Centre de Mathématiques Laurent Schwartz
ECOLE POLYTECHNIQUE

SEMINAIRE

Equations aux Dérivées Partielles

2009-2010

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Supercritical nonlinear Schrödinger equations: Quasi-periodic solutions and almost global existence

Séminaire É. D. P. (2009-2010), Exposé nº XXXII, 18 p.

<http://sedp.cedram.org/item?id=SEDP_2009-2010____A32_0>

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Séminaire *Équations aux dérivées partielles*
Centre de mathématiques Laurent Schwartz, 2009-2010
Exposé n° XXXII, 1-18

SUPERCRITICAL NONLINEAR SCHRÖDINGER EQUATIONS: QUASI-PERIODIC SOLUTIONS AND ALMOST GLOBAL EXISTENCE

W.-M. Wang

Abstract. We construct time quasi-periodic solutions and prove almost global existence for the energy supercritical nonlinear Schrödinger equations on the torus in arbitrary dimensions. The main new ingredient is a *geometric* selection in the Fourier space. This method is applicable to other nonlinear equations.

1. Introduction and time quasi-periodic solutions

We consider the nonlinear Schrödinger equation on the d-torus $\mathbb{T}^d = [0, 2\pi)^d$:

$$i\frac{\partial}{\partial t}u = -\Delta u + |u|^{2p}u + H(x, u, \bar{u}) \qquad (p \geq 1, p \in \mathbb{N}), \tag{1.1}$$

with periodic boundary conditions: $u(t, x) = u(t, x + 2j\pi)$, $x \in [0, 2\pi)^d$ for all $j \in \mathbb{Z}^d$, where $H(x, u, \bar{u})$ is analytic in $(x, u, \bar{u})$ and has the expansion:

$$H(x, u, \bar{u}) = \sum_{m=1}^{\infty} \alpha_m(x)|u|^{2p+2m}u,$$

where α_m as a function on $\mathbb{R}^d$ is $(2\pi)^d$ periodic and real and analytic in a strip of width $\mathcal{O}(1)$ for all m. The integer p in (1.1) is *arbitrary.*

Using Fourier series, the solutions to the linear equation:

$$i\frac{\partial}{\partial t}u = -\Delta u$$

are linear combinations of eigenfunction solutions of the form:

$$e^{-ij^2t}e^{ij\cdot x}, \quad j \in \mathbb{Z}^d,$$

where $j^2 = |j|^2$ and $\cdot$ is the usual inner product. These solutions are periodic in time.

It is natural to investigate the persistence of this type of solutions in the presence of nonlinearity. Therefore we first construct time quasi-periodic solutions to (1.1). Using a related construction, we then prove almost global existence for a class of smooth solutions to Cauchy problems.

Specializing to $H = 0$, it is well known from [B1] that (1.1) is locally well-posed in H^s for

$$s > \max(0, \frac{1}{2}(d - \frac{2}{p})).$$

This is derived by linearizing about the flow of the Laplacian and proving L^{2p+2} estimates of its eigenfunction solutions (Strichartz estimates).

For $d \geq 3$ and sufficiently large p, the local theory is in H^s for $s > 1$. For example, in dimension 4, the quintic nonlinear Schrödinger equation is locally well-posed in $H^{3/2}$, where there is no conservation law. These equations are therefore energy *supercritical* as there is no a priori global existence from patching up local solutions, not even for small solutions, as (1.1) is non-dispersive, i. e., $\|u\|_\infty$ cannot tend to 0 as $t \to \infty$ on the torus $\mathbb{T}^d$.

One of the central points of the new theory in [W3, 4] (cf. also [W5]), is that it does not make use of conservation laws. Instead it analyzes the geometry of the characteristics. A consequence is that the results hold both in the focusing and the defocusing cases.

The nonlinear Fourier series

To proceed, let $u^{(0)}$ be a solution of finite number of frequencies, b frequencies, to the linear equation:

$$i\frac{\partial}{\partial t}u^{(0)} = -\Delta u^{(0)}, \tag{1.2}$$

$$u^{(0)}(t, x) = \sum_{k=1}^{b} a_k e^{-ij_k^2 t} e^{ij_k\cdot x}.$$

For the nonlinear construction, it is useful to add a dimension for each frequency in time and view $u^{(0)}$ as a function on $\mathbb{T}^b \times \mathbb{T}^d = \mathbb{T}^{b+d} \supset \mathbb{T}^d$. Henceforth $u^{(0)}$ adopts the form:

$$\begin{aligned} u^{(0)}(t, x) &= \sum_{k=1}^{b} a_k e^{-ij_k^2 t} e^{ij_k\cdot x} \\ :&= \sum_{k=1}^{b} \hat{u}(-e_k, j_k) e^{-i(e_k\cdot\omega^{(0)})t} e^{ij_k\cdot x}, \end{aligned}$$

where $e_k = (0, 0, ...1, .., 0) \in \mathbb{Z}^b$ is a unit vector, with the only non-zero component in the kth direction, $\omega^{(0)} = \{j_k^2\}_{k=1}^b$ $(j_k \neq 0)$ and $\hat{u}(-e_k, j_k) = a_k$. Therefore $u^{(0)}$ has Fourier support

$$\text{supp } \hat{u}^{(0)} = \{(-e_k, j_k), k = 1, ..., b\} \subset \mathbb{Z}^{b+d}, \tag{1.3}$$

where $j_k \neq j_{k'}$ if $k \neq k'$.

For the nonlinear equation (1.1), we seek quasi-periodic solutions with b frequencies in the form of a *nonlinear* space-time Fourier series:

$$u(t,x) = \sum_{(n,j)} \hat{u}(n,j) e^{in\cdot\omega t} e^{ij\cdot x}, \qquad (n,j) \in \mathbb{Z}^{b+d}, \tag{1.4}$$

with the frequency $\omega \in \mathbb{R}^b$ to be determined. This is the well-known frequency-amplitude modulation fundamental to nonlinear equations. We note that the corresponding linear solution $u^{(0)}$ has *fixed* frequency $\omega = \omega^{(0)} = \{j_k^2\}_{k=1}^b \in \mathbb{R}^b$, which are eigenvalues of the Laplacian.

In the Fourier space $\mathbb{Z}^{b+d}$, the support of the solution in the form (1.4) to the linear equation (1.2) and its complex conjugate are by definition, the bi-characteristics $\mathcal{C}$:

$$\mathcal{C} = \{(n,j) \in \mathbb{Z}^{b+d} | \pm n \cdot \omega^{(0)} + j^2 = 0\}. \tag{1.5}$$

We further define

$$\begin{aligned} \mathcal{C}^+ &= \{(n,j) | n \cdot \omega^{(0)} + j^2 = 0, j \neq 0\} \cup \{(n,0) | n \cdot \omega^{(0)} = 0, n_1 \leq 0\}, \\ \mathcal{C}^- &= \{(n,j) | -n \cdot \omega^{(0)} + j^2 = 0, j \neq 0\} \cup \{(n,0) | n \cdot \omega^{(0)} = 0, n_1 > 0\}. \end{aligned} \tag{1.6}$$

So we have

$$\mathcal{C}^+ \cap \mathcal{C}^- = \emptyset, \quad \mathcal{C}^+ \cup \mathcal{C}^- = \mathcal{C}.$$

$\mathcal{C}$ is the support of the solution to the linear equation (1.2) in the form (1.4) and is the resonant or singular set for the nonlinear equation (1.1). We consider $\mathcal{C}$ as the restriction to $\mathbb{Z}^{b+d}$ of the corresponding manifold on $\mathbb{R}^{b+d}$. So $\mathcal{C}$ is a manifold of singularities and not just isolated points. Moreover since $\omega^{(0)}$ is an integer vector, $\mathcal{C}$ not only lacks convexity but also has null directions in n. We introduce the novel *geometric* concept of *generic* linear solutions to overcome these major difficulties.

Assume $u^{(0)}$ is *generic*, satisfying the genericity conditions (i-iv) near the end of this section. Here it suffices to mention that the genericity conditions pertain entirely to the spatial Fourier support of $u^{(0)}$: $\{j_k\}_{k=1}^b \in (\mathbb{R}^d)^b$ and are determined by the $|u|^{2p}u$ term in (1.1) only. These conditions are explicit and moreover the non-generic set Ω is of codimension 1 in $(\mathbb{R}^d)^b$.

The first result is

Theorem 1. *Assume*

$$u^{(0)}(t,x) = \sum_{k=1}^b a_k e^{-ij_k^2 t} e^{ij_k\cdot x},$$

a solution to the linear equation (1.2) is generic and $a = \{a_k\} \in (0, \delta]^b = \mathcal{B}(0, \delta)$. *Then there exist* C, $c > 0$, *such that for all* $\epsilon \in (0, 1)$, *there exists* $\delta_0 > 0$ *and for all* $\delta \in (0, \delta_0)$ *a Cantor set* $\mathcal{G}$ *with*

$$\text{meas}\,\{\mathcal{G} \cap \mathcal{B}(0, \delta)\}/\delta^b \geq 1 - C\epsilon^c. \tag{1.7}$$

For all $a \in \mathcal{G}$, *there is a quasi-periodic solution of* b *frequencies to the nonlinear Schrödinger equation (1.1):*

$$u(t, x) = \sum a_k e^{-i\omega_k t} e^{ij_k \cdot x} + \mathcal{O}(\delta^3), \tag{1.8}$$

with basic frequencies $\omega = \{\omega_k\}$ *satisfying*

$$\omega_k = j_k^2 + \mathcal{O}(\delta^{2p}).$$

The remainder $\mathcal{O}(\delta^3)$ *is in an analytic norm about a strip of width* $\mathcal{O}(1)$ *on* $\mathbb{T}^{b+d}$.

Remarks. 1. The Theorem also holds when there is in addition an overall phase, $m \neq 0$, corresponding to adding m to the right side of (1.1).

2. When $d = p = 1$, the non-generic set $\Omega = \emptyset$. *All* $u^{(0)}$ are generic and only amplitude selection is necessary. This is the well understood scenario after writing (1.1) as an infinite dimensional Hamiltonian equation [KP].

3. For the geometry of the cubic nonlinearity in any dimension and a reinterpretation of the integrability of the cubic nonlinearity in one dimension, see the Appendix.

This is the first existence results on quasi-periodic solutions to the nonlinear Schrödinger equation (1.1) in arbitrary dimensions and for arbitrary nonlinearity p. The main obstacle to the construction was the complete violation of the Kolmogorov non-degeneracy condition or its weaker versions. This is because the perturbation is about a linear system (the Laplacian) and *not* an integrable nonlinear system.

Furthermore, as mentioned earlier, the existence of a global flow for these equations are generally unknown. The concept of *generic* linear solutions $u^{(0)}$ and the ensuing geometric excision enable us to overcome these fundamental difficulties.

Previously, quasi-periodic solutions were constructed using partial Birkhoff normal forms for the cubic nonlinear Schrödinger equation in dimensions one and two [B3, GXY, KP]. These algebraic normal form constructions use in an essential way the specifics of the resonance geometry generated by the cubic nonlinearity, see the Appendix.

Moreover in dimension two the normalizing transform depends on translation invariance and is unstable under small perturbations of the form H. So we have kept the perturbation H in (1.1) to underline our different approach. We note that the cubic nonlinear Schrödinger equation has global flow in dimensions one and two.

The spectral gap

To understand the substance of the geometric and amplitude excisions in Theorem 1, it is useful to take $H = 0$ and note the perpetual existence of periodic solutions:

$$u = ae^{-i(j^2+|a|^{2p})t}e^{ij\cdot x}$$

to (1.1) for all $j \in \mathbb{Z}^d$ and $a \in \mathbb{C}$. This perpetual existence is because of a spectral gap. The excision in the general, quasi-periodic case is precisely to ensure the persistence of this spectral gap.

We comment beforehand that this gap is in the space-time L^2 sense and is created by the nonlinearity itself and *not* by eigenvalue variation of a linear operator, which is difficult to achieve in two dimensions and above due to the degeneracy of the Laplacian. The spectral gap here is geometric in origin and is hence robust and stable under small perturbations. The core of the construction of this spectral gap is a projection or variable reduction argument. This is general and should be applicable to other equations.

Once we have the initial spectral gap, we achieve amplitude-frequency modulation and the scheme of Bourgain [B3, 6] becomes avaliable. In [B3, 6], Bourgain took care of the geometry in the convex spatial-j direction by using the notion of separated clusters; here we control the null time-n direction by introducing the concept of generic linear solutions. Combining the spatial and time directions, we are then able to treat the original equation (1.1).

The analysis part of this scheme is based on the Lyapunov-Schmidt method and was first introduced by Craig and Wayne [CW] to construct periodic solutions for the wave equation in one dimension. It was inspired by the multi-scale analysis of Fröhlich and Spencer [FS]. The construction was further developed by Bourgain to embrace the full generality of quasi-periodic solutions and in arbitrary dimensions [B3, 6]. More recently, Eliasson and Kuksin [EK] developed a KAM theory in the Schrödinger context.

All the above results, however, pertain to parameter dependent tangentially non-resonant equations by imposing Diophantine time frequencies. So in particular, there is no null n-direction.

The preceeding Theorem 1 uses a Newton scheme to construct a type of global solutions with precise control over the Fourier coefficients. Moreover we have the following high frequency (semi-classical) analog, which is new to the KAM context:

Corollary 1. *Set $H=0$ in (1.1). Let*

$$u^{(0)}(t,x)=\sum_{k=1}^{b}a_k e^{-ij_k^2 t}e^{ij_k\cdot x}$$

be a solution to the linear equation (1.2), $\{j_k\}_{k=1}^b\in[K\mathbb{Z}^d]^b$, $K\in\mathbb{N}^+$ and $a=\{a_k\}\in(0,1]^b=\mathcal{B}(0,1)$. Assume $\{\frac{j_k}{K}\}_{k=1}^b\in[\mathbb{Z}^d]^b$ is generic. There exist C, $c>0$, such that for all $\epsilon\in(0,1)$, there exists $K_0>0$ and for all $K>K_0$ a Cantor set $\mathcal{G}$ with

$$\text{meas }\{\mathcal{G}\cap\mathcal{B}(0,1)\}\geq 1-C\epsilon^c.$$

For all $a\in\mathcal{G}$, there is a quasi-periodic solution of b frequencies to the nonlinear Schrödinger equation (1.1):

$$u(t,x)=\sum a_k e^{-i\omega_k t}e^{ij_k\cdot x}+\mathcal{O}(1/K^2),$$

with basic frequencies $\omega=\{\omega_k\}$ satisfying

$$\omega_k=j_k^2+\mathcal{O}(1).$$

The remainder $\mathcal{O}(1/K^2)$ is in an analytic norm about a strip of width $\mathcal{O}(1)$ in t and $\mathcal{O}(1/K)$ in x on $\mathbb{T}^{b+d}$.

Remarks. 1. In fact one could have the measure going to 1 as $K\to\infty$ by making the estimate in the first step K dependent. Using Borel-Cantelli, this then implies that as $K\to\infty$, on a set of full measure in the unit ball, generically, the nonlinear torus converges to the corresponding linear torus of size 1. Similar statement holds as $\delta\to 0$, but the convergence is to the origin.

2. These are quantitative, global, $\mathbb{L}^2$ size 1 and large kinetic energy solutions, which could be relevant to the compressible Euler equations.

Before we indicate some ideas of the proof, we mention that there is in addition a linear component to this theory. It concerns L^p estimates of L^2 eigenfunctions of the Schrödinger operator [W2], cf. also [W1]. The main impetus is to develop fine analysis of the geometry of the level sets (energy surfaces) in order to have a sharp L^p theory. This is related to the aim of the nonlinear component here. The right notion of convexity or separation again plays an essential role.

A sketch of the proof of Theorem 1

We write (1.1) in the Fourier space, it becomes

$$\text{diag }(n\cdot\omega+j^2)\hat{u}+(\hat{u}*\hat{v})^{*p}*\hat{u}+\sum_{m=1}^{\infty}\hat{\alpha}_m*(\hat{u}*\hat{v})^{*(p+m)}*\hat{u}=0,\tag{1.9}$$

where $(n, j) \in \mathbb{Z}^{b+d}$, $\hat{v} = \hat{\bar{u}}$, $\omega \in \mathbb{R}^b$ is to be determined and

$$|\hat{\alpha}_m(\ell)| \le C' e^{-c'|\ell|} \quad (C', c' > 0)$$

for all m. From now on we work with (1.9), for simplicity we drop the hat and write u for $\hat{u}$ and v for $\hat{v}$ etc. We seek solutions close to the linear solution $u^{(0)}$ of b frequencies, supp $u^{(0)} = \{(-e_k, j_k), k = 1, ..., b\}$, with frequencies $\omega^{(0)} = \{j_k^2\}_{k=1}^b$ $(j_k \neq 0)$ and small amplitudes $a = \{a_k\}_{k=1}^b$ satisfying $\|a\| = \mathcal{O}(\delta) \ll 1$.

We complete (1.9) by writing the equation for the complex conjugate. So we have

$$\begin{cases} \text{diag } (n \cdot \omega + j^2)u + (u * v)^{*p} * u + \sum_{m=1}^{\infty} \alpha_m * (u * v)^{*(p+m)} * u = 0, \\ \text{diag } (-n \cdot \omega + j^2)v + (u * v)^{*p} * v + \sum_{m=1}^{\infty} \alpha_m * (u * v)^{*(p+m)} * v = 0, \end{cases} \tag{1.10}$$

By supp, we will always mean the Fourier support, so we write supp $u^{(0)}$ for supp $\hat{u}^{(0)}$ etc. Let

$$\mathcal{S} = \text{supp } u^{(0)} \cup \text{supp } \bar{u}^{(0)}. \tag{1.11}$$

Denote the left side of (1.10) by $F(u, v)$. We make a Lyapunov-Schmidt decomposition into the P-equations:

$$F(u, v)|_{\mathbb{Z}^{b+d} \setminus \mathcal{S}} = 0,$$

and the Q-equations:

$$F(u, v)|_{\mathcal{S}} = 0.$$

We seek solutions such that $u|_{\mathcal{S}} = u^{(0)}$. The P-equations are infinite dimensional and determine u in the complement of supp $u^{(0)}$; the Q-equations are $2b$ dimensional and determine the frequency $\omega = \{\omega_k\}_{k=1}^b$.

We use a Newton scheme to solve the P-equations, with $u^{(0)}$ as the initial approximation. The major difference with [CW, B3, 6], cf. also [EK] is that (1.10) is completely resonant and there are *no* parameters at this initial stage. The frequency $\omega^{(0)}$ is an *integer* in $\mathbb{Z}^b$. So we need to proceed differently and first extract a parameter from $u^{(0)}$ and then use the established analysis acheme.

First recall the formal Newton scheme: the first correction

$$\Delta \begin{pmatrix} u^{(1)} \\ v^{(1)} \end{pmatrix} = \begin{pmatrix} u^{(1)} \\ v^{(1)} \end{pmatrix} - \begin{pmatrix} u^{(0)} \\ v^{(0)} \end{pmatrix} = [F'(u^{(0)}, v^{(0)}]^{-1} F(u^{(0)}, v^{(0)}), \tag{1.12}$$

where $\begin{pmatrix} u^{(1)} \\ v^{(1)} \end{pmatrix}$ is the next approximation and $F'(u^{(0)}, v^{(0)})$ is the linearized operator on $\ell^2(\mathbb{Z}^{b+d}) \times \ell^2(\mathbb{Z}^{b+d})$

$$F' = D + A, \tag{1.13}$$

where

$$D = \begin{pmatrix} \text{diag } (n \cdot \omega + j^2) & 0 \\ 0 & \text{diag } (-n \cdot \omega + j^2) \end{pmatrix} \tag{1.14}$$

and

$$\begin{aligned} A &= \begin{pmatrix} (p+1)(u * v)^{*p} & p(u*v)^{*p-1} * u * u \\ p(u*v)^{*p-1} * v * v & (p+1)(u*v)^{*p} \end{pmatrix} + \mathcal{O}(\delta^{2p+2}) \quad (p \geq 1), \\ &= A_0 + \mathcal{O}(\delta^{2p+2}). \end{aligned} \tag{1.15}$$

with $\omega = \omega^{(0)}$, $u = u^{(0)}$ and $v = v^{(0)}$.

Since we look at small data, $\|A\| = \mathcal{O}(\delta^{2p}) \ll 1$ and the diagonal: $\pm n \cdot \omega + j^2$ are integer valued, using the Schur complement reduction [S1, 2], the spectrum of F' around 0 is equivalent to that of a reduced operator on $\ell^2(\mathcal{C})$, where $\mathcal{C}$ is defined in (1.5) and to $\mathcal{O}(\delta^{2p+2})$ it is the same as the spectrum of A_0 on $\ell^2(\mathcal{C})$.

The genericity conditions

To define generic $u^{(0)}$, we need to analyze the convolution matrix A_0. We use the notation introduced in (1.4). Let

$$\Gamma^{++} = \text{supp } [|u^{(0)}|^{2p}] = \{(\Delta n, \Delta j)\} \subset \mathbb{Z}^{b+d},$$

with

$$\text{supp } u^{(0)} = \{(-e_k, j_k)\}_{k=1}^{b}, j_k \neq j_{k'} \text{ if } k \neq k'.$$

So

$$\begin{aligned} &\Delta n = -\sum p_{kk'}(e_k - e_{k'}), \\ &\Delta j = \sum p_{kk'}(j_k - j_{k'}), \\ &p_{kk'} \geq 0, \sum p_{kk'} \leq p, \\ &\text{where all sums are for } k, k' = 1, ..., b; \end{aligned} \tag{1.16}$$

and

$$\Gamma^{+-} = \text{supp } [|u^{(0)}|^{2(p-1)}[u^{(0)}]^2] = \{(\Delta n, \Delta j)\},$$

where

$$\begin{aligned} &\Delta n = -\sum p_{kk'}(e_k - e_{k'}) - (e_\kappa + e_{\kappa'}), \\ &\Delta j = \sum p_{kk'}(j_k - j_{k'}) + (j_\kappa + j_{\kappa'}), \\ &p_{kk'} \geq 0, \sum p_{kk'} \leq p-1, \ k, k', \kappa, \kappa' = 1, ..., b; \end{aligned} \tag{1.17}$$

and

$$\Gamma^{-+} = \text{supp } [|u^{(0)}|^{2(p-1)}[v^{(0)}]^2], \tag{1.18}$$

with $v^{(0)} = \bar{u}^{(0)}$ as before.

Let

$$\mathcal{A} = \bigcup \Gamma_1 \cdot ... \cdot \Gamma_{d+2} := \bigcup \prod_{i=1}^{d+2} \Gamma_i, \tag{1.19}$$

where Γ_i is Γ^{++}, Γ^{+-} or Γ^{-+}, and the multiplication $\cdot$ stands for multiplication of Fourier series and the union is over all choices of Γ_i with the difference of the number of factors of Γ^{+-} and Γ^{-+} in the $(d+2)$- fold product to be at most 1.

Elements of $\mathcal{A}$ are of the form:

$$\mathcal{A} \ni (\Delta n, \Delta j) = \sum_{i \leq d+2} (\Delta n^{(i)}, \Delta j^{(i)}), \tag{1.20}$$

where $(\Delta n^{(i)}, \Delta j^{(i)}) \in \Gamma^{++}$, Γ^{+-} or Γ^{-+}. Since $\Gamma^{++} \ni (0,0)$, any finite product with at most $d+2$ factors (under the above restrictions) is in $\mathcal{A}$. For σ in $\mathcal{A}$, write $|\sigma|$ for its length.

Let σ be a set of $d+1$ elements in $\mathcal{A}$: $\sigma \subset \mathcal{A} \backslash (0,0)$, $|\sigma| = d+1$ and

$$\sigma \subset \prod_{d+1} \Gamma^{++}. \tag{1.21}$$

Define

$$J = |\Delta j|^2 + \Delta n \cdot \omega^{(0)}, \tag{1.22}$$

for $(\Delta n, \Delta j) \in \sigma$.

Otherwise let σ be a set of $d+2$ elements in $\mathcal{A} \backslash (0,0)$, such that σ does not contain a $d+1$ element subset

$$\sigma' \subset \prod_{d+1} \Gamma^{++}. \tag{1.23}$$

So

$$\sigma \cap (\prod_{d} \Gamma^{++}) \Gamma^{+-} \neq \emptyset, \tag{1.24}$$

or

$$\sigma \cap (\prod_{d} \Gamma^{++}) \Gamma^{-+} \neq \emptyset, \tag{1.25}$$

where $(\prod_d \Gamma^{++})\Gamma^{+-}$ denotes the $(d+1)$-fold product with one factor of Γ^{+-} with all possible order and similarly for $(\prod_d \Gamma^{++})\Gamma^{-+}$. From symmetry it suffices to consider σ such that (1.24) holds.

Define

$$\mathbb{A} = (\prod_{d} \Gamma^{++}))\Gamma^{+-}.$$

From (1.24)

$$\sigma \cap \mathbb{A} \neq \emptyset. \tag{1.26}$$

Let

$$(a, a') \in \sigma \cap \mathbb{A}. \tag{1.27}$$

On $\sigma \backslash (a, a')$ define

$$\tilde{\Delta} j = \Delta j - a', \tag{1.28}$$

$$\tilde{\Delta} n = \Delta n - a, \tag{1.29}$$

$$J = |\tilde{\Delta} j|^2 + 2a' \cdot \tilde{\Delta} j - \tilde{\Delta} n \cdot \omega^{(0)}, \tag{1.30}$$

if the difference in the number of factors in Γ^{+-} and Γ^{-+} in the sum in (1.20) is 1. If the difference is 0, then use the definition of J in (1.22).

Definition. $u^{(0)}$ of b frequencies is *generic* if its Fourier support $\{(-e_m, j_m)\}_{m=1}^{b} \subset \mathbb{Z}^{b+d}$, where $j_k \neq j_{k'}$ if $k \neq k'$ satisfies:

(i) For all $(\Delta n, \Delta j) \in \mathcal{A} \backslash (0, 0)$,

$$\Sigma_{\pm} = |\Delta j|^2 \pm \Delta n \cdot \omega^{(0)} \neq 0,$$

where $\omega^{(0)} = \{j_k^2\}_{k=1}^{b}$.

(ii) For any $\sigma \subset \mathcal{A}$ with $|\sigma| = d+1$ satisfying (1.21) and $\Delta j \neq 0$ identically for any $(\Delta n, \Delta j) \in \sigma$, assume $\nexists j_k$, $k = 1, ..., b$, such that

$$\{\Delta j\} \subseteq \{j_{k'} - j_k, k' = 1, ..., b, k \neq k'\}.$$

Then the $(d+1) \times (d+1)$ determinant

$$D = \det[[2\Delta j, J]] \neq 0,$$

where

$$[[2\Delta j, J]] = \begin{pmatrix} 2(\Delta j)_1^{(1)} & 2(\Delta j)_1^{(2)} & \cdots & 2(\Delta j)_1^{(d)} & J_1 \\ 2(\Delta j)_2^{(1)} & 2(\Delta j)_2^{(2)} & \cdots & 2(\Delta j)_2^{(d)} & J_2 \\ \vdots & \cdots & \vdots & \cdots & \cdots \\ 2(\Delta j)_{d+1}^{(1)} & 2(\Delta j)_{d+1}^{(2)} & \cdots & 2(\Delta j)_{d+1}^{(d)} & J_{d+1} \end{pmatrix},$$

and for each $i = 1$ to $d+1$, $(\Delta j)_i$ and J_i are as defined in (1.16, 1.22).

(iii) For any $\sigma \subset \mathcal{A}$ with $|\sigma| = d+2$ satisfying (1.26) and $\Delta j \neq 0$ identically for any $(\Delta n, \Delta j) \in \sigma$, assume $\nexists j_k, j_{k'}$, not necessarily distinct, such that

$$a' = -j_k - j_{k'}$$

or

$$\{\Delta j\} \subseteq \{j_{k''} - j_k, k'' = 1, ..., b, k'' \neq k\},$$

and if $\{\tilde{\Delta} j\} \neq \emptyset$ also

$$\{\tilde{\Delta} j\} \subseteq \{j_{k'} - j_{k''}, k'' = 1, ..., b, k'' \neq k'\}.$$

Then the $(d+1) \times (d+1)$ determinant

$$D = \det[[2\bar{\Delta} j, J]] \neq 0,$$

where $\bar{\Delta} j$ stands for Δj or $\tilde{\Delta} j$ as defined in (1.28) and J as defined in (1.22, 1.30) accordingly.

(iv) For all j_m, $m = 1, ..., b$ and all $(\Delta n, \Delta j) \in \Gamma^{++}$, the functions

$$f = \Delta n \cdot \omega^{(0)} + 2 j_m \cdot \Delta j + |\Delta j|^2 \neq 0,$$

if $(\Delta n, \Delta j) \neq (-e_{k'} + e_m, j_{k'} - j_m)$, for all $k' = 1, ..., b$.

Remarks. (i) prevents pure translations in time. This is a recurrent condition, which is almost necessary. (iv) is for the analysis in the Newton scheme. It always holds for the cubic nonlinearity in any dimension, see the Appendix.

Let $(j_k - j_{k'})$, $k \neq k'$ be a factor present in $(\Delta n, \Delta j) \in \mathcal{A} \backslash (0,0)$, cf. (1.19, 1.16, 1.17). We note that $\partial \Sigma_\pm$ and ∂f are not identically zero, where ∂ is the directional derivative in $(j_k - j_{k'})$. If there is no factor of form $(j_k - j_{k'})$, then Δj contains a factor of form $(j_\kappa + j_{\kappa'})$ and Δn, $(e_\kappa + e_{\kappa'})$. Taking derivative in the j_κ direction, we reach the same conclusion. Therefore $\{\Sigma_\pm = 0\}$ and $\{f = 0\}$ are sets of codimension 1 in $(\mathbb{R}^d)^b$.

If $\exists \bar{\Delta} j \notin \{m(j_{k'} - j_k), k' = 1, ..., b, k' \neq k, m = \pm 1, ..., \pm p(d+2)\} = I$, then when setting this $\bar{\Delta} j = 0$, the corresponding $J \neq 0$ identically, where we also used (i). So the quadratic J is not reducible and D is not identically 0.

If all $\bar{\Delta} j \in I$, then it follows from the restrictions on $\bar{\Delta} j$ and a' in (ii, iii) that D is not identically 0. Therefore $\{D = 0\}$ gives a set of codimension 1 in $(\mathbb{R}^d)^b$.

Combining the above deliberations, we obtain the following important a priori ingredient for the construction.

Lemma. *The non-generic set*

$$(\mathbb{R}^d)^b \supset \Omega := \{\Sigma_\pm = 0\} \cup \{D = 0\} \cup \{f = 0\}$$

has codimension 1.

Below we briefly indicate the considerations that led to (i-iv). For more details, see [W3, 4].

Origins of the genericity conditions

To implement the Newton scheme using (1.12), we need to bound A_0^{-1}. From previous considerations, it suffices to consider A_0 restricted to $\mathcal{C}$. For $u^{(0)}$ satisfying (i-iii), it can be shown that $A_0|_{\mathcal{C}} = \oplus \mathcal{A}_0$, where $\mathcal{A}_0$ are Töplitz matrices of sizes at most $(2b+d) \times (2b+d)$. This can be seen by considering connected sets on $\mathcal{C}$.

Assume $(n, j) \in \mathcal{C}^+$ is connected to $(n', j') \in \mathcal{C}$ by the convolution operator A_0, then $n' = n + \Delta n$ and $j' = j + \Delta j$, where $(\Delta n, \Delta j) \in \text{supp } (u^{(0)} * v^{(0)})^{*p}$, if $(n', j') \in \mathcal{C}^+$ and

$$\begin{cases} (n \cdot \omega^{(0)} + j^2) = 0, \\ (n + \Delta n) \cdot \omega^{(0)} + (j + \Delta j)^2 = 0; \end{cases} \tag{1.31}$$

and if $(n', j') \in \mathcal{C}^-$, then $(\Delta n, \Delta j) \in \text{supp } (u^{(0)} * v^{(0)})^{*p-1} * u^{(0)} * u^{(0)}$ and

$$\begin{cases} (n \cdot \omega^{(0)} + j^2) = 0, \\ -(n + \Delta n) \cdot \omega^{(0)} + (j + \Delta j)^2 = 0. \end{cases} \tag{1.32}$$

(Clearly the situation is similar if $(n, j) \in \mathcal{C}^-$.)

(1.31, 1.32) define a system of polynomial equations. For $u^{(0)}$ satisfying (i-iii), we show in sect. 2 that the largest connected set is of size at most $\max(2b, d+2) \le 2b+d$. The connected sets of sizes at most $2b$ result from translation invariance. The other connected sets are of sizes at most $d+2$. The translation invariant sets correspond to degeneracy and are in fact the reason for requiring the leading nonlinear $\mathcal{O}(\delta^{2p+1})$ term in (1.1) to be independent of x. The x dependence of the higher order terms do not matter as they are treated as perturbations.

The invertibility of A_0 is then ensured by making an initial excision in a as 0 is typically *not* an eigenvalue of a finite matrix. So $\|{F'}^{-1}\| \asymp \|A_0^{-1}\| \le \mathcal{O}(\delta^{-2p})$. Let

$$F_0(u^{(0)}, v^{(0)}) = \begin{pmatrix} (u^{(0)} * v^{(0)})^{*p} * u^{(0)} \\ (u^{(0)} * v^{(0)})^{*p} * v^{(0)} \end{pmatrix}. \tag{1.33}$$

By requiring

$$\text{supp } F_0(u^{(0)}, v^{(0)}) \cap \{\mathcal{C} \backslash \mathcal{S}\} = \emptyset,$$

which amounts to condition (iv), we obtain from (1.12)

$$\|\Delta u^{(1)}\| = \|\Delta v^{(1)}\| \le \mathcal{O}(\delta^3)$$

for small δ. Inserting this into the Q-equations, which determine ω, we achieve amplitude-frequency modulation:

$$\|\Delta \omega^{(1)}\| \asymp \mathcal{O}(\delta^{2p})$$

$$\left| \det\left(\frac{\partial \omega^{(1)}}{\partial a}\right) \right| \asymp \mathcal{O}(\delta^{2p-1}) > 0$$

ensuring transversality and moreover Diophantine $\omega^{(1)}$ on a set of a of positive measure. The tangentially non-resonant perturbation theory in [B3, 6] becomes available.

The first iteration is therefore the key step and is the core of the present construction. The main new ingredient is the fine analysis of resonances via systems of polynomial equations, which provides the geometry to achieve modulated Diophantine frequency as input for the analysis part of the construction.

2. Almost global existence

We now consider the Cauchy problems for the nonlinear Schrödinger equation (1.1). Following the custom, we set the higher order terms H to be zero. This is also because for small data, the construction below carries over verbatim to $H \neq 0$.

It is convenient to add a parameter and consider initial data of size one. So we have the following Cauchy problem on $\mathbb{T}^d$:

$$\begin{cases} i\frac{\partial}{\partial t}u = -\Delta u + \delta|u|^{2p}u \qquad (p \geq 1, p \in \mathbb{N} \textit{ arbitrary}), \\ u(t=0) = u_0, \end{cases} \tag{2.1}$$

with periodic boundary conditions: $u(t,x) = u(t, x + 2j\pi)$, $x \in [0, 2\pi)^d$ for all $j \in \mathbb{Z}^d$ and $\delta \neq 0$ is the parameter.

In Theorem 1, we went one step further and analyzed the resonance geometry created by the nonlinear term $|u|^{2p}u$. Relying on the geometric information afforded by this analysis and linearizing about a suitable approximate quasi-periodic solutions, we prove the following:

Theorem 2. *Let $u_0 = u_1 + u_2$. Assume u_1 is generic satisfying (I. i-iv.) and $\|u_2\| = \mathcal{O}(\delta)$, where $\|\cdot\|$ is an analytic norm (about a strip of width $\mathcal{O}(1)$) on $\mathbb{T}^d$. Let $\mathcal{B}(0,1) = (0,1]^b$, where b is the dimension of the Fourier support of u_1. Then for all $A > 1$, there exist an open set $\mathcal{A} \subset \mathcal{B}(0,1)$ of positive measure and $\delta_0 > 0$, such that for all $\delta \in (-\delta_0, \delta_0)$, if $\{|\hat{u}_1|\} \in \mathcal{A}$, then (2.1) has a unique solution $u(t)$ for $|t| \leq \delta^{-A}$ satisfying $u(t=0) = u_0$ and $\|u(t)\| \leq \|u_0\| + \mathcal{O}(\delta)$. Moreover, if $u_2 = 0$, then meas $\mathcal{A} \to 1$ as $\delta \to 0$.*

Remarks. 1. It is essential that the set $\mathcal{A}$ is *open*, as we will need to establish an open mapping theorem to analyze Cauchy problems.

2. The geometric excision is essentially necessary here, in view of the growth of Sobolev norms exhibited in [CKSTT] for the cubic nonlinear Schrödinger equation in dimension 2 and its likely relation with the non-generic codimension 1 set Ω.

3. The situation here is completely different from seeking global solutions for energy subcritical equations in H^s for $s < 1$, as H^s for $s < 1$ is locally controlled by H^1, cf. [B4].

For perturbations of the $1d$ cubic nonlinear Schrödinger equation ($d = p = 1$), similar stability results are proven in [Ba, B5]. For parameter dependent equations see [BG, B2]. The equations treated in [Ba, BG, B2, 5] are either $\mathbb{L}^2$ or essentially $\mathbb{L}^2$ well-posed. So there is a priori global existence.

The equations treated in Theorem 2 are of a different nature, there is no a priori global existence from conservation laws. In fact existence is obtained via explicit construction. This is possible because the known invariant measure for smooth flow is supported on KAM tori. Linearizing about approximate quasi-periodic solutions to prove existence and uniqueness for a time arbitrarily longer than local existence time is the main novelty in Theorem 2.

As for quasi-periodic solutions, we also have the following high frequency (semi-classical) counterpart, providing quantitative, almost global, $\mathbb{L}^2$ size 1 and large kinetic energy solutions to Cauchy problems. These solutions could be relevant to Cauchy problems for the compressible Euler equations.

Corollary 2. *Set $\delta = 1$ in (2.1). Assume u_0 is generic with frequencies $\{j_k\}_{k=1}^b \in [K\mathbb{Z}^d]^b$, $K \in \mathbb{N}^+$. Let $\mathcal{B}(0,1) = (0,1]^b$. Then for all $A > 1$, there exist an open set $\mathcal{A} \subset \mathcal{B}(0,1)$ of positive measure and $K_0 > 0$, such that for all $K > K_0$, if $\{|\hat{u}_0|\} \in \mathcal{A}$, then (2.1) has a unique solution $u(t)$ for $|t| \leq K^A$ satisfying $u(t=0) = u_0$ and $\|u(t)\| \leq \|u_0\| + \mathcal{O}(1/K^2)$, where $\|\cdot\|$ is an analytic norm (about a strip of width $\mathcal{O}(1/K)$) on $\mathbb{T}^d$, moreover meas $\mathcal{A} \to 1$ as $K \to \infty$.*

Remark. The previous related results are only up to time $\mathcal{O}(K)$, by solving the associated Hamilton-Jacobi equations *before* the arrival of caustics, cf. [Ca].

Theorem 2 and Corollary 2 provide the first existence and uniqueness results for Cauchy problems for energy supercritical nonlinear Schrödinger equations beyond the local one. Combining with the results in Theorem 1 and Corollary 1, the following rather general picture emerges:

Let u_0 be a generic solution to the linear Schrödinger equation with finite number of frequencies. Then on a set of Fourier coefficients of positive measure, for infinite time, there is a solution to the nonlinear equation "close" to u_0 and for "arbitrary long" time, there is a unique solution u to the Cauchy problem satisfying $u(t=0) = u_0$.

A sketch of the proof of Theorem 2

The proof uses the good geometry constructed in Theorem 1 and adapts an analysis scheme in [B2]. Writing the first equation in (2.1) as $F(u) = 0$, for u_0 satisfying the conditions in Theorem 2, we first find an approximate solution v such that

$$\begin{cases} F(v) = \mathcal{O}(\delta^r), & (2.3)\\ v(t=0) - u_0 = \mathcal{O}(\delta^r), & (2.4)\end{cases}$$

where $r > A > 1$.

This approximate solution v is quasi-periodic with $\mathcal{O}(|\log \delta|)$ number of basic frequencies. Moreover at $t = 0$, v has the decomposition:

$$v(t=0) = u_1 + v_2$$

with u_1 as in Theorem 2 and $\|v_2\| = \mathcal{O}(\delta)$.

The construction of v comprises of two steps. The first step is to construct approximate quasi-periodic solutions of $\mathcal{O}(|\log \delta|)$ number of basic frequencies with the initial approximation the solution to the linear equation

$$u^{(0)} = u_1 + u_2 \tag{2.5}$$

for all $u^{(0)}$ such that u_1 is generic, using a *finitely* iterated Newton scheme.

The excision in $\{|\hat{u}_1|\}$ is essentially the same as in Theorem 1, ensuring the existence of a spectral gap. The amplitudes $\{|\hat{u}_2|\}$ are arbitrary as long as $\delta\|u_2\| = \mathcal{O}(\delta^2)$ is smaller than the spectral gap. The constructed solution u satisfies

$$F(u) = \mathcal{O}(\delta^r). \tag{2.6}$$

Since the above construction is valid on a open set, using the spectral gap to establish an open mapping theorem, we show that for all

$$\tilde{u}_0 = u_1 + \tilde{u}_2$$

of $\mathcal{O}(|\log \delta|)$ number of frequencies, there is

$$u^{(0)} = u_1 + v_2$$

of the same number of frequencies such that the corresponding quasi-periodic solution v satisfies

$$\begin{cases} F(v) = \mathcal{O}(\delta^r), \\ v(t=0) = \tilde{u}_0. \end{cases} \tag{2.7}$$

We then differentiate (2.6) with respect to the Fourier coefficients of $u^{(0)}$ in (2.5) and prove that the solutions to the linearized equation is a basis which spans $\mathbb{L}^2(\mathbb{T}^d)$, after a further excision of $\{|\hat{u}_1|\}$. Schematically this could be understood as follows.

Assume u is a solution satisfying the equation $F(u) = 0$ and that it depends on a parameter a, then $\partial u/\partial a$ is a solution to the linearized equation:

$$F'(u)(\frac{\partial u}{\partial a}) = 0.$$

(Here u represents both u and $\bar{u}$.) The main difficulty here is to control the coupling of u and $\bar{u}$. The fact that $\{\frac{\partial u}{\partial a}\}$ is a basis is a direct consequence of the separation property of the resonance geometry entailed by generic u_1. This basis in turn allows us to control the flow linearized about the v in (2.7). Using Duhamel's formula and the linearized flow to control the difference of (2.1) and (2.7), we conclude the proof of Theorem 2.

3. Appendix: the cubic nonlinearity

For simplicity we write u for $u^{(0)}$ and ω for $\omega^{(0)}$, the solutions and frequencies of the linear equation. The symbols of convolution for the cubic nonlinearity are $|u|^2$, u^2 and $\bar{u}^2$. Assume $(n, j) \in \mathcal{C}^+$ ($(n, j) \in \mathcal{C}^-$ works similarly). In order that (n, j) is connected to $(n'j') \in \mathcal{C}$, it is necessary that either

(a) $[u * v](n, j; n'j') \neq 0$ or

(b) $[u * u](n, j; n'j') \neq 0$.

Case (a): Since

$$n \cdot \omega + j^2 = 0,$$
$$n' \cdot \omega + {j'}^2 = 0,$$

subtracting the two equations gives immediately

$$(j_k - j'_k) \cdot (j + j_k) = 0, \tag{3.1}$$

where $j_k, j_{k'} \in \mathbb{Z}^d$ ($k, k' = 1, ..., b$) and $j_k \neq j_{k'}$ if $k \neq k'$, are the b Fourier components of u.

Case (b): Since

$$n \cdot \omega + j^2 = 0,$$
$$-n' \cdot \omega + {j'}^2 = 0,$$

adding the two equations gives immediately

$$(j + j_k) \cdot (j + j_{k'}) = 0, \tag{3.2}$$

where $j_k, j_{k'} \in \mathbb{Z}^d$ ($k, k' = 1, ..., b$) and $j_k \neq j_{k'}$ if $k \neq k'$, are the b Fourier components of u.

(3.1, 3.2) are precisely the well known resonant set for the partial Birkhoff normal form transform in [B3, GXY, KP]. (3.1, 3.2) describe rectangular type of geometry.

$$\text{supp } F_0(u, v) \cap \{\mathcal{C} \backslash \mathcal{S}\} = \emptyset$$

for the cubic nonlinearity in any d. When $d = 1$, (3.1, 3.2) reduce to a finite set of $2b$ lattice points in $\mathbb{Z}$: $\{j = \pm j_k,\ k = 1, ..., b\}$ and $\Omega = \emptyset$ in the Theorems and Corollaries.

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