SÉMINAIRE DE TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE

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Séminaire de topologie et géométrie différentielle, tome 1 (1957-1958), exp. nº 17, p. 1-12
<http://www.numdam.org/item?id=SE_1957-1958__1__A6_0>

Faculté des Sciences de Paris
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Séminaire de TOPOLOGIE
et de GÉOMÉTRIE DIFFÉRENTIELLE
(C. EHRESMANN)
Année 1957/1958
-:-:-:-

22 mai 1958

GENERALIZED TORSIONAL DERIVATION

by T.J. WILLMORE

1. Introduction.

In a recent paper A.G. WALKER [3] introduced a new derivation associated with an almost complex structure which he called torsional derivation. This derivation is determined by the almost complexe structure, which itself is defined by a tensor field $\underset{\sim}{h}$ of type (1,1) such that $\underset{\sim}{h}^2 = -\underset{\sim}{1}$. The dérivation is a mapping of tensor fields of type (p , q) into tensor fields of type (p , q + 2) , and has the property of annihilating all tensor fields when the almost complex structure has zero torsion. When applied to the torsion tensor itself, the derivation gives a non-trivial tensor of type (1,4) which appears as a new differential invariant associated with an almost complex structure.

A formula for the torsion $\underset{\sim}{H}$ of an almost complex structure was given, for example, by ECKMANN [1] , and it was subsequently shown by NIJENHUIS [2] that by slightly modifying this formula it was possible to establish the tensoriel nature of $\underset{\sim}{H}$ without using the relation $\underset{\sim}{h}^2 = -\underset{\sim}{1}$, i.e. the tensor $\underset{\sim}{H}$ defined for an arbitrary tensor $\underset{\sim}{h}$ became the torsion tensor of the almost complex structure when $\underset{\sim}{h}^2 = -\underset{\sim}{1}$. In [4] WALKER raised the interesting question whether it was possible to modify similarly his formulas defining new tensors (e.g. the new tensor of type (1,4)) in such a way that its tensorial nature could be established without using the relation $\underset{\sim}{h}^2 = -\underset{\sim}{1}$. NIJENHUIS has conjectured that $\underset{\sim}{H}$ is theonly essentially new tensor field which can be constructed from an arbitrary tensor field $\underset{\sim}{h}$. Walker's results show that this conjecture is false provided that $\underset{\sim}{h}$ satisfies the additional restriction $\underset{\sim}{h}^2 = -\underset{\sim}{1}$. If Walker's operation of torsional derivation could be generalized by relaxing the requirement $\underset{\sim}{h}^2 = -1$, then the conjecture of Nijenhuis would be disproved. In this paper we make some contribution towards obtaining a generalized torsional derivation.

2. Walker's operation.

In terms of the tensor $\underset{\sim}{h}$, the components of torsion tensor $\underset{\sim}{H}$ are defined by

(2.1) $$H^i_{jk} = \frac{1}{4}(h^i_p \partial_{[j} h^p_{k]} - h^p_{[j} \partial_{|p|} h^i_{k]}) ,$$

and the tensorial character of (2.1) is easily established without using the relation $\underset{\sim}{h}^2 = -\underset{\sim}{1}$. Following WALKER [4] we define the torsional derivative with respect to the almost complex structure h of a tensor field with components $T^{i\ldots}_{j\ldots}$ to be the tensor with components $T^{i\ldots}_{j\ldots}||_{rs}$, where

(2.2) $$T^{i\ldots}_{j\ldots}||_{rs} = H^p_{rs} \partial_p T^{i\ldots}_{j\ldots} + T^{p\ldots}_{j\ldots} h^i_{prs} + \ldots - T^{i\ldots}_{p\ldots} h^p_{jrs} - \ldots ,$$

and where the right hand member contains a term like $+ T^{p\ldots}_{j\ldots} h^i_{prs}$ corresponding to each contravariant suffix of $T^{i\ldots}_{j\ldots}$ and a term like $- T^{i\ldots}_{p\ldots} h^p_{jrs}$ corresponding to each covariant suffix, and where the symbols h^i_{prs} are defined by

(2.3) $$h^i_{jrs} = -\frac{1}{2} \partial_j H^i_{rs} + \frac{1}{2} h^i_p(h^q_j \partial_q H^p_{rs} - H^q_{rs} \partial_q h^p_j + H^p_{qs} \partial_r h^2_j - H^p_{qr} \partial_s h^q_j) .$$

It is readily verified by expressing the partial derivatives in (2.2) and (2.3) in terms of covariant derivatives with respect to some arbitrary symmetric connexion that $T^{i\ldots}_{j\ldots}||_{rs}$ defined by (2.2) are in fact the components of a tensor, but to establish this result use has to be made of the relation $\underset{\sim}{h}^2 = -\underset{\sim}{1}$.

Torsional derivation defined by (2.2), (2.3) has the usual properties of a derivation relative to addition, multiplication and contraction of tensors. Moreover, it is easily verified that

(2.4) $$\delta^i_j ||_{rs} = 0 ,$$

(2.5) $$h^i_j ||_{rs} = 0 .$$

Our problem is to obtain an operator analogous to Walker's operator without making the restriction $\underset{\sim}{h}^2 = -\underset{\sim}{1}$.

3. Extension of operators.

In what follows we shall assume that the differentiable manifold under consideration is of class ∞, and that all tensor fields, vector fields, etc. introduced are also of class ∞.

The operator $\mathfrak{D}$ of torsional derivation defined in paragraph 2 has the following properties :

i. it is linear over the real numbers i.e. if $\underset{\sim}{T}_1$, $\underset{\sim}{T}_2$ are two tensor fields and a , b arbitrary real numbers (constants) , then

$$\mathfrak{D}(a\underset{\sim}{T}_1 + b\underset{\sim}{T}_2) = a\,\mathfrak{D}(\underset{\sim}{T}_1) + b\,\mathfrak{D}(\underset{\sim}{T}_2) \quad ;$$

ii. it satisfies the usual product law, i.e.

$$\mathfrak{D}(\underset{\sim}{T}_1 \otimes \underset{\sim}{T}_2) = (\mathfrak{D}\underset{\sim}{T}_1) \otimes \underset{\sim}{T}_2 + \underset{\sim}{T}_1 \otimes (\mathfrak{D}\underset{\sim}{T}_2) \quad ;$$

iii. since $\mathfrak{D}(\delta^i_j) = 0$, $\mathfrak{D}$ commutes with the operation of contraction, e.g.

$$\mathfrak{D}(\delta^{j_3}_{i_1} T^{i_1 i_2}_{j_1 j_2 j_3}) = \delta^{j_3}_{i_1}\,\mathfrak{D}(T^{i_1 i_2}_{j_1 j_2 j_3}) \quad .$$

In particular, when restricted to contravariant vector fields $\underset{\sim}{u}$, $\underset{\sim}{v}$ and scalars f the operator satisfies

a. $\mathfrak{D}(a\underset{\sim}{u} + b\underset{\sim}{v}) = a\,\mathfrak{D}\underset{\sim}{u} + b\,\mathfrak{D}\underset{\sim}{v}$,

b. $\mathfrak{D}(f\underset{\sim}{u}) = f\,\mathfrak{D}\underset{\sim}{u} + \mathfrak{D}f \otimes \underset{\sim}{u}$.

Conversely, suppose that the operator $\mathfrak{D}$ had been defined only for contravariant vector fields and scalar fields such that (a) and (b) were satisfied. <u>Then the domain of the operator</u> $\mathfrak{D}$ <u>can be extended in a unique manner to include tensor fields of arbitrary type</u> (p , q) <u>so that conditions</u> (i), (ii) <u>and</u> (iii) <u>are satisfied</u>. This follows from precisely the same arguments which allow ordinary covariant differentiation defined first for contravariant vector fields and scalars to be extended to general tensor fields. More generally, any differential operator which is defined for contravariant vector fields and scalars so that conditions (a) and (b) are satisfied may be extended in a unique manner to operate on arbitrary tensor fields so that conditions (i) , (ii) and (iii) are satisfied. It follows that it is sufficient to define a generalized torsional derivation over vector fields and scalar fields provided that condition (a) and (b) are satisfied.

4. The Lie derivative.

This differential operator $\mathcal{L}_v$, determined by a given contravariant vector field $\underset{\sim}{v}$, maps tensor fields of type (p , q) into tensor fields of the same type. The effect of $\mathcal{L}_v$ on a contravariant vector field with components u^i is defined by

$$(4.1) \qquad (\mathcal{L}_v u)^i = v^j \partial_j u^i - u^j \partial_j v^i \quad ,$$

while the effect on a scalar f is given by

$$(4.2) \qquad \mathcal{L}_v f = v^j \partial_j f \quad .$$

It is readily verified that the operator $\mathcal{L}_v$ satisfies conditions (a) , (b) of paragraph 3, and it can therefore be extended uniquely to operate on arbitrary tensor fields of type (p , q) so that conditions (i) , (ii) , (iii) of paragraph 3 are satisfied. For a detailed study of Lie derivatives the reader is referred to the recent book by K. YANO [4] . However, except for the definition of the Lie derivative of a tensor field of type (p , q) , the only part of the theory required here concerns the invariant nature of the operation of Lie derivation. For the sake of completeness we give an alternative approach to the Lie derivative, due to A.G. WALKER, which does not seem to be included in the standard texts on the subject. For reasons of brevity we shall obtain a formula for the Lie derivative of a tensor field of type $(1 , 2)$, but the method can be used with a tensor field of general type (p , q) .

Let $T^i{}_{jk}$ be the components of a tensor field of type $(1,2)$. Let $L^i{}_{s\ell}$, $\Gamma^i{}_{s\ell}$ be the components of two arbitrary symmetric affine connexions L , Γ; then the symbols $X^i{}_{s\ell}$ defined by

$$(4.3) \qquad X^i{}_{s\ell} = L^i{}_{s\ell} - \Gamma^i{}_{s\ell}$$

are components of a symmetric tensor of type $(1 , 2)$. The covariabt derivatives with respect to L , Γ will be denoted by a comma and a bar respectively.

Then we have

$$(4.4) \quad T^i_{jk,\ell} = \partial_\ell T^i_{jk} + L^i_{s\ell} T^s_{jk} - L^s_{j\ell} T^i_{sk} - L^s_{k\ell} T^i_{js} \quad ,$$

$$(4.5) \quad T^i_{jk|\ell} = \partial_\ell T^i_{jk} + \Gamma^i_{s\ell} T^i_{jk} - \Gamma^s_{j\ell} T^i_{sk} - \Gamma^s_{k\ell} T^i_{js} \quad .$$

Subtracting (4.5) from (4.4) and using (4.3) we get

$$(4.6) \quad T^i_{jk,\ell} - T^i_{jk|\ell} = X^i_{s\ell} T^s_{jk} - X^s_{j\ell} T^i_{sk} - X^s_{k\ell} T^i_{js} \quad .$$

Now let v^i be the components of a given vector field $\underset{\sim}{v}$. Then we have

$$(4.7) \quad v^i{}_{,\ell} - v^i{}_{|\ell} = X^i{}_{s\ell} v^s \quad .$$

If we multiply (4.6) by v^ℓ, we obtain, on using the symmetry of $X^i{}_{s\ell}$,

$$(4.8) \quad (T^i_{jk,\ell} - T^i_{jk|\ell})v^\ell = T^s_{jk} X^i_{\ell s} v^\ell - T^i_{sk} X^s_{\ell j} v^\ell - T^i_{js} X^s_{\ell k} v^\ell .$$

Using (4.7) this equation may be written

$$(4.9) \quad T^i_{jk,\ell} v^\ell - T^s_{jk} v^i{}_{,s} + T^i_{sk} v^s{}_{,j} + T^i_{js} v^s{}_{,k}$$

$$= T^i_{jk|\ell} v^\ell - T^s_{jk} v^i{}_{|s} + T^i_{sk} v^s{}_{|j} + T^i_{js} v^s{}_{|k}$$

It follows that the value of left hand member of (4.9) is independent of the particular symmetric connexion used. In particular, by choosing the symmetric connexion whose components are all zero, the covariant derivatives in (4.9) may be replaced by partial derivatives. Thus we find that the mapping $\mathcal{L}_v : \underset{\sim}{T} \rightarrow \mathcal{L}_v \underset{\sim}{T}$ given by

$$(4.10) \quad (\mathcal{L}_v T)^i_{jk} = v^\ell \partial_\ell T^i_{jk} - T^s_{jk} \partial_s v^i + T^i_{sk} \partial_j v^s + T^i_{js} \partial_k v^s \quad ,$$

is an invariant operation. In the more general case we obtain similarly the formula

$$(4.11)\qquad (\mathcal{L}_{v} T)^{i_1\cdots i_p}_{j_1\cdots j_q} = v^{\ell}\,\partial_{\ell}\, T^{i_1\cdots i_p}_{j_1\cdots j_q} - T^{s i_2\cdots i_p}_{j_1 j_2\cdots j_q}\,\partial_s v^{i_1} - T^{i_1 s i_3\cdots i_p}_{j_1 j_2\cdots j_q}\,\partial_s v^{i_2} - \cdots + T^{i_1\cdots i_p}_{s j_2\cdots j_q}\,\partial_{j_1} v^s + T^{i_1\cdots i_p}_{j_1 s j_3\cdots j_q}\,\partial_{j_2} v^s + \cdots$$

In the particular case when T is a contravariant vector u or a scalar f, equation (4.11) reduces to (4.1) and (4.2) respectively. The operator defined by (4.11) satisfies conditions (i), (ii) and (iii) of paragraph 3, and hence is the extension to arbitrary tensor fields of the operator $\mathcal{L}_v$ defined for contravariant vectors and scalars by equations (4.1), (4.2).

Instead of using the notation $\mathcal{L}_v T$ for the Lie derivatives of T with respect to the vector field v, it will be more convenient to use the notation $[v, T]$. Indeed, we shall find it convenient to interpret $[v, T]$ as the result of operating on a <u>fixed</u> tensor field T by a <u>variable</u> vector field v, rather than the classical interpretation when v is fixed and T is variable tensor field.

5. <u>The operator</u> $\mathcal{R}$.

Let h, k be two tensor fields of type (1,1), and let ω be a tensor field of type (1,2). Define an operator $\mathcal{R}$ which maps fields of contravariant vectors into tensor fields of type (1,2) according to the law

$$(5.1)\qquad \mathcal{R}\, v = \frac{1}{4}\, h\,[kv\,,\omega] + \frac{1}{4} k\,[hv\,,\omega] + \frac{1}{4}(hk + kh)\,[v\,,\omega]\ .$$

Suppose that $\mathcal{R}$ also maps fields of scalars into tensor fields of type (0,2) according to the law

$$(5.2)\qquad (\mathcal{R}\, f)_{rs} = \omega^{a}_{rs}\,\partial_a f\ .$$

We seek conditions on h, k and ω such that conditions 3(a), 3(b) are satisfied for all v and f. The operator $\mathcal{R}$ is linear over the real numbers so 3(a) is satisfied. Condition 3(b) leads to the equation

$$(5.3)\qquad 2\,\omega^{a}_{rs}(h^{i}_{m} k^{m}_{p} + k^{i}_{m} h^{m}_{p} + 2\,\delta^{i}_{p}) - \delta^{q}_{r}\left\{h^{i}_{n}\,\omega^{n}_{bs}\,k^{b}_{p} + k^{i}_{n}\,\omega^{n}_{bs}\,h^{b}_{p} + (h^{i}_{m} k^{m}_{n} + k^{i}_{m} h^{m}_{n})\omega^{n}_{ps}\right\} - \delta^{a}_{s}\left\{h^{i}_{n}\,\omega^{n}_{rb}\,k^{b}_{p} + k^{i}_{n}\,\omega^{n}_{rb}\,h^{b}_{p} + (h^{i}_{m} k^{m}_{n} + k^{i}_{n} h^{m}_{n})\,\omega^{n}_{rp}\right\} = 0$$

This relation between $\underset{\sim}{h}$, $\underset{\sim}{k}$ and $\underset{\sim}{\omega}$ is seen to be a necessary and sufficient condition that the mapping $\textcircled{\omega}$ can be extended uniquely to tensor fields of arbitrary order so that conditions 3(i) , (ii) , (iii) are satisfied.

In particular this condition is evidently fulfilled when each term vanishes, and under these circumstances we obtain (when $\underset{\sim}{\omega} \neq \underset{\sim}{0}$) equations which in matrix form may be written

(5.4) $$\underset{\sim}{h}\,\underset{\sim}{k} + \underset{\sim}{k}\,\underset{\sim}{h} = -\underset{\sim}{2} ,$$

(5.5) $$\underset{\sim}{h}\,\underset{\sim}{\omega}\,\underset{\sim}{k} + \underset{\sim}{k}\,\omega\,\underset{\sim}{h} = 2\,\underset{\sim}{\omega} .$$

Condition (5.5) really represents two equations obtained by fixing in turn one of the two covariant suffixes of ω . However, when ω is skew-symmetric or symmetric in both covariant indices, then conditions (5.5) leads to a single equation.

In the particular case when $\underset{\sim}{h} = \underset{\sim}{k}$, equation (5.4) reduces to

(5.6) $$\underset{\sim}{h}^2 = -\underset{\sim}{1} ,$$

so the manifold admits an almost complex structure. In this case (5.5) reduces to the condition

(5.7) $$\underset{\sim}{h}\,\underset{\sim}{\omega} + \underset{\sim}{\omega}\,\underset{\sim}{h} = \underset{\sim}{0} .$$

Now it is easily verified that the torsion tensor $\underset{\sim}{H}$ of the almost complex structure derived from $\underset{\sim}{h}$ satisfies the condition (5.7), so we may take $\underset{\sim}{\omega} = \underset{\sim}{H}$. Equation (5.1) then becomes

(5.8) $$\textcircled{\omega}\, v = \frac{1}{2} h\,[hv , H] - \frac{1}{2}[v , H] ,$$

and (5.2) becomes

(5.9) $$\textcircled{\omega} f = H^{a}{}_{rs}\, \partial_a f .$$

Now it is easily verified that equations (5.8) , (5.9) are precisely the same equations as those obtained by applying Walker's operation of torsional derivation. It follows that Walker's operator is obtained from our operator $\textcircled{\omega}$ by taking the special solutions of (5.4) , (5.5) given by $\underset{\sim}{h} = \underset{\sim}{k}$, $\omega = \underset{\sim}{H}$. Our operator $\textcircled{\omega}$ is thus seen to be a natural generalization of Walker's operator.

6. More general solutions.

It is evident that equations (5.4) , (5.5) have many solutions besides the particular one which yields Walker's operator. I am grateful to Dr. Graham HIGMAN for pointing out to me that these equations assume a more natural form in terms of Jordan algebra. If we change the sign of $\underset{\sim}{k}$, the equations become

(6.1) $$\underset{\sim}{h}\,\underset{\sim}{k} + \underset{\sim}{k}\,\underset{\sim}{h} = 2 \quad ,$$

(6.2) $$\underset{\sim}{h}\,\underset{\sim}{\omega}\,\underset{\sim}{k} + \underset{\sim}{k}\,\underset{\sim}{\omega}\,\underset{\sim}{h} = -2\,\underset{\sim}{\omega} .$$

If we introduce the Jordan multiplication

$$\{ab\} = \frac{1}{2}(ab + ba) \quad ,$$

equation (6.1) implies that $\underset{\sim}{h}$ and $\underset{\sim}{k}$ are Jordan inverses. Moreover, the left hand side of (6.2) can be written as the Jordan polynomial

$$2\left[\left\{\left\{\underset{\sim}{\omega}\,\underset{\sim}{h}\right\}\underset{\sim}{k}\right\} + \left\{\left\{\underset{\sim}{\omega}\,\underset{\sim}{k}\right\}\underset{\sim}{h}\right\} - \left\{\underset{\sim}{\omega}\left\{\underset{\sim}{h}\,\underset{\sim}{k}\right\}\right\}\right] \quad ,$$

so, using (6.1) again, (6.2) can be written

$$\left\{\left\{\omega\,\underset{\sim}{h}\right\}\underset{\sim}{k}\right\} + \left\{\left\{\omega\,\underset{\sim}{k}\right\}\underset{\sim}{h}\right\} = 0 \quad ;$$

Thus the problem of solving equations (6.1) , (6.2) is equivalent to determining the (finite dimensional) special representations of the (infinite dimensional) Jordan algebra which has an identity, and is generated by W , H , K subject to the relations

(6.3) $$HK = 1 \quad ,$$

(6.4) $$(WH)K + (WK)H = 0 \quad .$$

Corresponding to each such representation there will be determined an operation which generalizes torsional derivation (1).

Returning now to equations (5.4) , (5.5) , let us assume now that $\underset{\sim}{h}$ is non-singular, and write

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(1) This was already known to A. NIJENHUIS.

(6.5) $$\underset{\sim}{k} = -\underset{\sim}{h}^{-1} + x \ .$$

Then equation (5.4) becomes

(6.6) $$\underset{\sim}{h}x + x\underset{\sim}{h} = 0 \ ,$$

and (5.5) becomes

(6.7) $$\underset{\sim}{h}(\underset{\sim}{\omega}\,\underset{\sim}{h}^{-1} + \underset{\sim}{h}^{-1}\underset{\sim}{\omega}) + \underset{\sim}{h}^{-1}(\underset{\sim}{\omega}\,\underset{\sim}{h} + \underset{\sim}{h}\,\underset{\sim}{\omega}) = \underset{\sim}{h}\,\underset{\sim}{\omega}\,x + x\,\underset{\sim}{\omega}\,\underset{\sim}{h} \ .$$

Write

(6.8) $$y = \underset{\sim}{\omega}\,\underset{\sim}{h}^{-1} + \underset{\sim}{h}^{-1}\underset{\sim}{\omega} \ ,$$

so that

(6.9) $$\underset{\sim}{h}y\underset{\sim}{h} = \underset{\sim}{\omega}\,\underset{\sim}{h} + \underset{\sim}{h}\,\underset{\sim}{\omega} \ .$$

Equations (5.4) , (5.5) thus become

(6.6) $$\underset{\sim}{h}x + x\underset{\sim}{h} = 0 \ ,$$

(6.10) $$\underset{\sim}{h}y + y\underset{\sim}{h} = \underset{\sim}{h}\,\underset{\sim}{\omega}\,x + x\,\underset{\sim}{\omega}\,\underset{\sim}{h} \ .$$

An obvious solution of (6.6) is $x = 0$, and an obvious solution of (6.10) is then $y = 0$. Equation (6.9) then gives

(6.11) $$\underset{\sim}{\omega}\,\underset{\sim}{h} + \underset{\sim}{h}\,\underset{\sim}{\omega} = 0 \ .$$

Thus, provided h is non-singular, any skew-symmetric tensor field $\underset{\sim}{\omega}$ which satisfies (6.11) will give rise to a suitable operator $\mathcal{G}$.

7. The Nijenhuis tensor.

An alternative procedure is to define ω to be the Nijenhuis tensor $N(h, k)$ associated with h and k , where

(7.1) $$8N^i_{jk} = h^i_p \partial_{[j} k^p_{k]} + k^i_p \partial_{[j} h^p_{k]} - h^p_{[j} \partial_{|p|} k^i_{k]} - k^p_{[j} \partial_{|p|} h^i_{k]} \ .$$

Let us denote by (5.3)' equation (5.3) where $\underset{\sim}{\omega}$ is replaced by the Nijenhuis tensor $N(h, k)$. Then any pair of tensor fields $\underset{\sim}{h}$, $\underset{\sim}{k}$ which satisfy (5.3)' will give rise to a generalized torsional derivation. When $\underset{\sim}{h}^2 = -\underset{\sim}{1}$, Walker's operator appears as the particular solution $\underset{\sim}{k} = \underset{\sim}{h}$ of equation (5.3)' .

It is natural to ask how the operation $\circledS$ is modified by choosing for $\underset{\sim}{\omega}$ a tensor of type (1,p) where $p \neq 2$, but retaining the laws of operation (5.1) , (5.2) . Evidently (5.3) would be replaced by a similar equation with p negatives terms instead of the 2 negative terms in (5.3). This equation would certainly be satisfied when conditions (5.4) , (5.5) are satisfied, but equation (5.5) would represent p conditions. In the particular case when ω is a vectorial p-form, the skew symmetry of the covariant suffixes would reduce (5.5) to a single condition.

8. Connexions associated with tensor fields.

The method which we have used to obtain generalised torsional derivatives may also be used to obtain connexions associated with given tensor fields. Suppose $\underset{\sim}{h}$, $\underset{\sim}{k}$, $\underset{\sim}{\ell}$ are three mixed tensor fields of class ∞ defined over a C^{∞}-manifold M . Consider the mapping θ which sends vector fields into tensor fields of type (1.1) according to the law

$$\theta v = \alpha \underset{\sim}{h}[\underset{\sim}{k}v , \underset{\sim}{\ell}] + \beta \underset{\sim}{k}[\underset{\sim}{h}v , \underset{\sim}{\ell}] + (\gamma \underset{\sim}{h}\underset{\sim}{k} + \delta \underset{\sim}{k}\underset{\sim}{h})[v , \underset{\sim}{\ell}]$$

where α , β, γ, δ are real numbers, at present unspecified. Let us assume also that θ maps scalar fields into their gradient vector fields, i.e.

(8.2) $$(\theta f)_i = \partial_i f \quad .$$

If we denote by $v^c_{\# j}$ the components of the tensor θv , then direct computation gives

(8.3) $$v^c_{\# j} = A^c_p \, \ell^a_j \, \partial_a v^p + B^c_p \, \partial_j v^p + L^c_{pj} v^p \quad ,$$

where

(8.4) $$A = -(\alpha + \gamma)\underset{\sim}{h}\underset{\sim}{k} - (\beta + \delta)\underset{\sim}{k}\underset{\sim}{h} \quad ,$$

(8.5) $$B = \alpha \underset{\sim}{h}\underset{\sim}{\ell}\underset{\sim}{k} + \beta \underset{\sim}{k}\underset{\sim}{\ell}\underset{\sim}{h} + \gamma \underset{\sim}{h}\underset{\sim}{k}\underset{\sim}{\ell} + \delta \underset{\sim}{k}\underset{\sim}{h}\underset{\sim}{\ell}$$

(8.6) $$L^c_{\cdot pj} = \alpha h^c_i(\ell^i_a \, \partial_j k^a_p + k^a_p \, \partial_a \ell^i_j - \ell^a_j \, \partial_a k^i_p) + \beta k^c_i(\ell^i_a \, \partial_j h^a_p + h^a_p \, \partial_a \ell^i_j - \ell^a_j \, \partial_a h^i_p) + \gamma h^c_a k^a_i \, \partial_p \ell^i_j + \delta k^c_a h^a_i \, \partial_p \ell^i_j \quad .$$

The operator θ is evidently linear over the real numbers. We now seek conditions satisfied by $\underset{\sim}{h}$, $\underset{\sim}{k}$, $\underset{\sim}{\ell}$ in order that

$$(8.7) \qquad \theta(fv) = f\theta(v) + \theta f \otimes \underset{\sim}{v}$$

We have

$$(fv^c)_{\#j} = f\, v^c_{\# j} + (A^c_p\, \ell^a_j + B^c_p\, \delta^a_j) v^p\, \partial_a f \quad ,$$

so (8.7) leads to the requirement

$$v^c\, \partial_j\, f = (A^c_p\, \ell^a_j + B^c_p\, \delta^a_j) v^p\, \partial_a\, f \quad ,$$

to be satisfied by all scalar fields f and all vector fields v.

This leads to the condition

$$(8.8) \qquad \delta^c_p\, \delta^a_j = A^c_p\, \ell^a_j + B^c_p\, \delta^a_j$$

corresponding to (5.3) of paragraph 5.

It follows that when the three vector fields $\underset{\sim}{h}$, $\underset{\sim}{k}$, $\underset{\sim}{\ell}$ satisfy (8.8), then the operator θ can be extended uniquely to arbitrary tensor fields in such a way that conditions (i), (ii) (iii) of paragraph 3 are satisifed. In this case θ maps tensor fields of type (p, q) into tensor fields of type $(p, q+1)$, and in particular θ maps a scalar field into its gradient.

Equations (8.3), (8.8) now give

$$v^c_{\# j} = (\delta^c_p\, \delta^a_j - B^c_p\, \delta^a_j)\, \partial_a\, v^p + B^c_p\, \partial_j\, v^p + L^c_{pj}\, v^p \quad ,$$

i.e.

$$(8.9) \qquad v^c_{\# j} = \partial_j\, \partial^c + L^c_{pj}\, v^p \quad .$$

It follows that the coefficients L^c_{pj} can be interpreted as connexion coefficients and the operation θ may then be regarded as covariant differentiation with respect to this connexion. Corresponding to any set of tensor fields $\underset{\sim}{h}$, $\underset{\sim}{k}$, $\underset{\sim}{\ell}$ related by (8.8) there is canonically associated a connexion whose coefficients are given explicitly by (8.6).

One obvious solution of (8.8) is obtained by taking

(8.10) $$A^c_p = 0 \ , \quad B^c_p = \delta^c_p \ .$$

This first condition can be satisfied by taking $\alpha + \gamma = 0$, $\beta + \gamma = 0$, and the second condition then becomes

(8.11) $$\alpha\, h(\ell\, k - k\, \ell) + \beta\, k(\ell\, h - h\, \ell) = 1$$

Provided that $(\ell k - k \ell)$ is a non-singular matrix, this equation can be satisfied by taking $\alpha = 1$, $\beta = 0$, $h = (\ell k - k \ell)^{-1}$.

Equation (8.6) then becomes

(8.12) $$L^c_{pj} = h^c_i(\ell^i_a\, \partial_j\, k^a_p + k^a_p\, \partial_a\, \ell^i_j - \ell^a_j\, \partial_a\, k^i_p - k^i_a\, \partial_p\, \ell^a_j)$$

Returning to the general case (8.8), the torsion tensor of the corresponding connexion given by (8.6) is found to be

(8.13) $$T^c_{pj} = \tfrac{1}{2}(L^c_{pj} - L^c_{jp}) = A^c_i\, \partial_{[j}\, \ell^i_{p]} + 8\, \alpha\, h^c_i\, N^i_{cp}(\ell, k) + 8\, \beta k^c_i\, N^i_{jp}(\ell, h)$$

For the particular connexion given by (8.12), $A^c_i = 0$, $\alpha = 1$, $\beta = 0$ so that

(8.14) $$T^c_{pj} = 8 h^c_i\, N^i_{jp}(\ell\, , k)$$

Thus the torsion tensor is <u>not</u> equal to the Nijenhuis tensor (as might have been expected) but is the inner product of this tensor with the tensor $(\ell k - k \ell)^{-1}$

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