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ON FIBRATION OF SPHERES BY SPHERES AND HOPF INVARIANT

by K. SRINIVASACHARYULU

1. Let $f : S_{2n-1} \rightarrow S_n$ be a map which we may assume to be simplicial. If x_1, x_2 are interior points of n -simplexes of S_n , then $z_i^{n-1} = f^{-1}(x_i)$, $i = 1, 2$, are $(n-1)$ cycles in S_{2n-1} . Since $H_{n-1}(S_{2n-1}) = 0$, the cycles are boundaries, say $z_i^{n-1} = \partial c_i$. The linking coefficient of z_1^{n-1} and z_2^{n-2} is called the Hopf number $H(f)$ of f .

REMARK. $H(f)$ is clearly the degree of the map $f|c_i^n$. We have the following properties of $H(f)$ [5] :

1) $H(f)$ depends only on the homotopy class of $f : S_{2n-1} \rightarrow S_n$. It follows that H is a homomorphism :

$$H : \Pi_{2n-1}(S_n) \rightarrow \mathbb{Z}.$$

2) If n is odd, $H(f)$ is zero.

H. Hopf has shown [5] that, for $n = 2, 4$ and 8 , there exist maps of Hopf invariant 1. He raised for what other values of n do maps $f : S_{2n-1} \rightarrow S_n$ with $H(f) = 1$ exist? Adams [1] has shown recently that these are the only values of n for which maps with Hopf invariant one exist.

This problem is intimately connected with the existence of fibrations of spheres by spheres over spheres. Suppose S_{2n-1} is a fibre space over S_n with fibre S_{n-1} . We have :

PROPOSITION 1. If $f : S_{2n-1} \rightarrow S_n$ is a fibre mapping, then $H(f) = 1$.

More generally, we consider the following situation [10] : suppose that E is a topological space aspherical in dimensions $\leq n$. If E is fibred by k -spheres S_k over S_n , then, from the homotopy-exact sequence, it follows that $k = n-1$. One has the following (cf, [3] also) :

PROPOSITION 1'. Let E be a topological space aspherical in dimensions $\leq n$. If $p : E \rightarrow S_n$ is a fibre space with fibre S_{n-1} , then $H(p) = 1$.

2. Let E be aspherical in dimensions $\leq n$; suppose that E is a $(n-1)$ sphere bundle over S_n . We have the following :

PROPOSITION 2. The fibre S_{n-1} is contractible to a point in E .

PROPOSITION 3. Let E be as above : if E is a $(n-1)$ sphere bundle over S_n , then n is a multiple of 4.

PROOF. If $(n-1)$ is even, then the inclusion $S_{n-1} \rightarrow E$ maps $\Pi_{n-1}(S_{n-1})$ isomorphically into $\Pi_{n-1}(E) = 0$ ([9] cf § 28. 3) a contradiction. If $n > 2$ and $n \equiv 1 \pmod{4}$, then the image of $\Pi_{n-1}(S_{n-1})$ in $\Pi_{n-1}(E)$ is either infinite cyclic or finite cyclic of even order ([9] cf § 28. 3). Thus $n = 1, 2$ or $n = 4m$.

REMARK. In fact, one can prove, by an argument similar to that used by J. Adam (cf [2] cor. 2. 3), that if such a fibration exists, then n is a power of 2.

Examples of such fibrations are furnished by the classical fibrations of Hopf [5].

PROPOSITION 4. Let E , aspherical in dimensions $\leq n$, be a fibre space over S_n with fibre S_{n-1} and any structure group; then $n = 1$ or even.

PROOF. Cf [9] § 28. 7 theorem.

REMARK. Proposition 3 restricts that the structure group is $O(n)$ while proposition 4 shows that n is only even for any structure group. If E is a fibre space (in the sense of Serre) over S_n with fibre S_{n-1} one can obtain the same results as here Cf [3].

3. In this section we determine completely the $O(n)$ fibrations of E by S_{n-1} over S_n using some results of Milnor [7]. For the sake of completeness we include them here.

THEOREM. 1. There exists an $O(m)$ -bundle ξ over S_n with the n dimensional Stiefel-Whitney class $w^n \neq 0$ only if $n = 1, 2, 4$ or 8 . We prove this using the following result of Wu.

THEOREM 2. (Wu). For any $O(m)$ -bundle ξ over a complex K , the Pontrjagin class p_k is related to the Stiefel-Whitney classes $w^1, \dots, w^{4k-1}$ by

$$(p_k)_4 = P(w^{2k}) + i_* f_k(w^1, \dots, w^{4k})$$

where $(p_k)_4$ is the image of p_k by the homomorphism $H^*(K; \mathbb{Z}) \rightarrow H^*(K; \mathbb{Z}_4)$ induced by $\mathbb{Z} \rightarrow \mathbb{Z}_4$, i denotes the inclusion $\mathbb{Z}_2 \rightarrow \mathbb{Z}_4$, P the Pontrjagin operation : $H^{2k}(K; \mathbb{Z}_2) \rightarrow H^{4k}(K; \mathbb{Z}_4)$ and f_k a polynomial with coefficients in $\mathbb{Z}_2$.

Following Hinzebruch we define the $4k$ -dimensional Pontrjagin class p_k of an $O(m)$ -bundle as $(-1)^k$ times the Chern class c_{2k} of the unitary extension induced by the inclusion $O(m) \rightarrow U(m)$.

PROOF. It is sufficient to consider the case of the universal bundle over the Grassmann space $G_m(k)$, with m sufficiently large. Consider the exact coefficient sequence

$$0 \rightarrow \mathbb{Z}_2 \xrightarrow{i} \mathbb{Z}_4 \xrightarrow{j} \mathbb{Z}_2 \rightarrow 0 : \text{ we have :}$$

$$i_* P(w) = w \cup w \quad \text{for any cohomology class } w.$$

Also
$$i_* ((p_k)_4) = (p_k)_2 = w^{2k} \cup w^{2k};$$

it follows that
$$(p_k)_4 - P(w^{2k}) \in \ker j_* = i_* H^{4k}(G_m(k); Z_2).$$

Since the cohomology ring $H^*(G_m(k); Z_2)$ is generated by the Stiefel-Whitney classes, the theorem follows.

COROLLARY. *If the Stiefel-Whitney classes $w^1, \dots, w^{4k-1}$ are zero, then $(p_k)_4 = i_* w^{4k}$.*

PROOF. We will show that the coefficient of w^{4k} in f_k is non-zero. Let Γ denote the universal $U(m)$ -bundle over $G_m(C)$, the complex Grassmann space. It is known that the cohomology ring $H^*(G_m(C); Z)$ is a polynomial ring generated by the Chern classes of Γ . The inclusion $U(m) \rightarrow O(2m)$ induces an $O(2m)$ bundle Γ_R over $G_m(C)$. Applying theorem 2 to Γ_R , the relations

$$w^{2r+1}(\Gamma_R) = 0, \quad w^{2r}(\Gamma_R) = (C^r(\Gamma))_2$$

$$\text{and} \quad p_k(\Gamma_R) = C_k(\Gamma)^2 - 2C_{k-1}(\Gamma)C_{k+1}(\Gamma) + \dots \pm 2C_0(\Gamma)C_{2k}(\Gamma)$$

show that the polynomial f_k must satisfy

$$f(0, w^2, 0, \dots, w^{4k}) = w^{2k-2}w^{2k+2} + \dots + w^0w^{4k}$$

In particular, we have $f_k(0, 0, \dots, 0, w^{4k}) = w^{4k}$.

We also make use of the following result of Bott :

THEOREM. 3.(Bott). *For any $O(m)$ -bundle ξ over S_{4k} , the Pontrjagin class $p_k \in H^{4k}(S_{4k}; Z)$ is always divisible by $(2k-1)!$. For a proof refer [6].*

PROOF OF THEOREM 1. According to Wu [12] such a bundle exists only if $n = 2^m$. Hence we can assume that $n = 4k$, $k > 2$. The identity :

$$(p_k)_4 = i_* w^{2k} \in H^{4k}(S_{4k}; Z_4) = Z_4$$

holds, since the lower dimensional Stiefel-Whitney classes are zero. Hence $w^{4k} = 0$ if and only if p_k is divisible by 4. But by Bott's theorem, p_k is divisible by $(2k-1)!$. Thus $w^{4k} = 0$ for $k > 2$. As a corollary we have the following :

THEOREM. *Let E be aspherical in dimensions $\leq n$. E is a $(n-1)$ -sphere bundle over S_n only if $n = 1, 2, 4$ or 8 .*

PROOF: $w^n = 0$ if and only if E admits a cross-section.

Taking $E = S_{2n-1}$, we have :

COROLLARY. *There do not exist fibrations of S_{2n-1} by S_{n-1} over S_n with structure group $O(n)$ for $n > 8$.*

4. Classification Theorem.

It is well-known that the equivalence classes of fibre spaces over S_n are in 1-1 correspondance with equivalence classes of $\Pi_{n-1}(O(n))$ by the operations of $\Pi_0(O(n))$. Since $\Pi_1(SO(2)) = Z$, there exist an infinity of inequivalent fibre spaces. From the homotopy sequence of the fibration $S_1 \rightarrow S_3 \rightarrow S_2$ we see that $\Pi_2(S_2) \approx \Pi_1(S_1)$ or the fibre space corresponds to 1. Now let E be a 1-sphere bundle over S_2 with $\Pi_i(E) = 0$, $i \leq 2$; by the same argument it follows that E also corresponds to 1. Thus the two bundles are equivalent and E is topologically homeomorphic to S_3 .

In the case $n = 4$, since $\Pi_3(SO(4)) = Z + Z$, we have a fibre space $B_{m,n}$ corresponding to each pair (m, n) . It is easy to see that the primary obstruction for the existence of a cross-section is $n\beta_3$ where β_3 is a generator of $\Pi_3(SO(4))$ defined by $\sigma: S_3 \rightarrow SO(4)$ ([9], § 22.6). Hence the fibre spaces $B_{m,0}$ have cross-sections. We will show that $B_{m,\pm 1}$ is equivalent to $S_3 \rightarrow S_1 \rightarrow S_4$. This follows from the fact that the characteristic map represents a generator of $\Pi_3(S_3) \approx \Pi_4(S_4)$. Moreover, the fibration $S_3 \rightarrow E \rightarrow S_4$ with $\Pi_i(E) = 0$ for $i < 4$, is equivalent to $B_{m,\pm 1}$. Therefore E is topologically homeomorphic to S_7 . For $n = 8$, there exist an infinity of inequivalent bundles $B_{m,n}$ corresponding to each pair (m, n) since $\Pi_7(SO(8)) = Z + Z$. It can be seen as above that the primary obstruction for the existence of a cross-section is $n\beta_8$, where β_8 is defined by $\beta_8(x)y = x \cdot y$, x, y being Cayley numbers. We will show that the classical fibration $S_7 \rightarrow S_{15} \rightarrow S_8$ corresponds to $m \pm 1$. Suppose it corresponds to (m, n) . Let $\tilde{B}_{m,n}$ be its associated principal bundle: if $y_0 \in S_7$, denote by y_0^* ; $\tilde{B}_{m,n} \rightarrow S_{15}$ the principal map, we have the following

$$\begin{array}{ccc} & \partial & \\ \Pi_8(S_8) & \xrightarrow{\quad} & \Pi_7(SO(8)) \\ \downarrow & & \downarrow y_0^* \\ \Pi_8(S_8) & \xrightarrow{\quad} & \Pi_7(S_7) \end{array}$$

By the definition of the index (m, n) ∂ maps a generator of $\Pi_8(S_8)$ into n times a generator of $\Pi_7(SO(8))$; but n times a generator of $\Pi_7(SO(8))$ is carried by y_0^* to n times a generator of $\Pi_7(S_7)$, a contradiction since $\Pi_8(S_8) \approx \Pi_7(S_7)$. By the same argument we can prove that the fibration $S_7 \rightarrow E \rightarrow S_8$ with $\Pi_i(E) = 0$, $i < 8$, is equivalent to $B_{m,\pm 1}$. Therefore we have again E is topologically homeomorphic to S_{15} . Thus we have:

THEOREM. Suppose that $S_{n-1} \rightarrow E \rightarrow S_n$ with $\Pi_i(E) = 0$, $i < n$, is a sphere-bundle. Then E is topologically homeomorphic to S_3 , S_7 or S_{15} .

REMARK. In fact it follows, from a remark of J. Milnor ([8] p 403), that E is differentially homeomorphic to S_3 , S_7 or S_{15} if $m = 0$.

Appendix.

1. An equally interesting problem is the following : is S_n a k -sphere bundle over some space B ?

Let $\tilde{B}$ be the principal bundle of $S_k \rightarrow S_m \rightarrow B$ and let $y : \tilde{B} \rightarrow S_m$ be the map defined by a point $y \in S_k$. Then the kernel of $i_* : \Pi_i(S_k) \rightarrow \Pi_i(S_m)$, induced by the inclusion of S_k in S_m , is contained in the image of $y'_* : \Pi_i(SO(k+1)) \rightarrow \Pi_i(S_k)$ where $y' = y|_{SO(k+1)}$ [9, 17, 13]. This means that $y'_* : \Pi_k(SO(k+1)) \rightarrow \Pi_k(S_k) \approx \mathbb{Z}$ is onto. Let $f : S_k \rightarrow SO(k+1)$ be a mapping which corresponds to a generator of $y'_* \Pi_k(SO(k+1))$; that means $y' : SO(k+1) \rightarrow S_k$ is such that $y'f = \text{identity}$. Since y' is topologically equivalent to the fibre mapping $SO(k+1) \rightarrow S_k$, this implies that S_k is parallelisable. Hence $k = 1, 3$ or 7 [7]. Thus we have the following :

PROPOSITION 1. *The sphere S_n is the bundle space of a k -sphere over some space B , only if $k = 1, 3$ or 7 .*

REMARK. Similarly let B be a differentiable manifold ; if E (a compact differentiable manifold) is a differentiable fibre space over B (locally trivial) with differentially contractible fibre F , it can be shown that F is parallelisable.

We observe the following :

PROPOSITION 2. *If S_m is a k -sphere bundle over B , then S_k is a H -space, that is, S_k admits a continuous multiplication with a two-sided identity.*

PROOF. Consider the homotopy sequence $\dots \rightarrow \Pi_k(SO(k+1)) \xrightarrow{p_*} \Pi_k(S_k) \xrightarrow{\partial} \Pi_{k-1}(SO(k)) \rightarrow \dots$ of the fibration $SO(k) \rightarrow SO(k+1) \rightarrow S_k$. Since S_k is parallelisable (1), we have $\partial(i) = 0$ ($i = \text{identity of } \Pi_k(S_k)$).

Let $J : \Pi_{k-1}(SO(k)) \rightarrow \Pi_{2k-1}(S_k)$ be the G. W. Whitehead homomorphism ; it is known that $J\partial = \neg[\ ,]$ where $[\ ,]$ is the Whitehead product by i ; thus $[i, i] = 0$, or equivalently S_k is a H -space.

REMARK. After Adams [1], S_k is a H -space only if $k = 1, 3$, or 7 .

2. In what follows we assume that B is a locally finite polyhedron. Since the fibre is contractible in S_m , we have :

$$\Pi_i(B) \approx \Pi_i(S_m) + \Pi_{i-1}(S_k).$$

Thus we have $\Pi_i(B) = 0$ for $i \leq k$. We assert that the dimension of the polyhedron B is $\geq k+1$; if the $\dim B \leq k$, then it can be seen easily that B is contractible, consequently $\Pi_i(B) = 0$ for all i , a contradiction. Hence $\dim B \geq k+1$ so that $m \geq 2k+1$.

(1) Dr. A. DOLD has informed me that if S_n , endowed of an arbitrary differentiable structure is parallelisable, then it is an H -space. However, it is not clear whether the tangent structure of S_n is independent of its differentiable structure.

It is known that m is odd (1). Thus we have the following :

THEOREM. *If S_m is a k -sphere bundle over a locally finite polyhedron B , then m is an odd number $\geq 2k + 1$ and $k = 1, 3$ or 7 .*

The following are the only known examples of such fibrations :

S_{2k+1} is a circle bundle over $P_k(C)$ the complex projective space.

S_{4k+3} is a 3-sphere bundle over the quaternion projective space $P_k(Q)$.

3. Classification theorem. Since the fibration $S_{2n+1} \rightarrow B, k > 1$, is a principal 3-universal bundle with group $SO(2)$, it follows from the classification theorem that the equivalence classes of bundles with group $SO(2)$ and base space a 2-complex K_2 are in 1-1 correspondance with the homotopy classes of maps $f : K_2 \rightarrow B$. Consider a mapping $f : K_2 \rightarrow B$ and let m_f be the integer such that $f_*(\alpha) = m_f \beta$ where α is a generator of $H^2(B)$ and β a generator of $H^2(K_2)$. Then there is a 1-1 correspondance between homotopy classes of maps $f : K_2 \rightarrow B$ and the integers Z . In fact, the bundle $S_{2n+1} \rightarrow B$ corresponds to $\pm 1 \in Z$. For if it corresponds to $p \in Z$, then the space $B_p = \frac{S_{2n+1}}{H_p}$

where H_p is a cyclic subgroup of $SO(2)$ of order p , is a bundle over B with group $SO(2)/H_p \cong SO(2)$. Since B is simply connected, this cannot happen unless $p = \pm 1$.

It follows from the homotopy exact sequence that $\Pi_1(B) = 0$, $\Pi_2(B) = \Pi_{2n+1}(B) = Z$, $\Pi_i(B) = 0$ for $2 < i < 2n+1$ and $\Pi_i(B) = \Pi_i(S_{2n+1})$ for $i > 2n+1$. Comparing this with the classical bundle S_{2n+1} over the complex projective space $P_n(C)$, one can see that B is of the same homotopy type as $P_n(C)$.

REMARK. We assume that B is a compact, connected metric space ; suppose that B admits a locally strongly transitive group of isometries [11]. Then it follows, from [11], that B is homeomorphic to a sphere, or complex or quaternion projective space. Thus we have the following :

THEOREM. *If S_{2n+1} is a k -sphere bundle over a compact, connected metric space B which admits a locally strongly transitive group of isometries, then $k = 1, 3$ or 7 and B is homeomorphic to a complex or quaternion projective space or to a sphere.*

(1) The fact that m is odd can be seen as follows : we obtain from Gysin's sequence $[4] \dots \rightarrow H^i(B) \rightarrow H^i(S_m) \rightarrow H^{i-k}(B) \rightarrow H^{i+1}(B) \rightarrow \dots$ that $H^{i-k}(B) \cong H^{i+1}(B)$, $0 < i < m-1$. Thus the Poincaré polynomial of B is of the form $P(B; t) = 1 + t^{k+1} + \dots + t^{m(k+1)}$ where $\dim B = m(k+1)$: hence the Euler characteristic $\chi(B) = P(B, -1) = \dim B/k+2 > 0$. Thus, if m is even, we obtain from $\chi(S_m) = \chi(S_k) \chi(B)$ that k is even, a contradiction to Prop. 1.

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