section IV.4.

If *(32)* holds true, Theorem 24 states the conclusions; in case *(33)*, Theorem 25 is valid. The only case not covered completely by previous results is when *(31)* holds true.

Let then G be a one-dimensional rigid group. If $G_x = G$ for some $x \in R$, we have Theorem 21. If we have two points x, y $(y \neq x)$ such that both G_x and G_y are infinite, the proof of Theorem 24 shows that G has the properties stated in the Main Theorem. Thus we may suppose presently

$$G_x \text{ is finite, for every } x \in R . \tag{34}$$

By the proof of Theorem 24, G^e has a neighborhood V of the neutral element e

such that every $g \in V$ is a topological translation. Theorem 22 applies to G^e and shows that $G^e = R^1$, and its transformations are given by (8) in appropriate coordinates ξ, η to be used from now on.

As G^e is a normal sub-group of G, we have $hgx_o = hgh^{-1}hx_o = g'(hx_o)$, thus

$$h\,G^e(x_o) = G^e(hx_o)$$

for any $h \in G$. Now $L = G^e(x_o)$ is the straight line $\eta = const.$ in the plane. Thus, if $hL \neq L$, h is a topological translation. Let us suppose now that we have a sequence h_n of topological translations such that

$$h_n L \neq L \tag{35}$$

$$\lim h_n x_o = x_o\,. \tag{36}$$

By Theorem 10, Section III.3 this implies, for an appropriate partial sequence,

$$\lim h_n = e\,. \tag{37}$$

and this is a contradiction to (35), because (37) means that, for n large, $h_n \in G^e$.

Let L_1 be $h_1 L$ such that there are no hL's in the strip L, $h_1 L$. The sequence of lines

$$\ldots,\ h_1^{-1}L,\ L,\ h_1 L\ ,\ldots \tag{38}$$

contains all sets of the form hL. Hence G is generated by G^e, h_1, and all transformations k for which

$$k\,L = L\,. \tag{39}$$

Clearly, the transformations k satisfying (39), if any, are topological rotations of angle π.

With these data, it is easy to introduce new coordinates and a Euclidean metric such that G be a group of isometries in the metric of question. This completes the proof of the Main Theorem.

VI. The Main Theorem for Groups Acting on Surfaces.

1. The Main Theorem can be formulated for surfaces (see Section IV.2, in particular Theorem 16, for the concept of surface). Let us give just one formulation of it.

THEOREM 26. *Let M be an orientable surface, and G a group of orientation preserving homeomorphisms of M onto M, which is equi-continuous in a metric σ extending to the one point compactification of M, in case M is not compact. Then M carries a conformal structure for which every $g \in G$ is a conformal map.*

In case $M = S^2$, theorems by Brouwer and Kerékjártó give this result. We suppose, in what follows that M is not homeomorphic to S^2. Using the universal covering surface R of M, and «lifting» the maps $g : M \to M$ into maps $\hat{g} : R \to R$, it is easy to see that the conditions of the Main Theorem are satisfied and that the conclusion follows. This program is carried out in the next Section.

2. We will use [3] for references on covering spaces. Standard hypotheses are the following :

(1) $\qquad X$ is a connected, locally connected Hausdorff space

(2) $\qquad X$ is locally simply connected

(3) $\qquad p : \hat{X} \to X$ is a covering map .

LEMMA 5. *Let us suppose that X satisfies $(1), (2)$, and (3) is the universal covering. Given a continuous map $f : X \to X$, there are maps $\hat{f} : \hat{X} \to \hat{X}$ such that*

$$\begin{tikzcd} \hat{X} \arrow[r, "\hat{f}"] \arrow[d, "p"'] & \hat{X} \arrow[d, "p"] \\ X \arrow[r, "f"'] & X \end{tikzcd}$$

be a commutative diagram. If $\hat{f}_1, \hat{f}_2$ are two such maps, $\gamma \hat{f}_2 = \hat{f}_1$ holds true for suitable covering transformation $\gamma : \hat{X} \to \hat{X}$.

PROOF. Set $P(\hat{x}, \hat{y}) = (p\hat{x}, p\hat{y})$, then $\hat{P} : \hat{X} \times \hat{X} \to X \times X$ is the map of a universal covering space. We consider the sub-space

$$(5) \qquad F = \{ (x, fx) : x \in X \}$$

of $X \times X$ (graph of the map f); $x \to (x, fx)$ is a homeomorphism of X onto F. Let $\hat{F}_o$ be a connected component of $P^{-1}(F)$. We claim that $\hat{F}_o$ is the graph of a map $\hat{f} : \hat{X} \to \hat{X}$ and has the properties formulated in the Lemma.

Let us show that for every $\hat{x} \in \hat{X}$ there is an unique pair $(\hat{x}, \hat{y}) \in \hat{F}_o$; here we will use the monodromy principle. Let us introduce then the map

$$(6) \qquad \hat{x} \to \{ p^{-1} f p \hat{x} \}$$

of $\hat{X}$ into the family of discrete sub-sets of $\hat{X}$. Given $\hat{y}$ belonging to the set on the right-hand side of (6) there is an evenly covered connected, open neighborhood W of $p\hat{y}$ such that $p \mid \hat{W}_o$ is a homeomorphism onto W, if $\hat{W}_o$ is the component of $p^{-1}W$ containing $\hat{y}$. Let V be a connected, evenly covered open neighborhood of $p\hat{x}$, such that $fV \subset W$, and $\hat{V}_o$ the component of $p^{-1}V$ containing $\hat{x}$. Then, clearly, the right-hand side of (6) is the set of values of a family of continuous functions defined on $\hat{V}_o$ when $\hat{x}_o \in \hat{V}_o$; the ranges of these functions are disjoint. Thus the conditions of the monodromy principle are satisfied, and the principle shows that $\hat{F}_o$ is the graph of a function to be denoted by .

Given $\hat{f}_1$ and $\hat{f}_2$ as in the Lemma 5, let us consider a point $\hat{x}_o$ and an automorphism γ such that

$$\gamma \hat{f}_2 \hat{x}_o = \hat{f}_1 x_o .$$

Now $1 \times \gamma$ acting on $\hat{X} \times \hat{X}$ maps a connected component of $P^{-1}F$ into another component, and this proves the last statement of the Lemma.

THEOREME 27. *Let M be a space satisfying conditions (1), (2); let us denote $p: R \to M$ the universal covering, and $\pi = \{\gamma : R \to R\}$ the group of covering transformations. Let σ be a metric on M, and G an equi-continuous group acting on M. If $\hat{G}$ denotes the set of all maps $\hat{g}$ obtained by lifting in virtue of Lemma 5 all maps in G in all possible ways, then $\hat{G}$ is, in a suitable metric $\hat{\sigma}$ of $\hat{X}$ an equi-continuous group acting on $\hat{X}$, and containing π.*

PROOF. Let us lift the function $\sigma : M \times M \to R^1$ into a function $\hat{\sigma}_o : R \times R \to R^1$. It is easy to see that there is a metric $\hat{\sigma}$ which coincides with $\hat{\sigma}_o$ for nearbv points (i.e. defining the same uniform structure). In this metric, the equi-continuity modulus of $\hat{G}$ is the same as the modulus of G; in particular, π is a group of isor tries of $\hat{\sigma}$. The other statements of the Theorem are immediate.

3. PROOF OF THEOREM 26. We use the notations of the Theorem; we suppose that M is not S^2, hence the universal covering is the plane R. We apply Theorem 27, and we get a group $\hat{G}$ acting on R; as $\hat{G}$ is equi-continuous in the metric $\hat{\sigma}$, it will be equi-continuous in the spherical metric σ of R (see Section III.1). By Theorem 3, R carries a complex structure such that every $\hat{g} \in \hat{G}$ is an analytical map. As $\pi \subset \hat{G}$ the covering transformations $\gamma \in \pi$ are complex analytical. Thus the quotient space R/π carries an analytical structure preserved by all maps $g \in G$. This completes the proof of Theorem 26.

References.

[1] BROUWER. Beweis des ebenen Translations Satzes, Mathematische Annalen, 72 (1912), pp. 37-54.

[2] CARATHEODORY. Conformal Representation, Cambridge Tracts N° 28, Cambridge University Press, 1952.

[3] CHEVALLEY. Theory of Lie Groups, I, Princeton, University Press, 1946.

[4] VAN DANTZIG and VAN DER WAERDEN. Über metrisch homogene Raume, Abhandlungen Hamburg, 6 (1928), pp. 367-376.

[5] DIEUDONNE. On topological groups of homeomorphisms, Amer. Journ. of Math., 70 (1948), pp. 659-680.

[6] FARY. Sur les groupes d'homéomorphismes également continus du plan, Comptes rendus de l'Académie, Paris, 228 (1949), pp. 534-536.

[7] FARY. Sur la dimension des groupes d'homéomorphismes également continus du plan, Comptes rendus de l'Académie, Paris, 228 (1949), pp. 801-803.

[8] HUREWICZ and WALLMAN. Dimension Theory, Princeton University Press, 1948.

[9] HILBERT. Anhang IV, Über die Grundlagen der Geometrie, in Grundlagen der Geometrie, 4 th ed., Teubner, Berlin, 1913.

[10] KELLEY. General Topology, New York, van Nostrand, 1955.

[11] KEREKJARTO. Vorlesungen über Topologie, I, Springer, Berlin 1923.

[12] KEREKJARTO. Über die fixpunktfreien Abbildungen der Ebene, Acta Szeged 6 (1934), pp. 226-234.

[13] KEREKJARTO. Topologische Characterisierung der linearen Abbildungen, Acta Szeged, 6 (1934), pp. 235-262.

[14] KEREKJARTO. Sur la structure des transformations topologiques des surfaces en elles mêmes, L'enseignement mathématique, 35 (1936), pp. 297-316.

[15] NEWMAN. A theorem on periodic transformations of spaces, Quart. Journ. of Math., Oxford Series, 2 (1931), pp. 1-8.

[16] PONTRYAGIN. Topological Groups, Princeton University Press, 1939.

[17] RIESZ. Über einen Satz der Analysis Situs, Math. Annalen, 59 (1904), pp. 409-415.

[18] TERASAKA. Ein Beweis des Brouwerschen ebenen Translationssatzes, Japanese Journ. of Math. 7 (1930), pp. 61-69.

[19] ZIPPIN. Transformation Groups, Lectures in Topology, University of Michigan Press, Ann Arbor, Mich, 1940, pp. 191-221.