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Abstract

We formulate two results on controllability properties of the 3D Navier–Stokes (NS) system. They concern the approximate controllability and exact controllability in finite-dimensional projections of the problem in question. As a consequence, we obtain the existence of a strong solution of the Cauchy problem for the 3D NS system with an arbitrary initial function and a large class of right-hand sides. We also discuss some qualitative properties of admissible weak solutions for randomly forced NS equations.

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1 Main results

Let $D \subset \mathbb{R}^3$ be a bounded domain with $C^2$-smooth boundary $\partial D$. Consider 3D Navier–Stokes (NS) equations

$$
\dot{u} + (u, \nabla)u - \nu \Delta u + \nabla p = f(t, x), \quad \text{div } u = 0, \quad x \in D,
$$

where $u = (u_1, u_2, u_3)$ and $p$ are unknown velocity and pressure fields, $\nu > 0$ is the viscosity, and $f(t, x)$ is an external force. We introduce the spaces

$$
H = \{ u \in L^2(D, \mathbb{R}^3) : \text{div } u = 0 \text{ in } D, \langle u, n \rangle|_{\partial D} = 0 \},
$$

$$
V = H^1_0(D, \mathbb{R}^3) \cap H, \quad U = H^2(D, \mathbb{R}^3) \cap V,
$$

where $n$ stands for the outward unit normal to $\partial D$ and $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^3$. It is well known (e.g., see [Tem79]) that $H$ is a closed vector
space in $L^2(D, \mathbb{R}^3)$, and we denote by $\Pi$ the orthogonal projection in $L^2(D, \mathbb{R}^3)$ onto $H$. Equations (1) are equivalent to the following evolution equation in $H$:

$$u + \nu L u + B(u) = f.$$  

(2)

Here $L = -\Pi \Delta$, $B(u) = B(u, u)$, $B(u, v) = \Pi\{(u, \nabla)v\}$, and we use the same notation for the right-hand side of (1) and its projection to $H$. Equation (2) is supplemented with the initial condition

$$u(0) = u_0,$$  

(3)

where $u_0 \in V$. Let us assume that the right-hand side of (2) is represented in the form

$$f(t, x) = h(t, x) + \eta(t, x),$$  

(4)

where $h \in L^2_{\text{loc}}(\mathbb{R}^+, H)$ is a given function and $\eta$ is a control taking on values in a finite-dimensional subspace. To formulate the main results, we introduce some notation.

Define the space $\mathcal{X}_T = C(J_T, V) \cap L^2(J_T, U)$, where $J_T = [0, T]$. For any $T > 0$, $h \in L^2(J_T, H)$, and $u_0 \in V$, we denote by $\Theta_T(h, u_0)$ the set of functions $\eta \in L^2(J_T, H)$ for which problem (2) – (4) has a unique solution $u \in \mathcal{X}_T$. It follows from the implicit function theorem that

$$\mathcal{D}_T := \{(u_0, \eta) \in V \times L^2(J_T, H) : \eta \in \Theta_T(h, u_0)\}$$  

(5)

is an open subset of $V \times L^2(J_T, H)$, and the operator $\mathcal{R}$ taking $(u_0, \eta) \in \mathcal{D}_T$ to the solution $u \in \mathcal{X}_T$ of (2) – (4) is locally Lipschitz continuous. We denote by $\mathcal{R}_t$ the restriction of $\mathcal{R}$ to the time $t \in J_T$. Let $E \subset U$ and $F \subset H$ be finite-dimensional subspaces, let $\mathcal{P}_F : H \rightarrow H$ be the orthogonal projection onto $F$, and let $X \subset L^2(J_T, E)$ be a vector space, not necessarily closed. We denote by $B_F(R)$ the closed ball in $F$ of radius $R$ centred at origin.

**Definition 1.** Equations (2), (4) with $\eta \in X$ are said to be **approximately controllable in time $T$** if for any $u_0, \hat{u} \in V$ and any $\varepsilon > 0$ there is a control $\eta \in \Theta_T(h, u_0) \cap X$ such that

$$\|\mathcal{R}_T(u_0, \eta) - \hat{u}\|_V < \varepsilon.$$  

(6)

Equations (2), (4) with $\eta \in X$ are said to be **$F$-controllable in time $T$** if for any $u_0 \in V$ and $\hat{u} \in F$ there is $\eta \in \Theta_T(h, u_0) \cap X$ such that

$$\mathcal{P}_F \mathcal{R}_T(u_0, \eta) = \hat{u}.$$  

(7)

Equations (2), (4) with $\eta \in X$ are said to be **solidly $F$-controllable in time $T$** if for any $u_0 \in V$ and any $R > 0$ there is a constant $\delta > 0$ and a compact set $C$ in a finite-dimensional subspace $Y \subset X$ such that $C \subset \Theta_T(h, u_0)$, and for any continuous mapping $\Phi : C \rightarrow F$ satisfying the inequality

$$\sup_{\eta \in C} \|\Phi(\eta) - \mathcal{P}_F \mathcal{R}_T(u_0, \eta)\|_F \leq \delta,$$  

(8)

we have $\Phi(C) \supset B_F(R)$.

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For any finite-dimensional subspace \( G \subset U \), we denote by \( \mathcal{F}(G) \) the largest vector space \( G_1 \subset U \) such that any element \( \eta_1 \in G_1 \) is representable in the form
\[
\eta_1 = \eta - \sum_{j=1}^{k} \lambda_j B(\zeta^j),
\]
where \( \eta, \zeta^1, \ldots, \zeta^k \in G \) are some vectors and \( \lambda_1, \ldots, \lambda_k \) are non-negative constants. Since \( B \) is a quadratic operator continuous from \( U \) to \( V \), we see that \( \mathcal{F}(G) \subset U \) is a well-defined vector space of finite dimension. Also note that \( \mathcal{F}(G) \supset G \).

We now define a sequence of subspaces \( E_k \subset U \) by the rule
\[
E_0 = E, \quad E_k = \mathcal{F}(E_{k-1}) \quad \text{for } k \geq 1, \quad E_\infty = \bigcup_{k=1}^{\infty} E_k. \tag{9}
\]

The following theorem established in [Shi06a, Shi06b].

**Theorem 2.** Let \( E \subset U \) be a finite-dimensional subspace such that \( E_\infty \) is dense in \( H \). Then the following assertions take place for any \( T > 0, \nu > 0 \), and \( h \in L^2(J_T, H) \).

(i) Equations (2), (4) with \( \eta \in C^\infty(J_T, E) \) are approximately controllable in time \( T \).

(ii) Equations (2), (4) with \( \eta \in C^\infty(J_T, E) \) are solidly \( \mathcal{F} \)-controllable in time \( T \) for any finite-dimensional subspace \( \mathcal{F} \subset H \).

In the general case, it is difficult to verify whether a subspace \( E \subset U \) satisfies the conditions of Theorem 2. However, if \( D \) is a torus in \( \mathbb{R}^3 \), then one can obtain a sufficient condition under which \( E_\infty \) is dense in \( H \).

### 2 Case of a torus

In this subsection, we study controlled Navier–Stokes equations with periodic boundary conditions. More precisely, let us fix a vector \( q = (q_1, q_2, q_3) \) with positive components and set \( T^3_q = \mathbb{R}^3/2\pi \mathbb{Z}^3_q \), where
\[
\mathbb{Z}^3_q = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_i/q_i \in \mathbb{Z} \text{ for } i = 1, 2, 3\}.
\]

Consider the Navier–Stokes system on \( T^3_q \). In other words, we consider Eqs. (1) with \( D = \mathbb{R}^3 \) and assume that all functions are periodic of period \( 2\pi q_i \) with respect to \( x_i, i = 1, 2, 3 \). To simplify notation, we shall assume, without loss of generality, that the mean values of \( u, h, \) and \( \eta \) with respect to \( x \in T^3_q \) are zero. As in the case of a bounded domain with Dirichlet boundary condition, one can reduce (1) to an evolution equation in an appropriate Hilbert space. Namely, we set
\[
H = \left\{ u \in L^2(T^3_q, \mathbb{R}^3) : \text{div } u \equiv 0, \int_{T^3_q} u(x) \, dx = 0 \right\}
\]
and denote by $\Pi : L^2(T^3_q, \mathbb{R}^3) \to H$ the orthogonal projection in $L^2(T^3_q, \mathbb{R}^3)$ onto the closed subspace $H$. Define the spaces

\[ V = H^1(T^3_q, \mathbb{R}^3) \cap H, \quad U = H^2(T^3_q, \mathbb{R}^3) \cap H. \]

Projecting (1) to the space $H$, we obtain Eq. (2) in which $L = -\Delta$ is the Stokes operator with the domain $D(L) = U$ and $B(u) = \Pi \{ (u, \nabla u) \}$. Theorem 2, which was formulated for the Dirichlet boundary condition, remains valid in this case as well. Our aim is to describe explicitly a finite-dimensional subspace $E \subset U$ for which the hypothesis of Theorem 2 is fulfilled.

To this end, we first construct an orthogonal basis in $H$ formed of the eigenfunctions of $L$. For $x, y \in \mathbb{R}^3$, let

\[ \langle x, y \rangle_q = \sum_{i=1}^{3} q_i^{-1} x_i y_i, \quad \langle x, y \rangle = \sum_{i=1}^{3} x_i y_i, \quad |x| = \sum_{i=1}^{3} |x_i|. \]

We set $\mathbb{Z}^3_1 = \mathbb{Z}^3 \setminus \{0\}$ and $\mathbb{R}^3_1 = \mathbb{R}^3 \setminus \{0\}$. For $a \in \mathbb{R}^3_1$, denote by $a^\perp$ the two-dimensional subspace in $\mathbb{R}^3$ defined by the equation $\langle x, a \rangle_q = 0$. Note that $a^\perp = (-a)^\perp$. For any $m \in \mathbb{Z}^3_1$, let us choose a vector $\ell(m) \in m^\perp$ so that $\{ \ell(m), \ell(-m) \}$ is an orthonormal basis in $m^\perp$ with respect to the scalar product $\langle \cdot, \cdot \rangle$. We now set

\[ c_m(x) = \ell(m) \cos \langle m, x \rangle_q, \quad s_m(x) = \ell(m) \sin \langle m, x \rangle_q \quad \text{for} \quad m \in \mathbb{Z}^3_1. \]

It is a matter of direct verification to show that $c_m$ and $s_m$ are eigenfunctions of $L$ and that $\{ c_m, s_m, m \in \mathbb{Z}^3_1 \}$ is an orthogonal basis in $H$. For a finite family of functions $A$, we denote by $\text{span} \, A$ the vector space spanned by $A$.

**Theorem 3.** For any vector $q = (q_1, q_2, q_3)$ with positive components there is an integer $d \geq 4$ such that if

\[ E = \text{span}\{c_m, s_m, |m| \leq d\}, \]

then the vector space $E_\infty$ defined in (9) is dense in $H$.

Theorems 2 and 3 imply the following result on controllability of the NS system by a force of finite dimension.

**Corollary 4.** Let $E \subset U$ be the subspace defined in Theorem 3. Then for any finite-dimensional subspace $F \subset H$ and arbitrary constants $T > 0$ and $\nu > 0$ the Navier–Stokes equations (2), (4) with $\eta \in C^\infty(J_T, E)$ are approximately controllable and solidly $F$-controllable in time $T$.

The proofs of the above results are based on a development of a general approach introduced by Agrachev and Sarychev in the case of 2D Navier–Stokes equations (see [AS05, AS06]).
3 Applications

Our first application concerns the Cauchy problem for (2). Let $G \subset H$ be a closed vector space. For any $u_0 \in V$, $T > 0$, and $\nu > 0$, let $\Xi_{T,\nu}(G, u_0)$ be the set of functions $f \in L^2(J_T, G)$ for which problem (2), (3) has a unique solution $u \in X_T$. If $E \subset G$ is a closed subspace, then we denote by $G \ominus E$ the orthogonal complement of $E$ in $G$ and by $Q(T, G, E)$ the orthogonal projection in $L^2(J_T, G)$ onto the subspace $L^2(J_T, G \ominus E)$. The following result is established in [Shi06a].

**Theorem 5.** Let $E \subset U$ be a finite-dimensional subspace such that $E_\infty$ is dense in $H$ and let $G \subset H$ be a closed subspace containing $E$. Then $\Xi_{T,\nu}(G, u_0)$ is a non-empty open subset of $L^2(J_T, G)$ such that

$$Q(T, G, E)\Xi_{T,\nu}(G, u_0) = L^2(J_T, G \ominus E)$$

for any $T > 0$, $\nu > 0$, $u_0 \in V$.

Our second application concerns the case in which Navier–Stokes equations are perturbed by a random force. Namely, suppose that

$$f(t, x) = h(x) + \eta(t, x),$$

(10)

where $h \in H$ is a deterministic function and $\eta$ is an $H$-valued random process satisfying the following condition.

**(C)** There is an orthonormal basis $\{f_k\}$ in $V$ and a sequence of standard independent Brownian motions $\{\beta_j(t), t \geq 0\}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ such that

$$\eta(t) = \frac{\partial}{\partial t} \zeta(t), \quad \zeta(t) = \sum_{j,k=1}^{\infty} b_{jk} \beta_j(t) f_k,$$

where $\{b_{jk}\}$ is a family of real constants satisfying the condition

$$B := \sum_{j,k=1}^{\infty} b_{jk}^2 < \infty.$$

Let us recall the concepts of an admissible weak solution and of a stationary measure for (2), (10). Define an Ornstein–Uhlenbeck process by the formula

$$z(t) = \int_0^t e^{-\nu(t-s)L} d\zeta(s).$$

It is well known that if Condition (C) is fulfilled, then $z$ is a Gaussian process whose almost every trajectory belongs to the space $C(\mathbb{R}_+, V) \cap L^2_{\text{loc}}(\mathbb{R}_+, U)$ and satisfies the Stokes equation

$$\dot{u} + \nu Lu = \eta(t).$$

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Definition 6. An $H$-valued random process $u(t)$ is called an admissible weak solution for (2), (10) if it is representable in the form

$$u(t) = v(t) + z(t),$$

where $v(t)$ is an $H$-valued $F_t$-adapted random process whose almost every trajectory belongs to the space $L^2_{\text{loc}}(\mathbb{R}^+, V) \cap L^\infty_{\text{loc}}(\mathbb{R}^+, H)$ and satisfies the equation

$$\dot{v} + \nu Lv + B(v + z) = h$$

in the sense of distributions and the energy inequality

$$\frac{1}{2} \|v(t)\|^2 + \nu \int_0^t \|v(s)\|^2 \, ds + \int_0^t (B(v + z), v) \, ds$$

$$\leq \frac{1}{2} \|v(0)\|^2 + \int_0^t (h, v) \, ds, \quad t \geq 0,$$

where $(\cdot, \cdot)$ denotes the scalar product in $H$.

Definition 7. An admissible weak solution $u(t)$ for (2), (10) is said to be stationary if its distribution does not depend on $t$: $D(u(t)) = \mu$ for all $t \geq 0$.

In this case, $\mu$ is called a stationary measure for (2), (10).

Existence of admissible weak stationary solutions for 3D Navier–Stokes equations was established in [VF88, FG95]. Moreover, the construction of these works implies that

$$\int_H \|v\|^2_{V} \mu(dv) < \infty. \quad (11)$$

Let us denote by $Q$ the vector space of functions $v \in V$ that are representable in the form

$$v = \sum_{j,k=1}^{\infty} b_{jk} u_j f_k,$$

where $\{u_j\}$ is a sequence of real numbers such that $\sum_j u_j^2 < \infty$. Recall that the vector space $E_\infty$ is defined in (9). For a finite-dimensional space $F$, denote by $\ell_F$ the Lebesgue measure on $F$. The following theorem established in [Shi06c] provides some qualitative properties of stationary measures for (2), (10) (see also [AKSS06]).

Theorem 8. Let $\eta$ be a stationary process satisfying Condition (C), let $E \subset U$ be a finite-dimensional vector space for which $\tilde{E}_\infty$ is dense in $H$, and let $\mu$ be a stationary measure for (2), (10) such that (11) holds. Suppose that $Q \supset E$. Then the following assertions take place.

(i) The support of $\mu$ coincides with $H$.

(ii) Let $F \subset H$ be a finite-dimensional subspace and let $\mu_F$ be the projection of $\mu$ to $F$. Then there is a function $\rho_F \in C(F)$ such that $\mu_F \geq \rho_F \ell_F$ and $\rho_F(x) > 0$ for $\ell_F$-almost every $x \in F$.
References


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