



Centre de
Mathématiques
Laurent Schwartz



ÉCOLE
POLYTECHNIQUE

SEMINAIRE

**Equations aux
Dérivées
Partielles**

2009-2010

Toan Nguyen and Kevin Zumbrun

Long-time stability of noncharacteristic viscous boundary layers

Séminaire É. D. P. (2009-2010), Exposé n° VI, 15 p.

<http://sedp.cedram.org/item?id=SEDP_2009-2010____A6_0>

U.M.R. 7640 du C.N.R.S.
F-91128 PALAISEAU CEDEX

Fax : 33 (0)1 69 33 49 49

Tél : 33 (0)1 69 33 49 99

cedram

*Exposé mis en ligne dans le cadre du
Centre de diffusion des revues académiques de mathématiques*
<http://www.cedram.org/>

LONG-TIME STABILITY OF NONCHARACTERISTIC VISCOUS BOUNDARY LAYERS

TOAN NGUYEN AND KEVIN ZUMBRUN

ABSTRACT. We report our results on long-time stability of multi-dimensional non-characteristic boundary layers of a class of hyperbolic-parabolic systems including the compressible Navier-Stokes equations with inflow [outflow] boundary conditions, under the assumption of strong spectral, or uniform Evans, stability. Evans stability has been verified for small-amplitude layers by Guès, Métivier, Williams, and Zumbrun. For large-amplitudes, it may be checked numerically, as done in one-dimensional case for isentropic gas by Costanzino, Humpherys, Nguyen, and Zumbrun.

1. INTRODUCTION

We consider a boundary layer, or stationary solution,

$$(1.1) \quad \tilde{U} = \bar{U}(x_1), \quad \lim_{x_1 \rightarrow +\infty} \bar{U}(x_1) = U_+, \quad \bar{U}(0) = \bar{U}_0$$

of a system of conservation laws on the quarter-space

$$(1.2) \quad \tilde{U}_t + \sum_j F^j(\tilde{U})_{x_j} = \sum_{jk} (B^{jk}(\tilde{U})\tilde{U}_{x_k})_{x_j}, \quad x \in \mathbb{R}_+^d = \{x_1 > 0\}, \quad t > 0,$$

$\tilde{U}, F^j \in \mathbb{R}^n$, $B^{jk} \in \mathbb{R}^{n \times n}$, with initial data $\tilde{U}(x, 0) = \tilde{U}_0(x)$ and Dirichlet type boundary conditions specified in (1.5), (1.6) below.

Our studies to boundary layers are restricted to the case that the layers are assumed to be *noncharacteristic*, that is, the matrix dF_{11}^1 in the hyperbolic equations of \tilde{u} is either strictly positive (inflow case) or strictly negative (outflow case). Roughly speaking, the noncharacteristicity limits the signals to be transmitted into or out of but not along the boundary. In the context of gas dynamics or MHD, this corresponds to the situation of a porous boundary with prescribed inflow or outflow conditions accomplished by suction or blowing, a scenario that has been suggested as a means to reduce drag along an airfoil by stabilizing laminar flow; see Example 1.1 below.

A fundamental question is whether or not such boundary layer solutions are *stable* in the sense of PDE, i.e., whether or not a sufficiently small (initial and boundary) perturbation of \tilde{U} remains close to \tilde{U} , or converges time-asymptotically to \tilde{U} , under the evolution of (1.2). Purpose of this note is to report our results in [NZ2], addressing this time-asymptotic stability question.

1.1. Equations and assumptions. We consider the general hyperbolic-parabolic system of conservation laws (1.2) in conserved variable \tilde{U} , with

$$\tilde{U} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ b_1^{jk} & b_2^{jk} \end{pmatrix},$$

Date: June 4, 2010.

This work was supported in part by the National Science Foundation award number DMS-0300487.

$\tilde{u} \in \mathbb{R}^{n-r}$, and $\tilde{v} \in \mathbb{R}^r$, where

$$\Re \sigma \sum_{jk} b_2^{jk} \xi_j \xi_k \geq \theta |\xi|^2 > 0, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

Following [MaZ4, Z3, Z4], we assume that equations (1.2) can be written, alternatively, after a triangular change of coordinates

$$(1.3) \quad \tilde{W} := \tilde{W}(\tilde{U}) = \begin{pmatrix} \tilde{w}^I(\tilde{u}) \\ \tilde{w}^{II}(\tilde{u}, \tilde{v}) \end{pmatrix},$$

in the *quasilinear, partially symmetric hyperbolic-parabolic form*

$$(1.4) \quad \tilde{A}^0 \tilde{W}_t + \sum_j \tilde{A}^j \tilde{W}_{x_j} = \sum_{jk} (\tilde{B}^{jk} \tilde{W}_{x_k})_{x_j} + \tilde{G},$$

where

$$\tilde{A}^0 = \begin{pmatrix} \tilde{A}_{11}^0 & 0 \\ 0 & \tilde{A}_{22}^0 \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} \tilde{A}^{11} & \tilde{A}^{12} \\ \tilde{A}^{21} & \tilde{A}^{22} \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{b} \end{pmatrix}, \quad \tilde{G} = \begin{pmatrix} 0 \\ \tilde{g} \end{pmatrix}$$

and, defining $\tilde{W}_+ := \tilde{W}(U_+)$,

(A1) $\tilde{A}^j(\tilde{W}_+)$, \tilde{A}^0 , \tilde{A}_{11}^1 are symmetric, \tilde{A}^0 block diagonal, $\tilde{A}^0 \geq \theta_0 > 0$,

(A2) for each $\xi \in \mathbb{R}^d \setminus \{0\}$, no eigenvector of $\sum_j \xi_j \tilde{A}^j (\tilde{A}^0)^{-1} (\tilde{W}_+)$ lies in the kernel of $\sum_{jk} \xi_j \xi_k \tilde{B}^{jk} (\tilde{A}^0)^{-1} (\tilde{W}_+)$,

(A3) $\sum \tilde{b}^{jk} \xi_j \xi_k \geq \theta |\xi|^2$, and $\tilde{g}(\tilde{W}_x, \tilde{W}_x) = \mathcal{O}(|\tilde{W}_x|^2)$.

Along with the above structural assumptions, we make the following technical hypotheses:

(H0) $F^j, B^{jk}, \tilde{A}^0, \tilde{A}^j, \tilde{B}^{jk}, \tilde{W}(\cdot), \tilde{g}(\cdot, \cdot) \in C^{s+1}$, with $s \geq [(d-1)/2] + 4$ in our analysis of linearized stability, and $s \geq s(d) := [(d-1)/2] + 7$ in our analysis of nonlinear stability.

(H1) \tilde{A}_1^{11} is either strictly positive or strictly negative, that is, either $\tilde{A}_1^{11} \geq \theta_1 > 0$, or $\tilde{A}_1^{11} \leq -\theta_1 < 0$. (We shall call these cases the *inflow case* or *outflow case*, correspondingly.)

(H2) The eigenvalues of $dF^1(U_+)$ are distinct and nonzero.

(H3) The eigenvalues of $\sum_j dF_+^j \xi_j$ have constant multiplicity with respect to $\xi \in \mathbb{R}^d$, $\xi \neq 0$.

(H4) The set of branch points of the eigenvalues of $(\tilde{A}^1)^{-1} (i\tau \tilde{A}^0 + \sum_{j \neq 1} i\xi_j \tilde{A}^j)_+$, $\tau \in \mathbb{R}$, $\tilde{\xi} \in \mathbb{R}^{d-1}$ is the (possibly intersecting) union of finitely many smooth curves $\tau = \eta_q^+(\tilde{\xi})$, on which the branching eigenvalue has constant multiplicity s_q (by definition ≥ 2).

Condition (H1) corresponds to hyperbolic-parabolic noncharacteristicity, while (H2) is the condition for the hyperbolicity at U_+ of the associated first-order hyperbolic system obtained by dropping second-order terms. The assumptions (A1)-(A3) and (H0)-(H2) are satisfied for gas dynamics and MHD with van der Waals equation of state under inflow or outflow conditions; see discussions in [MaZ4, CHNZ, GMWZ5, GMWZ6]. Condition (H3) holds always for gas dynamics, but fails always for MHD in dimension $d \geq 2$. Condition (H4) is a technical requirement of the analysis introduced in [Z2]. It is satisfied always in dimension $d = 2$ or for rotationally invariant systems in dimensions

including the related issues of matched asymptotic expansion, multi-dimensional effects, and more general boundary configurations, see [GMWZ5].

Example 1.2. Alternatively, we may consider the compressible Navier–Stokes equations (1.7) with *blowing-type* boundary conditions

$$u_j(0, x_2, \dots, x_d) = 0, \quad j \neq 1 \quad \text{and} \quad u_1(0, x_2, \dots, x_d) = V(x) > 0,$$

and prescribed temperature and pressure

$$T(0, x_2, \dots, x_d) = T_{wall}(\tilde{x}), \quad p(0, x_2, \dots, x_d) = p_{wall}(\tilde{x})$$

(equivalently, prescribed temperature and density). Under the standard assumptions $p_\rho, T_e > 0$ on the equation of state (alternatively, van der Waals gas assumptions), this can be seen to satisfy hypotheses (A1)–(A3), (H0)–(H4), (B) in the *inflow case* (1.5).

1.2. The Evans condition and strong spectral stability. The linearized equations of (1.2), (B) about \bar{U} are

$$(1.8) \quad U_t = LU := \sum_{j,k} (B^{jk} U_{x_k})_{x_j} - \sum_j (A^j U)_{x_j}$$

with initial data $U(0) = U_0$ and boundary conditions in (linearized) \tilde{W} -coordinates of

$$W(0, \tilde{x}, t) := (w^I, w^{II})^T(0, \tilde{x}, t) = h$$

for the inflow case, and

$$w^{II}(0, \tilde{x}, t) = h$$

for the outflow case, with $x = (x_1, \tilde{x}) \in \mathbb{R}^d$, where $W := (\partial \tilde{W} / \partial U)(\bar{U})U$. Here, $B^{jk} := B^{jk}(\bar{U}(x_1))$ and $A^j U := dF^j(\bar{U}(x_1))U - [dB^{j1}(\bar{U}(x_1))U]\bar{U}_{x_1}(x_1)$.

A necessary condition for linearized stability is weak spectral stability, defined as nonexistence of unstable spectra $\Re \lambda > 0$ of the linearized operator L about the wave. This is equivalent to nonvanishing for all $\tilde{\xi} \in \mathbb{R}^{d-1}$, $\Re \lambda > 0$ of the *Evans function*

$$D_L(\tilde{\xi}, \lambda)$$

a Wronskian associated with the Fourier-transformed eigenvalue ODE.

Definition 1.3. We define *strong spectral stability as uniform Evans stability*:

$$(D) \quad |D_L(\tilde{\xi}, \lambda)| \geq \theta(C) > 0$$

for $(\tilde{\xi}, \lambda)$ on bounded subsets $C \subset \{\tilde{\xi} \in \mathbb{R}^{d-1}, \Re \lambda \geq 0\} \setminus \{0\}$.

For the class of equations we consider, this is equivalent to the uniform Evans condition of [GMWZ5, GMWZ6], which includes an additional high-frequency condition that for these equations is always satisfied (see Proposition 3.8, [GMWZ5]). A fundamental result proved in [GMWZ5] is that small-amplitude noncharacteristic boundary-layers are always strongly spectrally stable.²

Proposition 1.4 ([GMWZ5]). *Assuming (A1)–(A3), (H0)–(H3), (B) for some fixed endstate (or compact set of endstates) U_+ , boundary layers with amplitude*

$$\|\bar{U} - U_+\|_{L^\infty[0, +\infty]}$$

sufficiently small satisfy the strong spectral stability condition (D).

²The result of [GMWZ5] applies also to more general types of boundary conditions and in some situations to systems with variable multiplicity characteristics, including, in some parameter ranges, MHD.

As demonstrated in [SZ], stability of large-amplitude boundary layers may fail for the class of equations considered here, even in a single space dimension, so there is no such general theorem in the large-amplitude case. Stability of large-amplitude boundary layers may be checked efficiently by numerical Evans computations as in [BDG, Br1, Br2, BrZ, HuZ, BHRZ, HLZ, CHNZ, HLYZ1, HLYZ2].

1.3. Main results. Our main results are as follows.

Theorem 1.5 (Linearized stability). *Assuming (A1)-(A3), (H0)-(H4), (B), and strong spectral stability (D), we obtain asymptotic $L^1 \cap H^{[(d-1)/2]+5} \rightarrow L^p$ stability of (1.8) in dimension $d \geq 2$, and any $2 \leq p \leq \infty$, with rate of decay*

$$(1.9) \quad \begin{aligned} |U(t)|_{L^2} &\leq C(1+t)^{-\frac{d-1}{4}} |U_0|_{L^1 \cap H^3}, \\ |U(t)|_{L^p} &\leq C(1+t)^{-\frac{d}{2}(1-1/p)+1/2p} |U_0|_{L^1 \cap H^{[(d-1)/2]+5}}, \end{aligned}$$

provided that the initial perturbations U_0 are in $L^1 \cap H^3$ for $p = 2$, or in $L^1 \cap H^{[(d-1)/2]+5}$ for $p > 2$, and zero boundary perturbations $h = 0$.

Theorem 1.6 (Nonlinear stability). *Assuming (A1)-(A3), (H0)-(H4), (B), and strong spectral stability (D), we obtain asymptotic $L^1 \cap H^s \rightarrow L^p \cap H^s$ stability of \bar{U} as a solution of (1.2) in dimension $d \geq 2$, for $s \geq s(d)$ as defined in (H0), and any $2 \leq p \leq \infty$, with rate of decay*

$$(1.10) \quad \begin{aligned} |\tilde{U}(t) - \bar{U}|_{L^p} &\leq C(1+t)^{-\frac{d}{2}(1-1/p)+1/2p} |U_0|_{L^1 \cap H^s} \\ |\tilde{U}(t) - \bar{U}|_{H^s} &\leq C(1+t)^{-\frac{d-1}{4}} |U_0|_{L^1 \cap H^s}, \end{aligned}$$

provided that the initial perturbations $U_0 := \tilde{U}_0 - \bar{U}$ are sufficiently small in $L^1 \cap H^s$ and zero boundary perturbations $h = 0$.

Remark 1.7. Nonzero boundary perturbations are also treated in [NZ2]. However, for simplicity, we only report here the case of zero boundary perturbations.

Combining Theorem 1.6 and Proposition 1.4, we obtain the following small-amplitude stability result, applying in particular to the motivating situation of Example 1.1.

Corollary 1.8. *Assuming (A1)-(A3), (H0)-(H4), (B) for some fixed endstate (or compact set of endstates) U_+ , boundary layers with amplitude*

$$\|\bar{U} - U_+\|_{L^\infty[0,+\infty]}$$

sufficiently small are linearly and nonlinearly stable in the sense of Theorems 1.5 and 1.6.

Remark 1.9. The obtained rate of decay in L^2 may be recognized as that of a $(d-1)$ -dimensional heat kernel, and the obtained rate of decay in L^∞ as that of a d -dimensional heat kernel. We believe that the sharp rate of decay in L^2 is rather that of a d -dimensional heat kernel and the sharp rate of decay in L^∞ dependent on the characteristic structure of the associated inviscid equations, as in the constant-coefficient case [HoZ1, HoZ2].

Remark 1.10. In one dimension, strong spectral stability is necessary for linearized asymptotic stability; see Theorem 1.6, [NZ1]. However, in multi-dimensions, it appears likely that, as in the shock case [Z3], there are intermediate possibilities between strong and weak spectral stability for which linearized stability might hold with degraded rates of decay. In any case, the gap between the necessary weak spectral and the sufficient

strong spectral stability conditions concerns only pure imaginary spectra $\Re\lambda = 0$ on the boundary between strictly stable and unstable half-planes, so this should not interfere with investigation of physical stability regions.

1.4. Discussion and open problems. Asymptotic stability, without rates of decay, has been shown for small amplitude noncharacteristic “normal” boundary layers of the isentropic compressible Navier–Stokes equations with outflow boundary conditions and vanishing transverse velocity in [KK], using energy estimates. Corollary 1.8 recovers this existing result and extends it to the general arbitrary transverse velocity, outflow or inflow, and isentropic or nonisentropic (full compressible Navier–Stokes) case, in addition giving asymptotic rates of decay. Also, the type of boundary layer relevant to the drag-reduction strategy discussed in Examples 1.1–1.2 is a noncharacteristic “transverse” type with constant normal velocity, complementary to the normal type considered in [KK].

The large-amplitude asymptotic stability result of Theorem 1.6 extends to multi dimensions corresponding one-dimensional results of [YZ, NZ1], reducing the problem of stability to verification of a numerically checkable Evans condition. See also the related, but technically rather different, work on the small viscosity limit in [MZ, GMWZ5, GMWZ6]. By a combination of numerical Evans function computations and asymptotic ODE estimates, spectral stability has been checked for *arbitrary amplitude* noncharacteristic boundary layers of the one-dimensional isentropic compressible Navier–Stokes equations in [CHNZ]. Extensions to the nonisentropic and multi-dimensional case should be possible by the methods used in [HLyZ1] and [HLyZ2] respectively to treat the related shock stability problem.

This (investigation of large-amplitude spectral stability) would be a very interesting direction for further investigation. In particular, note that it is large-amplitude stability that is relevant to drag-reduction at flight speeds, since the transverse relative velocity (i.e., velocity parallel to the airfoil) is zero at the wing surface and flight speed outside a thin boundary layer, so that variation across the boundary layer is substantial.

Our method of analysis follows the basic approach introduced in [Z2, Z3, Z4] for the study of multi-dimensional shock stability and we are able to make use of much of that analysis without modification. However, there are some new difficulties to be overcome in the boundary-layer case.

The main new difficulty is that the boundary-layer case is analogous to the *undercompressive shock* case rather than the more favorable *Lax shock* case emphasized in [Z3], in that $G_{y_1} \not\sim t^{-1/2}G$ as in the Lax shock case but rather $G_{y_1} \sim (e^{-\theta|y_1|} + t^{-1/2})G$, $\theta > 0$, as in the undercompressive case. This is a significant difficulty; indeed, for this reason, the undercompressive shock analysis was carried out in [Z3] only in nonphysical dimensions $d \geq 4$. On the other hand, there is no translational invariance in the boundary layer problem, so no zero-eigenvalue and no pole of the resolvent kernel at the origin for the one-dimensional operator, and in this sense G is somewhat better in the boundary layer than in the shock case.

Thus, the difficulty of the present problem is roughly intermediate to that of the Lax and undercompressive shock cases. Though the undercompressive shock case is still open in multi-dimensions for $d \leq 3$, the slight advantage afforded by lack of pole terms allows us to close the argument in the boundary-layer case. Specifically, thanks to the absence of pole terms, we are able to get a slightly improved rate of decay in $L^\infty(x_1)$ norms, though our $L^2(x_1)$ estimates remain the same as in the shock case. By keeping track of these improved sup norm bounds throughout the proof, we are able to close the

argument without using detailed pointwise bounds as in the one-dimensional analyses of [HZ, RZ].

Other difficulties include the appearance of boundary terms in integrations by parts, which makes the auxiliary energy estimates by which we control high-frequency effects considerably more difficult in the boundary-layer than in the shock-layer case, and the treatment of boundary perturbations. In terms of the homogeneous Green function G , boundary perturbations lead by a standard duality argument to contributions consisting of integrals on the boundary of perturbations against various derivatives of G , and these are a bit too singular as time goes to zero to be absolutely integrable. Following the strategy introduced in [YZ, NZ1], we instead use duality to convert these to less singular integrals over the whole space, that *are* absolutely integrable in time. However, we make a key improvement here over the treatment in [YZ, NZ1], integrating against an exponentially decaying test function to obtain terms of exactly the same form already treated for the homogeneous problem. This is necessary for us in the multi-dimensional case, for which we have insufficient information about individual parts of the solution operator to estimate them separately as in [YZ, NZ1], but makes things much more transparent also in the one-dimensional case.

Among physical systems, our hypotheses appear to apply to and essentially only to the case of compressible Navier–Stokes equations with inflow or outflow boundary conditions. However, the method of analysis should apply, with suitable modifications, to more general situations such as MHD; see an extension to MHD case via a different approach [N2], and see also the recent results on the related small-viscosity problem in [GMWZ5, GMWZ6].

Finally, as pointed out in Remark 1.10, the strong spectral stability condition does not appear to be necessary for asymptotic stability. It would be interesting to develop a refined stability condition similarly as was done in [SZ, Z2, Z3, Z4] for the shock case.

2. LINEARIZED ESTIMATES

We first establish estimates on the linearized inhomogeneous problem

$$(2.1) \quad U_t - LU = f$$

with initial data $U(0) = U_0$ and Dirichlet boundary conditions as usual in \tilde{W} -coordinates:

$$(2.2) \quad W(0, \tilde{x}, t) := (w^I, w^{II})^T(0, \tilde{x}, t) = h$$

for the inflow case, and

$$(2.3) \quad w^{II}(0, \tilde{x}, t) = h$$

for the outflow case, with $x = (x_1, \tilde{x}) \in \mathbb{R}^d$.

Let us define low- and high-frequency parts of the linearized solution operator $\mathcal{S}(t)$ of the linearized problem with homogeneous boundary and forcing data, $f, h \equiv 0$, as

$$(2.4) \quad \mathcal{S}_1(t) := \frac{1}{(2\pi i)^d} \int_{|\tilde{\xi}| \leq r} \oint_{\Gamma^{\tilde{\xi}} \cap \{|\lambda| \leq r\}} e^{\lambda t + i\tilde{\xi} \cdot \tilde{x}} (L_{\tilde{\xi}} - \lambda)^{-1} d\lambda d\tilde{\xi}$$

and $\mathcal{S}_2(t) := e^{Lt} - \mathcal{S}_1(t)$.

Then we obtain the following linearized estimates.

Proposition 2.1 (Low-frequency estimate). *Under the hypotheses of Theorem 1.6, for $\beta = (\beta_1, \beta')$ with $\beta_1 = 0, 1$,*

$$(2.5) \quad \begin{aligned} |\mathcal{S}_1(t)\partial_x^\beta f|_{L_x^2} &\leq C(1+t)^{-(d-1)/4-|\beta|/2}|f|_{L_x^1} + C\beta_1(1+t)^{-(d-1)/4}|f|_{L_{\tilde{x},x_1}^{1,\infty}}, \\ |\mathcal{S}_1(t)\partial_x^\beta f|_{L_{\tilde{x},x_1}^{2,\infty}} &\leq C(1+t)^{-(d+1)/4-|\beta|/2}|f|_{L_x^1} + C\beta_1(1+t)^{-(d+1)/4}|f|_{L_{\tilde{x},x_1}^{1,\infty}}, \\ |\mathcal{S}_1(t)\partial_x^\beta f|_{L_{\tilde{x},x_1}^\infty} &\leq C(1+t)^{-d/2-|\beta|/2}|f|_{L_x^1} + C\beta_1(1+t)^{-d/2}|f|_{L_{\tilde{x},x_1}^{1,\infty}}, \end{aligned}$$

where $|\cdot|_{L_{\tilde{x},x_1}^{p,q}}$ denotes the norm in $L^p(\tilde{x}; L^q(x_1))$.

According to [Z4, Corollary 4.11], we can write

$$(2.6) \quad \begin{aligned} \mathcal{S}_2(t)f &= \frac{1}{(2\pi i)^d} \text{P.V.} \int_{-\theta_1-i\infty}^{-\theta_1+i\infty} \int_{\mathbb{R}^{d-1}} \chi_{|\tilde{\xi}|^2+|\Im m\lambda|^2 \geq \theta_1+\theta_2} \\ &\quad \times e^{i\tilde{\xi}\cdot\tilde{x}+\lambda t} (\lambda - L_{\tilde{\xi}})^{-1} \hat{f}(x_1, \tilde{\xi}) d\tilde{\xi} d\lambda, \end{aligned}$$

and we obtain the following.

Proposition 2.2 (High-frequency estimate). *Given (A1)-(A2), (H0)-(H2), (D), and homogeneous boundary conditions (B), for $0 \leq |\alpha| \leq s-3$, s as in (H0),*

$$(2.7) \quad \begin{aligned} |\mathcal{S}_2(t)f|_{L_x^2} &\leq C e^{-\theta_1 t} |f|_{H_x^3}, \\ |\partial_x^\alpha \mathcal{S}_2(t)f|_{L_x^2} &\leq C e^{-\theta_1 t} |f|_{H_x^{|\alpha|+3}}. \end{aligned}$$

2.1. Resolvent bounds. Our first step of proving the linearized estimates is to estimate solutions of the resolvent equation with homogeneous boundary data $\hat{h} \equiv 0$.

Proposition 2.3 (High-frequency bounds). *Given (A1)-(A2), (H0)-(H2), and homogeneous boundary conditions (B), for some R, C sufficiently large and $\theta > 0$ sufficiently small,*

$$(2.8) \quad |(L_{\tilde{\xi}} - \lambda)^{-1} \hat{f}|_{\hat{H}^1(x_1)} \leq C |\hat{f}|_{\hat{H}^1(x_1)},$$

and

$$(2.9) \quad |(L_{\tilde{\xi}} - \lambda)^{-1} \hat{f}|_{L^2(x_1)} \leq \frac{C}{|\lambda|^{1/2}} |\hat{f}|_{\hat{H}^1(x_1)},$$

for all $|(\tilde{\xi}, \lambda)| \geq R$ and $\Re \lambda \geq -\theta$, where \hat{f} is the Fourier transform of f in variable \tilde{x} and $|\hat{f}|_{\hat{H}^1(x_1)} := |(1 + |\partial_{x_1}| + |\tilde{\xi}|)\hat{f}|_{L^2(x_1)}$.

Proof. The proposition follows easily by applying a Laplace-Fourier transformed version with respect to variables (λ, \tilde{x}) of the nonlinear energy estimate in Section 3.1 with $s = 1$, carried out on the linearized equations written in W -coordinates. See [NZ2] for all the details. \square

We next have the following:

Proposition 2.4 (Mid-frequency bounds). *Given (A1)-(A2), (H0)-(H2), and strong spectral stability (D),*

$$(2.10) \quad |(L_{\tilde{\xi}} - \lambda)^{-1}|_{\hat{H}^1(x_1)} \leq C, \quad \text{for } R^{-1} \leq |(\tilde{\xi}, \lambda)| \leq R \text{ and } \Re \lambda \geq -\theta,$$

for any R and $C = C(R)$ sufficiently large and $\theta = \theta(R) > 0$ sufficiently small, where $|\hat{f}|_{\hat{H}^1(x_1)}$ is defined as in Proposition 2.3.

Proof. Immediate, by compactness of the set of frequencies under consideration together with the fact that the resolvent $(\lambda - L_{\tilde{\xi}})^{-1}$ is analytic with respect to H^1 in $(\tilde{\xi}, \lambda)$; see Proposition 4.8, [Z4]. \square

We next obtain the following resolvent bound for low-frequency regions as a direct consequence of pointwise bounds on the resolvent kernel.

Proposition 2.5 (Low-frequency bounds). *Under the hypotheses of Theorem 1.6, for $\lambda \in \Gamma^{\tilde{\xi}}$ and $\rho := |(\tilde{\xi}, \lambda)|$, θ_1 sufficiently small, there holds the resolvent bound*

$$(2.11) \quad |(L_{\tilde{\xi}} - \lambda)^{-1} \partial_{x_1}^{\beta} \hat{f}|_{L^p(x_1)} \leq C \gamma_2 \rho^{-2/p} \left[\rho^{\beta} |\hat{f}|_{L^1(x_1)} + \beta |\hat{f}|_{L^{\infty}(x_1)} \right]$$

for all $2 \leq p \leq \infty$, $\beta = 0, 1$, where

$$(2.12) \quad \gamma_2 := 1 + \sum_j \left[\rho^{-1} |\Im m \lambda - \eta_j^+(\tilde{\xi})| + \rho \right]^{1/s_j - 1},$$

and $s_j, \eta_j^+(\tilde{\xi})$ are as defined in (H4).

Proof. Applying the following pointwise bounds on the resolvent kernel deliberately constructed in [Z3] and recalled in [NZ2], Proposition 2.5,

$$|\partial_{y_1}^{\beta} G_{\tilde{\xi}, \lambda}(x_1, y_1)| \leq C \gamma_2 (\rho^{\beta} + \beta e^{-\theta y_1}) e^{-\theta \rho^2 |x_1 - y_1|},$$

and using the convolution inequality $|g * h|_{L^p} \leq |g|_{L^p} |h|_{L^1}$, we obtain

$$\begin{aligned} & |(L_{\tilde{\xi}} - \lambda)^{-1} \partial_{x_1}^{\beta} \hat{f}|_{L^p(x_1)} \\ &= \left| \int \partial_{y_1}^{\beta} G_{\tilde{\xi}, \lambda}(x_1, y_1) \hat{f}(y_1, \tilde{\xi}) dy_1 \right|_{L^p(x_1)} + \beta |G_{\tilde{\xi}, \lambda}(x_1, 0) \hat{f}(0, \tilde{\xi})|_{L^p(x_1)} \\ &\leq \left| \int C \gamma_2 (\rho^{\beta} + \beta e^{-\theta y_1}) e^{-\theta \rho^2 |x_1 - y_1|} |\hat{f}(y_1, \tilde{\xi})| dy_1 \right|_{L^p} + C \gamma_2 \beta |\hat{f}(0, \tilde{\xi})| e^{-\theta \rho^2 x_1} |_{L^p(x_1)} \\ &\leq C \gamma_2 \rho^{-2/p} \left[\rho^{\beta} |\hat{f}|_{L^1(x_1)} + \beta |\hat{f}|_{L^{\infty}(x_1)} \right] \end{aligned}$$

as claimed. \square

Remark 2.6. The above L^p bounds may alternatively be obtained directly by the argument of Section 12, [GMWZ1], using quite different Kreiss symmetrizer techniques, again omitting pole terms arising from vanishing of the Evans function at the origin, and also the auxiliary problem construction of Section 12.6 used to obtain sharpened bounds in the Lax or overcompressive shock case (not relevant here). See also [N2] in this direction with treatment of the boundary layer case.

2.2. Estimates on homogeneous solution operators. We sketch the proof of Propositions 2.1 and 2.2.

Proof of Proposition 2.1. The proof will follow closely the treatment of the shock case in [Z3]. Let $\hat{u}(x_1, \tilde{\xi}, \lambda)$ denote the solution of $(L_{\tilde{\xi}} - \lambda) \hat{u} = \hat{f}$, where $\hat{f}(x_1, \tilde{\xi})$ denotes Fourier transform of f , and

$$u(x, t) := \mathcal{S}_1(t) f = \frac{1}{(2\pi i)^d} \int_{|\tilde{\xi}| \leq r} \oint_{\Gamma^{\tilde{\xi}} \cap \{|\lambda| \leq r\}} e^{\lambda t + i \tilde{\xi} \cdot \tilde{x}} (L_{\tilde{\xi}} - \lambda)^{-1} \hat{f}(x_1, \tilde{\xi}) d\lambda d\tilde{\xi}.$$

Recalling the resolvent estimates in Proposition 2.5, we have

$$|\hat{u}(x_1, \tilde{\xi}, \lambda)|_{L^p(x_1)} \leq C \gamma_2 \rho^{-2/p} |\hat{f}|_{L^1(x_1)} \leq C \gamma_2 \rho^{-2/p} |f|_{L^1(x)}$$

where γ_2 is as defined in (2.12).

Therefore, using Parseval's identity, Fubini's theorem, and the triangle inequality, we may estimate

$$\begin{aligned}
 |u|_{L^2(x_1, \tilde{x})}^2(t) &= \frac{1}{(2\pi)^{2d}} \int_{x_1} \int_{\tilde{\xi}} \left| \oint_{\Gamma^{\tilde{\xi}} \cap \{|\lambda| \leq r\}} e^{\lambda t} \hat{u}(x_1, \tilde{\xi}, \lambda) d\lambda \right|^2 d\tilde{\xi} dx_1 \\
 &= \frac{1}{(2\pi)^{2d}} \int_{\tilde{\xi}} \left| \oint_{\Gamma^{\tilde{\xi}} \cap \{|\lambda| \leq r\}} e^{\lambda t} \hat{u}(x_1, \tilde{\xi}, \lambda) d\lambda \right|_{L^2(x_1)}^2 d\tilde{\xi} \\
 &\leq \frac{1}{(2\pi)^{2d}} \int_{\tilde{\xi}} \left| \oint_{\Gamma^{\tilde{\xi}} \cap \{|\lambda| \leq r\}} e^{\Re \lambda t} |\hat{u}(x_1, \tilde{\xi}, \lambda)|_{L^2(x_1)} d\lambda \right|^2 d\tilde{\xi} \\
 &\leq C |f|_{L^1(x)}^2 \int_{\tilde{\xi}} \left| \oint_{\Gamma^{\tilde{\xi}} \cap \{|\lambda| \leq r\}} e^{\Re \lambda t} \gamma_2 \rho^{-1} d\lambda \right|^2 d\tilde{\xi}.
 \end{aligned}$$

Specifically, parametrizing $\Gamma^{\tilde{\xi}}$ by

$$\lambda(\tilde{\xi}, k) = ik - \theta_1(k^2 + |\tilde{\xi}|^2), \quad k \in \mathbb{R},$$

and observing that by (2.12),

$$\begin{aligned}
 \gamma_2 \rho^{-1} &\leq (|k| + |\tilde{\xi}|)^{-1} \left[1 + \sum_j \left(\frac{|k - \tau_j(\tilde{\xi})|}{\rho} \right)^{1/s_j - 1} \right] \\
 (2.13) \quad &\leq (|k| + |\tilde{\xi}|)^{-1} \left[1 + \sum_j \left(\frac{|k - \tau_j(\tilde{\xi})|}{\rho} \right)^{\epsilon - 1} \right],
 \end{aligned}$$

where $\epsilon := \frac{1}{\max_j s_j}$ ($0 < \epsilon < 1$ chosen arbitrarily if there are no singularities), we estimate

$$\begin{aligned}
 \int_{\tilde{\xi}} \left| \oint_{\Gamma^{\tilde{\xi}} \cap \{|\lambda| \leq r\}} e^{\Re \lambda t} \gamma_2 \rho^{-1} d\lambda \right|^2 d\tilde{\xi} &\leq \int_{\tilde{\xi}} \left| \int_{\mathbb{R}} e^{-\theta_1(k^2 + |\tilde{\xi}|^2)t} \gamma_2 \rho^{-1} dk \right|^2 d\tilde{\xi} \\
 &\leq \int_{\tilde{\xi}} e^{-2\theta_1|\tilde{\xi}|^2 t} |\tilde{\xi}|^{-2\epsilon} \left| \int_{\mathbb{R}} e^{-\theta_1 k^2 t} |k|^{\epsilon-1} dk \right|^2 d\tilde{\xi} \\
 &\quad + \sum_j \int_{\tilde{\xi}} e^{-2\theta_1|\tilde{\xi}|^2 t} |\tilde{\xi}|^{-2\epsilon} \left| \int_{\mathbb{R}} e^{-\theta_1 k^2 t} |k - \tau_j(\tilde{\xi})|^{\epsilon-1} dk \right|^2 d\tilde{\xi} \\
 &\leq \int_{\tilde{\xi}} e^{-2\theta_1|\tilde{\xi}|^2 t} |\tilde{\xi}|^{-2\epsilon} \left| \int_{\mathbb{R}} e^{-\theta_1 k^2 t} |k|^{\epsilon-1} dk \right|^2 d\tilde{\xi} \\
 &\leq C t^{-(d-1)/2}.
 \end{aligned}$$

In the same way as above, we also obtain similar estimates for $|u|_{L_{\tilde{x}, x_1}^{2, \infty}}$ and $|u|_{L_{\tilde{x}, x_1}^{\infty}}$. The x_1 -derivative bounds follow similarly by using the resolvent bounds in Proposition 2.5 with $\beta_1 = 1$. The \tilde{x} -derivative bounds are straightforward by the fact that $\widehat{\partial_{\tilde{x}}^{\beta} f} = (i\tilde{\xi})^{\beta} \hat{f}$.

Finally, each of the above integrals is bounded by $C|f|_{L^1(x)}$ as the product of $|f|_{L^1(x)}$ times the integral quantities $\gamma_2 \rho^{-1}$, γ_2 over a bounded domain, hence we may replace t by $(1+t)$ in the above estimates. \square

Proof of Proposition 2.2. The proof starts with the following resolvent identity, using analyticity on the resolvent set $\rho(L_{\tilde{\xi}})$ of the resolvent $(\lambda - L_{\tilde{\xi}})^{-1}$, for all $f \in \mathcal{D}(L_{\tilde{\xi}})$,

$$(2.14) \quad (\lambda - L_{\tilde{\xi}})^{-1} f = \lambda^{-1} (\lambda - L_{\tilde{\xi}})^{-1} L_{\tilde{\xi}} f + \lambda^{-1} f.$$

Using this identity and (2.6), we estimate

$$\begin{aligned}
 \mathcal{S}_2(t)f &= \frac{1}{(2\pi i)^d} \text{P.V.} \int_{-\theta_1-i\infty}^{-\theta_1+i\infty} \int_{\mathbb{R}^{d-1}} \chi_{|\tilde{\xi}|^2+|\Im m \lambda|^2 \geq \theta_1+\theta_2} \\
 &\quad \times e^{i\tilde{\xi}\cdot\tilde{x}+\lambda t} \lambda^{-1} (\lambda - L_{\tilde{\xi}})^{-1} L_{\tilde{\xi}} \hat{f}(x_1, \tilde{\xi}) d\tilde{\xi} d\lambda \\
 (2.15) \quad &+ \frac{1}{(2\pi i)^d} \text{P.V.} \int_{-\theta_1-i\infty}^{-\theta_1+i\infty} \int_{\mathbb{R}^{d-1}} \chi_{|\tilde{\xi}|^2+|\Im m \lambda|^2 \geq \theta_1+\theta_2} \\
 &\quad \times e^{i\tilde{\xi}\cdot\tilde{x}+\lambda t} \lambda^{-1} \hat{f}(x_1, \tilde{\xi}) d\tilde{\xi} d\lambda \\
 &=: \mathcal{S}_1 + \mathcal{S}_2,
 \end{aligned}$$

where, by Plancherel's identity and Propositions 2.2 and 2.4, we have

$$\begin{aligned}
 |\mathcal{S}_1|_{L^2(\tilde{x}, x_1)} &\leq C \int_{-\theta_1-i\infty}^{-\theta_1+i\infty} |\lambda|^{-1} |e^{\lambda t}| |(\lambda - L_{\tilde{\xi}})^{-1} L_{\tilde{\xi}} \hat{f}|_{L^2(\tilde{\xi}, x_1)} |d\lambda| \\
 &\leq C e^{-\theta_1 t} \int_{-\theta_1-i\infty}^{-\theta_1+i\infty} |\lambda|^{-3/2} \left| (1 + |\tilde{\xi}|) |L_{\tilde{\xi}} \hat{f}|_{H^1(x_1)} \right|_{L^2(\tilde{\xi})} |d\lambda| \\
 &\leq C e^{-\theta_1 t} |f|_{H_x^3}
 \end{aligned}$$

and

$$\begin{aligned}
 |\mathcal{S}_2|_{L_x^2} &\leq \frac{1}{(2\pi)^d} \left| \text{P.V.} \int_{-\theta_1-i\infty}^{-\theta_1+i\infty} \lambda^{-1} e^{\lambda t} d\lambda \int_{\mathbb{R}^{d-1}} e^{i\tilde{x}\cdot\tilde{\xi}} \hat{f}(x_1, \tilde{\xi}) d\tilde{\xi} \right|_{L^2} \\
 (2.16) \quad &+ \frac{1}{(2\pi)^d} \left| \text{P.V.} \int_{-\theta_1-ir}^{-\theta_1+ir} \lambda^{-1} e^{\lambda t} d\lambda \int_{\mathbb{R}^{d-1}} e^{i\tilde{x}\cdot\tilde{\xi}} \hat{f}(x_1, \tilde{\xi}) d\tilde{\xi} \right|_{L^2} \\
 &\leq C e^{-\theta_1 t} |f|_{L_x^2},
 \end{aligned}$$

by direct computations, noting that the integral in λ in the first term is identically zero. This completes the proof of the first inequality stated in the proposition. Derivative bounds follow similarly. \square

2.3. Proof of linearized stability.

Proof of Theorem 1.5. Applying estimates on low- and high-frequency operators $\mathcal{S}_1(t)$ and $\mathcal{S}_2(t)$, we obtain

$$\begin{aligned}
 |U(t)|_{L^2} &\leq |\mathcal{S}_1(t)U_0|_{L^2} + |\mathcal{S}_2(t)U_0|_{L^2} \\
 (2.17) \quad &\leq C(1+t)^{-\frac{d-1}{4}} |U_0|_{L^1} + C e^{-\eta t} |U_0|_{H^3} \\
 &\leq C(1+t)^{-\frac{d-1}{4}} |U_0|_{L^1 \cap H^3}
 \end{aligned}$$

and

$$\begin{aligned}
 |U(t)|_{L^\infty} &\leq |\mathcal{S}_1(t)U_0|_{L^\infty} + |\mathcal{S}_2(t)U_0|_{L^\infty} \\
 (2.18) \quad &\leq C(1+t)^{-\frac{d}{2}} |U_0|_{L^1} + C |\mathcal{S}_2(t)U_0|_{H^{[(d-1)/2]+2}} \\
 &\leq C(1+t)^{-\frac{d}{2}} |U_0|_{L^1} + C e^{-\eta t} |U_0|_{H^{[(d-1)/2]+5}} \\
 &\leq C(1+t)^{-\frac{d}{2}} |U_0|_{L^1 \cap H^{[(d-1)/2]+5}}.
 \end{aligned}$$

These prove the bounds as stated in the theorem for $p = 2$ and $p = \infty$. For $2 < p < \infty$, we use the interpolation inequality between L^2 and L^∞ . \square

3. NONLINEAR STABILITY

3.1. Auxiliary energy estimates. For the analysis of nonlinear stability, we need the following energy estimate adapted from [MaZ4, NZ1, Z4]. Define the nonlinear perturbation variables $U = (u, v)$ by

$$(3.1) \quad U(x, t) := \tilde{U}(x, t) - \bar{U}(x_1).$$

Proposition 3.1. *Under the hypotheses of Theorem 1.6, let $U_0 \in H^s$ and $U = (u, v)^T$ be a solution of (1.2) and (3.1). Suppose that, for $0 \leq t \leq T$, the $W_x^{2, \infty}$ norm of the solution U remains bounded by a sufficiently small constant $\zeta > 0$. Then*

$$(3.2) \quad |U(t)|_{H^s}^2 \leq C e^{-\theta t} |U_0|_{H^s}^2 + C \int_0^t e^{-\theta(t-\tau)} |U(\tau)|_{L^2}^2 d\tau$$

for all $0 \leq t \leq T$.

Proof. The proof uses the Goodmann-weighted and Kawashima-type energy estimates adapted from [MaZ4, Z4] for the shock case. See [NZ2] for details. \square

3.2. Proof of nonlinear stability. Defining the perturbation variable $U := \tilde{U} - \bar{U}$, we obtain the nonlinear perturbation equations

$$(3.3) \quad U_t - LU = \sum_j Q^j(U, U_x)_{x_j},$$

where $Q^j(U, U_x) = \mathcal{O}(|U||U_x| + |U|^2)$ so long as $|U|$ remains bounded.

Applying the Duhamel formula (see Lemma 3.9, [NZ2]) to (3.3), we obtain

$$(3.4) \quad U(x, t) = \mathcal{S}(t)U_0 + \int_0^t \mathcal{S}(t-s) \sum_j \partial_{x_j} Q^j(U, U_x) ds$$

where $U(x, 0) = U_0(x)$.

Proof of Theorem 1.6. Define

$$(3.5) \quad \begin{aligned} \zeta(t) := & \sup_s \left(|U(s)|_{L_x^2} (1+s)^{\frac{d-1}{4}} + |U(s)|_{L_x^\infty} (1+s)^{\frac{d}{2}} \right. \\ & \left. + (|U(s)| + |U_x(s)| + |\partial_x^2 U(s)|)_{L_{\bar{x}, x_1}^{2, \infty}} (1+s)^{\frac{d+1}{4}} \right). \end{aligned}$$

We shall prove here that for all $t \geq 0$ for which a solution exists with $\zeta(t)$ uniformly bounded by some fixed, sufficiently small constant, there holds

$$(3.6) \quad \zeta(t) \leq C(|U_0|_{L^1 \cap H^s} + E_0 + \zeta(t)^2).$$

This bound together with continuity of $\zeta(t)$ implies that

$$(3.7) \quad \zeta(t) \leq 2C(|U_0|_{L^1 \cap H^s} + E_0)$$

for $t \geq 0$, provided that $|U_0|_{L^1 \cap H^s} + E_0 < 1/4C^2$. This would complete the proof of the bounds as claimed in the theorem, and thus give the main theorem.

By standard short-time theory/local well-posedness in H^s , and the standard principle of continuation, there exists a solution $U \in H^s$ on the open time-interval for which $|U|_{H^s}$ remains bounded, and on this interval $\zeta(t)$ is well-defined and continuous. Now, let $[0, T)$ be the maximal interval on which $|U|_{H^s}$ remains strictly bounded by some

fixed, sufficiently small constant $\delta > 0$. By Proposition 3.1, and the Sobolev embedding inequality $|U|_{W^{2,\infty}} \leq C|U|_{H^s}$, we have

$$(3.8) \quad \begin{aligned} |U(t)|_{H^s}^2 &\leq C e^{-\theta t} |U_0|_{H^s}^2 + C \int_0^t e^{-\theta(t-\tau)} |U(\tau)|_{L^2}^2 d\tau \\ &\leq C(|U_0|_{H^s}^2 + \zeta(t)^2)(1+t)^{-(d-1)/2}. \end{aligned}$$

and so the solution continues so long as ζ remains small, with bound (3.7), yielding existence and the claimed bounds.

Thus, it remains to prove the claim (3.6). First by (3.4), we obtain

$$(3.9) \quad \begin{aligned} |U(t)|_{L^2} &\leq |\mathcal{S}(t)U_0|_{L^2} + \int_0^t |\mathcal{S}_1(t-s)\partial_{x_j}Q^j(s)|_{L^2} ds + \int_0^t |\mathcal{S}_2(t-s)\partial_{x_j}Q^j(s)|_{L^2} ds \\ &\leq I_1 + I_2 + I_3 \end{aligned}$$

where

$$\begin{aligned} I_1 &:= |\mathcal{S}(t)U_0|_{L^2} \leq C(1+t)^{-\frac{d-1}{4}} |U_0|_{L^1 \cap H^3}, \\ I_2 &:= \int_0^t |\mathcal{S}_1(t-s)\partial_{x_j}Q^j(s)|_{L^2} ds \\ &\leq C \int_0^t (1+t-s)^{-\frac{d-1}{4}-\frac{1}{2}} |Q^j(s)|_{L^1} + (1+s)^{-\frac{d-1}{4}} |Q^j(s)|_{L^1_{\tilde{x},x_1}} ds \\ &\leq C \int_0^t (1+t-s)^{-\frac{d-1}{4}-\frac{1}{2}} |U|_{H^1}^2 + (1+t-s)^{-\frac{d-1}{4}} \left(|U|_{L^2_{\tilde{x},x_1}}^2 + |U_x|_{L^2_{\tilde{x},x_1}}^2 \right) ds \\ &\leq C(|U_0|_{H^s}^2 + \zeta(t)^2) \int_0^t \left[(1+t-s)^{-\frac{d-1}{4}-\frac{1}{2}} (1+s)^{-\frac{d-1}{2}} \right. \\ &\quad \left. + (1+t-s)^{-\frac{d-1}{4}} (1+s)^{-\frac{d+1}{2}} \right] ds \\ &\leq C(1+t)^{-\frac{d-1}{4}} (|U_0|_{H^s}^2 + \zeta(t)^2) \end{aligned}$$

and

$$\begin{aligned} I_3 &:= \int_0^t |\mathcal{S}_2(t-s)\partial_{x_j}Q^j(s)|_{L^2} ds \leq \int_0^t e^{-\theta(t-s)} |\partial_{x_j}Q^j(s)|_{H^3} ds \\ &\leq C \int_0^t e^{-\theta(t-s)} (|U|_{L^\infty} + |U_x|_{L^\infty}) |U|_{H^5} ds \leq C \int_0^t e^{-\theta(t-s)} |U|_{H^s}^2 ds \\ &\leq C(|U_0|_{H^s}^2 + \zeta(t)^2) \int_0^t e^{-\theta(t-s)} (1+s)^{-\frac{d-1}{2}} ds \\ &\leq C(1+t)^{-\frac{d-1}{2}} (|U_0|_{H^s}^2 + \zeta(t)^2). \end{aligned}$$

Combining these above estimates yields

$$(3.10) \quad |U(t)|_{L^2} (1+t)^{\frac{d-1}{4}} \leq C(|U_0|_{L^1 \cap H^s} + \zeta(t)^2).$$

Similarly, we can obtain estimates for $|U(t)|_{L^2_{\tilde{x},x_1}}$, $|U_x(t)|_{L^2_{\tilde{x},x_1}}$, $|U_{\tilde{x}\tilde{x}}|_{L^2_{\tilde{x},x_1}}$, and $|U(t)|_{L^\infty_x}$, completing the proof of claim (3.6), and the main theorem. \square

REFERENCES

- [BHRZ] B. Barker, J. Humpherys, K. Rudd, and K. Zumbrun. *Stability of viscous shocks in isentropic gas dynamics*, to appear, Comm. Math. Phys.

- [Bra] Braslow, A.L., *A history of suction-type laminar-flow control with emphasis on flight research*, NSA History Division, Monographs in aerospace history, number 13 (1999).
- [BDG] T. J. Bridges, G. Derks, and G. Gottwald, *Stability and instability of solitary waves of the fifth-order KdV equation: a numerical framework*, Phys. D, 172(1-4):190–216, 2002.
- [Br1] L. Q. Brin. *Numerical testing of the stability of viscous shock waves*, PhD thesis, Indiana University, Bloomington, 1998.
- [Br2] L. Q. Brin. *Numerical testing of the stability of viscous shock waves*, Math. Comp., 70(235):1071–1088, 2001.
- [BrZ] L. Q. Brin and K. Zumbrun. *Analytically varying eigenvectors and the stability of viscous shock waves*, Mat. Contemp., 22:19–32, 2002, Seventh Workshop on Partial Differential Equations, Part I (Rio de Janeiro, 2001).
- [CHNZ] N. Costanzino, J. Humpherys, T. Nguyen, and K. Zumbrun, *Spectral stability of noncharacteristic boundary layers of isentropic Navier–Stokes equations*, to appear, Arch. Ration. Mech. Anal.
- [GR] Grenier, E. and Rousset, F., *Stability of one dimensional boundary layers by using Green’s functions*, Comm. Pure Appl. Math. 54 (2001), 1343–1385.
- [GMWZ1] O. Guès, G. Métivier, M. Williams, and K. Zumbrun. *Multidimensional viscous shocks I: degenerate symmetrizers and long time stability*, J. Amer. Math. Soc. 18 (2005), no. 1, 61–120.
- [GMWZ5] O. Guès, G. Métivier, M. Williams, and K. Zumbrun. *Existence and stability of noncharacteristic hyperbolic-parabolic boundary-layers*. Preprint, 2008.
- [GMWZ6] O. Guès, G. Métivier, M. Williams, and K. Zumbrun. *Viscous boundary value problems for symmetric systems with variable multiplicities* J. Differential Equations 244 (2008) 309–387.
- [HZ] P. Howard and K. Zumbrun, *Stability of undercompressive viscous shock waves*, in press, J. Differential Equations 225 (2006), no. 1, 308–360.
- [HLZ] J. Humpherys, O. Lafitte, and K. Zumbrun. *Stability of viscous shock profiles in the high Mach number limit*, (Preprint, 2007).
- [HLyZ1] Humpherys, J., Lyng, G., and Zumbrun, K., *Spectral stability of ideal-gas shock layers*, Preprint (2007).
- [HLyZ2] Humpherys, J., Lyng, G., and Zumbrun, K., *Multidimensional spectral stability of large-amplitude Navier-Stokes shocks*, in preparation.
- [HoZ1] D. Hoff and K. Zumbrun, *Multi-dimensional diffusion waves for the Navier-Stokes equations of compressible flow*, Indiana Univ. Math. J. 44 (1995), no. 2, 603–676.
- [HoZ2] D. Hoff and K. Zumbrun, *Pointwise decay estimates for multidimensional Navier-Stokes diffusion waves*, Z. Angew. Math. Phys. 48 (1997), no. 4, 597–614.
- [HuZ] J. Humpherys and K. Zumbrun. *An efficient shooting algorithm for Evans function calculations in large systems*, Physica D, 220(2):116–126, 2006.
- [KK] Y. Kagei and S. Kawashima *Stability of planar stationary solutions to the compressible Navier-Stokes equations in the half space*, Comm. Math. Phys. 266 (2006), 401–430.
- [KNZ] S. Kawashima, S. Nishibata, and P. Zhu, *Asymptotic stability of the stationary solution to the compressible Navier-Stokes equations in the half space*, Comm. Math. Phys. 240 (2003), no. 3, 483–500.
- [MaZ3] C. Mascia and K. Zumbrun. *Pointwise Green function bounds for shock profiles of systems with real viscosity*. Arch. Ration. Mech. Anal., 169(3):177–263, 2003.
- [MaZ4] C. Mascia and K. Zumbrun. *Stability of large-amplitude viscous shock profiles of hyperbolic-parabolic systems*. Arch. Ration. Mech. Anal., 172(1):93–131, 2004.
- [MN] Matsumura, A. and Nishihara, K., *Large-time behaviors of solutions to an inflow problem in the half space for a one-dimensional system of compressible viscous gas*, Comm. Math. Phys., 222 (2001), no. 3, 449–474.
- [MZ] Métivier, G. and Zumbrun, K., *Viscous Boundary Layers for Noncharacteristic Nonlinear Hyperbolic Problems*, Memoirs AMS, 826 (2005).
- [N2] T. Nguyen, *On asymptotic stability of noncharacteristic viscous boundary layers*, SIAM J. Math. Analysis, to appear.
- [NZ1] T. Nguyen and K. Zumbrun, *Long-time stability of large-amplitude noncharacteristic boundary layers for hyperbolic-parabolic systems*, J. Maths. Pures et Appliquées, to appear.
- [NZ2] T. Nguyen and K. Zumbrun, *Long-time stability of multi-dimensional noncharacteristic viscous boundary layers*, Preprint, 2008

- [RZ] M. Raoofi and K. Zumbrun, *Stability of undercompressive viscous shock profiles of hyperbolic-parabolic systems* Preprint, 2007.
- [S] H. Schlichting, *Boundary layer theory*, Translated by J. Kestin. 4th ed. McGraw-Hill Series in Mechanical Engineering. McGraw-Hill Book Co., Inc., New York, 1960.
- [SZ] Serre, D. and Zumbrun, K., *Boundary layer stability in real vanishing-viscosity limit*, Comm. Math. Phys. 221 (2001), no. 2, 267–292.
- [YZ] S. Yarahmadian and K. Zumbrun, *Pointwise Green function bounds and long-time stability of large-amplitude noncharacteristic boundary layers*, Preprint (2008).
- [Z2] K. Zumbrun. Multidimensional stability of planar viscous shock waves. In *Advances in the theory of shock waves*, volume 47 of *Progr. Nonlinear Differential Equations Appl.*, pages 307–516. Birkhäuser Boston, Boston, MA, 2001.
- [Z3] K. Zumbrun. Stability of large-amplitude shock waves of compressible Navier-Stokes equations. In *Handbook of mathematical fluid dynamics. Vol. III*, pages 311–533. North-Holland, Amsterdam, 2004. With an appendix by Helge Kristian Jenssen and Gregory Lyng.
- [Z4] K. Zumbrun. Planar stability criteria for viscous shock waves of systems with real viscosity. In *Hyperbolic systems of balance laws*, volume 1911 of *Lecture Notes in Math.*, pages 229–326. Springer, Berlin, 2007.

INSTITUT DE MATHÉMATIQUES DE JUSSIEU, UNIVERSITÉ PIERRE ET MARIE CURIE (PARIS 6)

E-mail address: `nguyent@math.jussieu.fr`

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, IN 47402

E-mail address: `kzumbrun@indiana.edu`