Centre de Mathématiques Laurent Schwartz
Ecole Polytechnique

SEMINAIRE

**Equations aux Dérivées Partielles**

**2009-2010**

Mario Pulvirenti

**On the Plasma-Charge problem**

*Séminaire É. D. P.* (2009-2010), Exposé n° XIV, 9 p.

<http://sedp.cedram.org/item?id=SEDP_2009-2010____A14_0>

U.M.R. 7640 du C.N.R.S.
F-91128 PALAISEAU CEDEX
Fax : 33 (0)1 69 33 49 49
Tél : 33 (0)1 69 33 49 99

Séminaire *Équations aux dérivées partielles*
Centre de mathématiques Laurent Schwartz, 2009-2010
Exposé n$^o$ XIV, 1-9

# ON THE PLASMA-CHARGE PROBLEM

MARIO PULVIRENTI

Contents



## 1. Introduction

This short report is a review on recent results of S. Caprino, C. Marchioro, E. Miot and the author on the initial value problem associated to the evolution of a continuous distribution of charges (plasma) in presence of a finite number of point charges. More precisely we are interested in the following modification of the usual Vlasov-Poisson equation

$$\partial_t f + v \cdot \nabla_x f + (E + F) \cdot \nabla_v f = 0 \tag{1.1}$$

where $f = f(x, v; t)$ is the probability distribution of the plasma particle, $(x, v) \in \mathbb{R}^{2d}$ denotes its position and velocity, $d$ is the dimension of the physical space, $E$ is the electric field generated by the plasma:

$$E(x, t) = \int dy \rho(y, t) K(x - y) \tag{1.2}$$

and

$$\rho(x, t) = \int dv f(x, v, t) \tag{1.3}$$

is the spatial density.

Moreover

$$F(x, t) = \sigma \sum_{\alpha=1}^{N} K(x - \xi_\alpha(t)) \tag{1.4}$$

is the electric field generated by the point charges whose positions and velocities are denoted by $\xi_\alpha(t), \eta_\alpha(t)$, $\alpha = 1 \dots N$. Eq.n (1. 2,3,4) is complemented by the ordinary differential equations governing the motion of the charges:

$$\dot{\xi}_\alpha = \eta_\alpha, \quad \dot{\eta}_\alpha = \sum_{\beta=1;\beta\neq\alpha}^{N} K(\xi_\alpha - \xi_\beta) + \sigma E(\xi_\alpha(t), t) \tag{1.5}$$

Here the point particles are assumed to have unitary charges of the same sign, $\sigma = \pm 1$ according whether the interaction between the plasma and the point particle is repulsive or attractive and

$$K(x) = -\nabla g(x) \tag{1.6}$$

where

$$g(x) = -\log |x|, \quad g(x) = \frac{1}{|x|} \tag{1.7}$$

for $d = 2$ and $d = 3$ respectively.

There are an increasing difficulty in solving the above initial value problem passing from d=2 to d=3 and from $\sigma = 1$ to $\sigma = -1$. The attractive case is particularly interesting from a physical point of view describing, for instance, a continuum of negative charges (electrons) with a finite number of positive point charges (nuclei or ions). Also the completely attractive case (gravitational interaction) which we do not discuss here, can be considered as well.

The easiest case ($d = 2$, $\sigma = 1$) was solved in [1]. Here we mostly focus our analysis in the three dimensional repulsive case with a single point charge, following [6]. The case in which we have a finite number of point charges requires a nontrivial, but essentially technical generalization. See [6] for details.

We conclude this introduction by formulating precisely the problem. We find convenient to do this in terms of ordinary differential problem.

**Problem 1.** Find a flow in $\mathbb{R}^6$, $(x, v) \to (X(t, x, v), V(t, x, v))$ such that

$$\begin{cases} \dot{X}(t, x, v) = V(t, x, v) \\ \dot{V}(t, x, v) = \frac{(X(t,x,v) - \xi(t))}{|X(t,x,v)-\xi(t)|^3} + E(X(t, x, v), t) \\ (X, V)(0, x, v) = (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3, \end{cases} \tag{1.8}$$

where the charge $(\xi(t), \eta(t))$ evolves according to the second-order ODE

$$\begin{cases} \dot{\xi}(t) = \eta(t), \quad \dot{\eta}(t) = E(t, \xi(t)) \\ (\xi, \eta)(0) = (\xi_0, \eta_0). \end{cases} \tag{1.9}$$

Here the electric field $E$ is related to the density $f$, which satisfies

$$f(X(t, x, v), V(t, x, v), t) = f_0(x, v),$$

via the identity

$$E(x, t) = \int_{\mathbb{R}^3} \frac{(x - y)}{|x - y|^3} \rho(y, t)\, dy, \quad \rho(x, t) = \int_{\mathbb{R}^3} f(x, v, t)\, dv.$$

## 2. The classical Vlasov-Poisson problem

The Vlasov-Poisson equation in dimension two, has been solved since many years (see for instance [7], [3] and references quoted therein) We sketch the idea.

We start by considering an initial probability distibution $f_0$ compactly supported in velocity. Let $V_0$ be the maximal velocity at time 0, namely

$$f_0(x, v) = 0$$

if $|v| > V_0$. Then, denoting by $\bar{V}(t)$ the radius of the minimal sphere containing the support in velocity of $f(t)$, we have

$$\frac{d}{dt}\bar{V}(t) \leq \|E(t)\|_{L^\infty}.$$

On the other hand

$$\|E(t)\|_{L^\infty} \leq C \int \frac{\rho(y)}{|x-y|^{d-1}}.$$

Splitting the integral into the two regions $|x-y| < M$ and its complement, we readly arrive, after having optimized on $M$, to the estimate

$$\|E(t)\|_{L^\infty} \leq C \|\rho(t)\|_{L^\infty}^{1-\frac{1}{d}}.$$

Finally, using that

$$\|\rho(t)\|_{L^\infty} \leq C\bar{V}(t)^d$$

(remind that $\|f(t)\|_{L^\infty} = \|f(0)\|_{L^\infty}$) we arrive to

$$\frac{d}{dt}\bar{V}(t) \leq C\bar{V}(t)^{d-1}. \tag{2.1}$$

Eq.n (2.1) yields a global in time control of the spatial density in dimension two only. The control of the density is enough to get an existence and uniqueness of the solution to the Vlasov-Poisson problem in terms of characteristics.

In dimension three Eq.n (2.1) can be improved by using the energy conservation. Indeed the quantity

$$H(f(t)) = \frac{1}{2}\int dxdvv^2 f(x,v,t) + \frac{1}{2}\int dxdy \frac{\rho(x,t)\rho(y,t)}{|x-y|} = H_0 \tag{2.2}$$

is constant in time. It implies $\|\rho\|_{L^{5/3}} \leq K_1$. In fact

$$\rho \leq \int_{|v|<M} f + \frac{1}{M^2}\int_{|v|\geq M} dvv^2 f(t). \leq M^3\|f\|_{L^\infty} + \frac{\int dvv^2 f}{M^2}$$

Optimizing in $M$ we get the bound.

Moreover

$$\|E(t)\|_{L^\infty} \leq \int dy \frac{\rho(y,t)}{|x-y|^2} \leq \int_{|x-y|\leq M} dy \frac{\rho(y,t)}{|x-y|^2} + \int_{|x-y|>M} dy \frac{\rho(y,t)}{|x-y|^2}.$$

Using Hölder inequality and the 5/3 bound, optimizing on $M$, we conclude that

$$\|E(t)\|_{L^\infty} \leq C\|\rho\|_{L^\infty}^{4/9} \leq K_2 \bar{V}^{4/3}.$$

In conclusion:

$$\frac{d}{dt}\bar{V}(t) \leq C\bar{V}(t)^{4/3} \tag{2.3}$$

which is better than Eq.n (2.1) but still not enough to conclude.

In the nineties the three-dimensional Vlasov-Poisson problem was solved by Pfaffelmoser [8] (see also [9] [10] and [3]) by controlling the characteristics (Lagrangian point of view) and by Lions and Perthame [5], by using the equation to control the moments of $f$ (Eulerian point of view).

Here we find convenient to work with the trajectories so that we recall the basic ideas in [8] to see how to generalize that approach to the present context.

The basic idea in [8] is that the time average of the electric field is better than its maximum. Then, fixed an arbitrary time $T$, we split the time interval according to the following partition

$$(0,T] = \cup_{i=1}^{n-1}(t_{i-1},t_i],$$

where

$$|t_i - t_{i-1}| = \Delta T = P^{-1}.$$

Here

$$P = P(T) = \sup_{t\in(0,T]} V(t) + C,$$

for some large constant $C$. By Liouville theorem

$$\int_{t_{i-1}}^{t_i} dt|E(X(t)| \le \int_{t_{i-1}}^{t_i} dt \int dy \frac{\rho(y,t)}{|X(t)-y|^2}$$
$$= \int_{t_{i-1}}^{t_i} dt \int dy \int dw \frac{f(y,w;t_{i-1})}{|X(t)-Y(t)|^2}.$$

Then, setting $R = P^{3/4}$, we observe that, if $|w| \le 2R$ or $|v-w| \le 2R$, the integration yields a good bound. Indeed using the same argument yielding the $V(t)^{4/3}$ bound (eq.n (2.3)), we conclude that

$$\int_{t_{i-1}}^{t_i} \int dy \int dw (\chi(|w| \le 2R) + \chi(|v-w| \le 2R)) \frac{f(y,w;t_{i-1})}{|X(t)-Y(t)|^2} \le CP$$

Here $\chi(A)$ denotes the indicator of the event $A$.

Otherwise, if $|w| > 2R$ and $|v-w| > 2R$ we invoke the following stability property

$$|V(t) - W(t)| \ge |v-w| - 2P^{4/3}\Delta T \ge 2R - R = R \tag{2.4}$$

In this situation we estimate almost explicitly ([10], [9], [6] for instance)

$$\int_{t_{i-1}}^{t_i} \frac{dt}{|X(t)-Y(t)|^2} \le \frac{1}{\ell R} \tag{2.5}$$

being $\ell$ is the minimal distance between $X(t)$ and $Y(t)$ in this time interval. We assume that $\ell > P^{-2}$. Note that if $\ell \le P^{-2}$, the integration $dxdv$ gives us a linear bound on $P$ and this is straightforward. Finally

$$\int_{t_{i-1}}^{t_i} |E(X(t)| \le \frac{1}{\ell R} \int dy \int_{|w|>R} dw f(y,w;t_{i-1}) \tag{2.6}$$
$$\le \frac{P^2}{R^3 \Delta T}\Delta T = P^{3/4}\Delta T.$$

The physical meaning of this step is transparent. If two trajectories have large relative velocity they stay close each other (when the interaction is strong) for a very short time and the scattering angle is very small.

The control of $P$ allows us to state

**Theorem 1** ([8], [9], [10], [3])**.** *Given a probability distribution $f_0 \in L^\infty$, compactly supported in $x, v$, there exists a unique probability distribution $f(x,v;t)$ such that $\rho \in L^\infty$ and*

$$f(X(t), V(t), t) = f_0(x,v), \tag{2.7}$$

*where*

$$\dot{X} = V, \quad \dot{V} = E(X,t). \tag{2.8}$$

**Remark.** We are working in a minimal regularity setting. The unique weak solution we obtain in this way is such that $\rho \in L^\infty([0,T]; L^\infty(\mathbb{R}^3))$ for which the characteristic system (2.8) can be uniquely solved. Assuming further regularity on the initial datum $f_0$ we can obtain easily classical solutions as well.

## 3. The plasma-charge problem

Caprino and Marchioro solved the two-dimensional version of Problem 1 in case of repulsive interaction between the charge and the plasma [1]. The key idea was the introduction of the energy of a single plasma trajectory (see definition (3.2) below) which controls the motion of a plasma particle and prevents its approach to the point charge. Combining this with static estimates on the electric field, one can prove that there exists a unique solution to our problem.

The three-dimensional problem is in fact much more involved because static (a priori) estimates are not enough, as we have seen from the above analysis of the usual Vlasov-Poisson equation. When a charge is present, the situation described in section 2 changes drastically. Indeed because of the presence of a strong external field, that produced by the point charge, the stability property (2.4) is lost and new ideas are needed.

Before discussing the strategy of the proof, we first formulate the result.

We assume $f_0 \in L^\infty$ compactly supported in $(x, v)$ and such that

$$\inf |x - \xi_0| \geq \delta > 0.$$

**Theorem 2** ([6])**.** *Under the above hypotheses, there exists a unique solution to Problem 1.*

The main ideas of the proof are the following. As for the usual Vlasov-Poisson problem we need to control both large velocity and the distance between the plasma trajectories and the point charge (from below) which is initially strictly positive. This can be done by proving that the function $h$ (defined in (3.2)) stays bounded. At this point the existence and uniqueness proof is rather straightforward and we omit the details.

First we note that the energy:

$$H = \frac{1}{2}\int dxdv|v|^2 f + \frac{1}{2}|\eta|^2 + \frac{1}{2}\int dxdy\frac{\rho(x)\rho(y)}{|x-y|} + \int dx\frac{\rho(x)}{|x-\xi|} \tag{3.1}$$

is formally constant. Therefore $|\eta|$ bounded.

Following [1] we define:

$$h(x, v; t) = \frac{1}{2}|v - \eta(t)|^2 + \frac{1}{|x - \xi(t)|} + C \tag{3.2}$$

$C > 0$ some large constant.

Differentiating along the trajectory we find:

$$\frac{d}{dt}h(X(t), V(t); t) = (V(t) - \eta(t)) \cdot (E(X(t)) - E(\xi(t))) \tag{3.3}$$

Note that the singular field does not appear. As a consequence $h$ is stable (for small times we expect a little change) while $V$ is not. As a consequence we find convenient to work with $h$. By (3.3) we get

$$|\frac{d}{dt}\sqrt{h}(X,V;t)| \leq |E(X)| + |E(\xi)| \tag{3.4}$$

Next we define

$$Q = \sup\{\sqrt{h}(X(t),V(t);t)|(x,v) \in suppf(t_0), t \in [0,T]\}. \tag{3.5}$$

Splitting

$$(0,T] = \cup_{i=1}^{n-1}(t_{i-1},t_i],$$

where

$$|t_i - t_{i-1}| = \Delta T = CQ^{-1},$$

we define

$$Q_i = \sup\{\sqrt{h}(X,V;t)|(x,v) \in suppf(t_{i-1}), t \in (t_{i-1},t_i)\} \tag{3.6}$$

Our goal is to prove that

$$Q_i \leq Q_{i-1} + CQ\Delta T \tag{3.7}$$

where $C$ is a constant depending on the conserved quantities only. Indeed by Eq.n (3.7) we obtain

$$Q \leq Q_0 + CQT. \tag{3.8}$$

Thus we control $Q$ for $T$ small. But we can iterate because the smallness of $T$ depends on conserved quantities only.

Furthermore, to prove (3.7) by

$$|\frac{d}{dt}\sqrt{h}(X,V;t)| \leq |E(X)| + |E(\xi)| \tag{3.9}$$

we have to control

$$\int_{t_{i-1}}^{t_i} |E(X(t)| \leq CQ\Delta T; \qquad \int_{t_{i-1}}^{t_i} |E(\xi(t)| \leq CQ\Delta T \tag{3.10}$$

Setting $R_i = Q_i^{3/4}$ the second integral in (3.10) is bounded by

$$\int_{t_{i-1}}^{t_i} \int_{\sqrt{h}(y,w,t_{i-1}) \leq R_i} dydwf(y,w,t_{i-1}) \frac{1}{|Y(t)-\xi(t)|^2} \tag{3.11}$$
$$+ \int_{t_{i-1}}^{t_i} \int_{\sqrt{h}(y,w,t_{i-1}) > R_i} dydwf(y,w,t_{i-1}) \frac{1}{|Y(t)-\xi(t)|^2}$$

The first integral in Eq.n (3.11) is correctly bounded by the usual $V^{4/3}$ argument.

For the second we can show that

$$\int_{t_{i-1}}^{t_i} \frac{1}{|Y(t)-\xi(t)|^2} \leq CQ_i \tag{3.12}$$

By (3.12) the second integral is bounded by

$$C\frac{Q_i}{R_i^2} \int dy \int dwf(y,w,t_{i-1})h(y,w,t_{i-1}) \leq CQ\Delta T. \tag{3.13}$$

Finally to show (3.12) we define

$$\ell = |Y - \xi|$$

and compute ($\dot{Y} = W$)

$$\dot{\ell} = (W - \eta) \cdot \frac{(Y - \xi)}{|Y - \xi|}$$

$$\ddot{\ell} \geq \frac{1}{\ell^2} - CQ_i^{4/3}$$

$$\int_{t_{i-1}}^{t_i} \frac{1}{\ell^2} \leq \dot{\ell}(t_i) - \dot{\ell}(t_{i-1}) + CQ_i^{4/3}\Delta T \leq CQ_i.$$

It remains to control the first integral in (3.10):

$$\int_{t_{i-1}}^{t_i} |E(X(t)| \leq \int_{t_{i-1}}^{t_i} \int dydwf(y, w, t_{i-1}) \frac{1}{|X(t) - Y(t)|^2}$$

We have different situations:

1- $X$ is close to $\xi$: scattering plasma-charge i.e. $\inf |X(t) - \xi(t)| \leq \delta_i$

2- Both $X$ and $Y$ are far from $\xi$: scattering plasma-plasma i.e. $\inf |X(t) - \xi(t)| > \delta_i$ and $\inf |Y(t) - \xi(t)| > \delta_i$.

3- $X$ is far from $\xi$ but $Y$ is close to $\xi$: scattering plasma-charge i.e $\inf |X(t) - \xi(t)| > \delta_i$ and $\inf |Y(t) - \xi(t)| \leq \delta_i$.

Here we choose $\delta_i = Q_i^{-\frac{7}{8}}$.

For point 1. we can show that the set

$$J = \{t \in (t_{i-1}, t_i) | |X - \xi| \leq \delta_i\} \tag{3.14}$$

is connected and its measure $|J|$ can be bounded by

$$|J| \leq Q_i^{-15/8}. \tag{3.15}$$

The idea of the proof is the following. If $X$ is a very energetic trajectory i.e. $\sqrt{h}(X) \approx Q_i$. Then $X$ spends a very short time close to $\xi$ and then goes away. To show this we can use Virial Theorem, namely we compute the second derivative of $I = \frac{1}{2}|X - \xi|^2$.

By using (3.14) and (3.15) we get

$$\int_J |E(X(t)|dt \leq CQ_i^{4/3} Q_i^{-15/8} \leq CQ\Delta T$$

For point 2: we see that the field generated by $\xi$ outside a sphere of radius $\delta_i$ times $\Delta T$ is bounded by

$$\frac{1}{\delta_i^2}\Delta T = CQ_i^{7/4} Q_i^{-1} = CQ_i^{3/4}.$$

This is good for the stability condition (2.4) . Therefore we can proceed as explained above ([8], [9], [10], [3]).

For point 3, $X$ and $Y$ are relatively far each other.

$$|X - Y| \geq |X - \xi| - |Y - \xi| \geq \frac{\delta_i}{2} = CQ_i^{-\frac{7}{8}}.$$

We can assume that $Y$ is moderately energetic i.e. $\sqrt{h}(Y) \geq Q_i^{3/4}$. Otherwise the contribution can be controlled by a straightforward phase-space integration.

Then, by the same argument as in (3.14) and (3.15)

$$|J| \leq Q_i^{-13/8}. \tag{3.16}$$

Therefore:

$$\int_J |E(X(t)|dt \le CQ_i^{7/4} \int_J \int_{\sqrt{h}(y,w)>Q_i^{3/4}} dydwf(y,w,t_{i-1}) \tag{3.17}$$
$$\le CQ_i^{7/4} Q_i^{-13/8} Q_i^{-3/2} \le CQ_i^{-11/8} \le CQ\Delta T.$$

This concludes our quick description of the result.

## 4. the 2-D attractive case

The problem can be formulated in the following way.

**Problem 2.** Find a flow in $\mathbb{R}^4$, $(x,v) \to (X(t,x,v), V(t,x,v))$ such that

$$\begin{cases} \dot{X}(t,x,v) = V(t,x,v) \\ \dot{V}(t,x,v) = -\frac{(X(t,x,v)-\xi(t))}{|X(t,x,v)-\xi(t)|^2} + E(t, X(t,x,v)) \\ (X,V)(0,x,v) = (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3, \end{cases} \tag{4.1}$$

where the charge $(\xi(t), \eta(t))$ evolves according to the second-order ODE

$$\begin{cases} \dot{\xi}(t) = \eta(t), \quad \dot{\eta}(t) = -E(t, \xi(t)) \\ (\xi, \eta)(0) = (\xi_0, \eta_0). \end{cases} \tag{4.2}$$

Here the electric field $E$ is related to the density $f$, which satisfies

$$f(X(t,x,v), V(t,x,v), t) = f_0(x,v),$$

via the identity

$$E(t,x) = \int_{\mathbb{R}^2} \frac{(x-y)}{|x-y|^2} \rho(t,y)\, dy, \quad \rho(t,x) = \int_{\mathbb{R}^2} f(t,x,v)\, dv.$$

Assuming $f_0 \in L^\infty$ compactly supported in $x, v$ and such that

$$\inf |x - \xi_0| \ge \delta > 0,$$

then

**Theorem 3** ([2]). *Under the above hypotheses, there exists a (possibly not unique) solution to Problem 2.*

We give a very rough idea of the proof. Once more we introduce the function:

$$h(x,v,t) = \frac{1}{2}|v - \eta(t)|^2 + \ln |x - \xi(t)|. \tag{4.3}$$

and

$$\mathcal{H}(t) = \sup_{s\in[0,t]} \sup_{(x,v)\in S_0} |h(X_(t;x,v), V(t;x,v), t)| + C$$

where $C > 1$ is a suitable large constant and

$$\mathcal{H} := \mathcal{H}(T).$$

Assume that:

$$\text{supp}(f_0) := S_0 \subset \left\{(x,v) \in \mathbb{R}^2 \times \mathbb{R}^2 \;:\; \alpha_0 \le |x - \xi| \le \alpha_1,\ |v| \le \nu_0 \right\}. \tag{4.4}$$

for some positive constants $\alpha_0, \alpha_1, \nu_0$.

We prove Theorem 2 along the following steps. First we prove the static estimates

$$\|E(t)\|_{L^\infty} \le C\sqrt{\mathcal{H}(t)}. \tag{4.5}$$

and

$$|E(x,t) - E(y,t)| \le C\varphi(|x-y|)\big(\mathcal{H}(t) + \ln_- |x-y|\big), \tag{4.6}$$

where

$$\varphi(r) = r(\ln_- r + 1), \qquad \ln_- r = -\ln r \chi(r < 1).$$

By using (4.5) (4.6) and other considerations, differentiating $h$ along the trajectories, one can prove that

$$\mathcal{H}(t) \le C. \tag{4.7}$$

Note that, being $h$ not positive, we do not get a uniform bound neither on the maximal velocity nor on the minimal distance to the point charge of the generic plasma trajectory, Nevertheless the bound (4.7) is enough to show that almost all trajectory do not reach the point charge and that the maximal velocity and the inverse of the minimal distance to the point charge can grow at most logarithmically with this distance. This is enough to show the existence of a (possibly not unique) weak solution and, by virtue of (4.5) and (4.6), also the existence and uniqueness for the ordinary differential system (4.1) (4.2).

## References

[1] S. Caprino and C. Marchioro, *On the plasma-charge model*, Kinetic and Related Models **3** (2010), no. 2, 241-254 (2010).

[2] S. Caprino and C. Marchioro, , E. Miot and M. Pulvirenti, *In preparation*

[3] T. Glassey, *The Cauchy problem in kinetic theory*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1996.

[4] E. Horst, *On the asymptotic growth of the solutions of the Vlasov-Poisson system*, Math. Methods Appl. Sci. **16** (1993), no. 2, 75-86.

[5] P.-L. Lions and B. Perthame, *Propagation of moments and regularity for the* 3-*dimensional Vlasov-Poisson system*, Invent. Math. **105** (1991), no. 2, 415-430.

[6] C. Marchioro, E. Miot and M. Pulvirenti, *The Cauchy problem for the 3-D Vlasov-Poisson system with point charges*, preprint (2010)

[7] S. Okabe and T. Ukai, *On classical solutions in the large in time of the two-dimensional Vlasov equation*, Osaka J. Math. **15** (1978), 245-261.

[8] K. Pfaffelmoser, *Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data*, Jour. Diff. Eq. **95** (1992), 281-303.

[9] J. Schaeffer, *Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions*, Comm. Partial Differential Equations **16** (1991), no. 8-9, 1313-1335.

[10] S. Wollman, *Global in time solution to the three-dimensional Vlasov-Poisson system*, Jour. Math. Anal. Appl. **176 (1)** (1996), 76-81.

Dipartimento di Matematica "G. Castelnuovo", Università di Roma "La Sapienza", Italy

*E-mail address*: pulvirenti@mat.uniroma1.it