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A WIENER ALGEBRA FOR THE FEFFERMAN-PHONG INEQUALITY

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This text is a written version of a talk given at the Ecole Polytechnique PDE seminar on March 21, 2006. A complete version is given in [LM].

1. Introduction and statement of the result

1.1. The Fefferman-Phong inequality. We consider a function $a \in C^\infty(\mathbb{R}^{2n})$ bounded as well as all its derivatives. The (semi-classical) Fefferman-Phong inequality states that, if a is a nonnegative function, there exists C such that, for all $u \in L^2(\mathbb{R}^n)$ and all $h \in (0,1)$

$$\mathrm{Re}\langle a(x,hD)u,u\rangle_{L^2} + Ch^2 \|u\|_{L^2}^2 \geq 0,$$

or equivalently (with an a priori different constant C)

$$a(x,h\xi)^w + Ch^2 \geq 0.$$

The constants C above depend only a finite number of derivatives of a. Let us ask our first question:

Q1: *How many derivatives of a are needed to control C?*

From the proof by Fefferman and Phong ([FP]), it is clear that the number N of derivatives of a needed to control C should be

$$N = 4 + \nu(n).$$

Since the proof is using an induction on the dimension, it is not completely obvious to answer to our question with a reasonably simple ν. We remark that, with a unitary equivalence,

$$h^{-2}a(x,h\xi)^w \equiv h^{-2}a(xh^{1/2}, h^{1/2}\xi)^w.$$

Defining $A(x,\xi) = h^{-2}a(xh^{1/2}, h^{1/2}\xi)$, we see that the following property holds:

$$(\sharp) \qquad A(x,\xi) \geq 0, \qquad A^{(k)} \text{ is bounded for } k \geq 4.$$

Bony proved in 1998 ([Bo1]) that

$$(\sharp) \Longrightarrow A^w + C \geq 0.$$

Naturally, from the above identities, this implies the Fefferman-Phong inequality. This result shows a twofold phenomenon:

- · Only derivatives with order ≥ 4 are needed.
- · The control of these derivatives is quite weak, of type $S^0_{0,0}$. In particular, the derivatives of large order do not get small (the class $S^0_{0,0}$ does not have an asymptotic calculus).

Our second question is

Q2: *Is it possible to relax* $(\sharp)$ *by asking only* $A^{(4)} \in \mathcal{A}$,

where $\mathcal{A}$ is a suitable Banach algebra containing $S^0_{0,0}$? We shall in fact prove a result involving a Wiener-type algebra introduced by Sjöstrand in [S1]. To formulate this, we need first to introduce that algebra.

1.2. The Sjöstrand algebra. Let $\mathbb{Z}^{2n}$ be the standard lattice in $\mathbb{R}^{2n}_X$ and let $1 = \sum_{j\in\mathbb{Z}^{2n}} \chi_0(X-j), \chi_0 \in C^\infty_{\mathrm{c}}(\mathbb{R}^{2n})$, be a partition of unity. We note $\chi_j(X) = \chi_0(X-j)$.

Definition. Let $a \in \mathcal{S}'(\mathbb{R}^{2n})$. We shall say that a belongs to $\mathcal{A}$ whenever $\omega_a \in L^1(\mathbb{R}^{2n})$, with $\omega_a(\Xi) = \sup_{j\in\mathbb{Z}^{2n}} |\mathcal{F}(\chi_j a)(\Xi)|$. $\mathcal{A}$ is a Banach algebra for the multiplication with the norm $\|a\|_{\mathcal{A}} = \|\omega_a\|_{L^1(\mathbb{R}^{2n})}$.

The next three lemmas are propositions 1.2.1, 1.2.3 and lemma A.2.1 in [LM].

Lemma 1. *We have $S^0_{0,0} \subset S^0_{0,0;2n+1} \subset \mathcal{A} \subset C^0(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n})$, where $S_{0,0;2n+1}$ is the set of functions defined on $\mathbb{R}^{2n}$ such that $|(\partial^\alpha_\xi \partial^\beta_x a)(x,\xi)| \leq C_{\alpha\beta}$ for $|\alpha| + |\beta| \leq 2n+1$. The algebra $\mathcal{A}$ is stable by change of quantization, i.e. for all t real, $a \in \mathcal{A} \Longleftrightarrow J^t a = \exp(2i\pi t D_x \cdot D_\xi) a \in \mathcal{A}$.*

We recall that $(a_1 \sharp a_2)^w = a_1^w a_2^w$ with

$$(a_1 \sharp a_2)(X) = 2^{2n} \iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} a_1(Y_1) a_2(Y_2) e^{-4i\pi[X-Y_1, X-Y_2]} dY_1 dY_2.$$

Lemma 2. *The bilinear map $a_1, a_2 \mapsto a_1 \sharp a_2$ is defined on $\mathcal{A} \times \mathcal{A}$ and continuous valued in $\mathcal{A}$, which is a (noncommutative) Banach algebra for $\sharp$. The maps $a \mapsto a^w, a(x,D)$ are continuous from $\mathcal{A}$ to $\mathcal{L}(L^2(\mathbb{R}^n))$.*

Lemma 3. *Let b be a function in $\mathcal{A}$ and $T \in \mathbb{R}^{2n}, t \in \mathbb{R}$. Then the functions $\tau_T b$, b_t defined by $\tau_T b(X) = b(X-T), b_t(X) = b(tX)$ belong to $\mathcal{A}$ and*

$$\sup_{T\in\mathbb{R}^{2n}} \|\tau_T b\|_{\mathcal{A}} \leq C \|b\|_{\mathcal{A}}, \qquad \|b_t\|_{\mathcal{A}} \leq (1+|t|)^{2n} C \|b\|_{\mathcal{A}}.$$

Comments on the Wiener Lemma. The standard Wiener's lemma states that if $a \in$ for some $b \in$ Sjöstrand has proven several types of Wiener lemmas for $\mathcal{A}$ ([S2]). First a commutative version, saying that if $a \in \mathcal{A}$ and $1/a$ is a bounded function, then $1/a$ belongs to $\mathcal{A}$. Next, a noncommutative version of the Wiener lemma for the algebra $\mathcal{A}$: if an operator a^w with $a \in \mathcal{A}$ is invertible as a continuous operator on L^2, then the inverse operator is b^w with $b \in \mathcal{A}$. In a paper by Gröchenig and Leinert ([GL]), the authors prove several versions of the noncommutative Wiener lemma, and their definition of the twisted convolution is indeed very close to (a discrete version of) the composition formula above. It would be interesting to compare the methods used to prove these noncommutative versions of the Wiener lemma.

1.3. The main result.

Theorem I. *There exists a constant C such that, for all nonnegative functions a defined on $\mathbb{R}^{2n}$ satisfying $a^{(4)} \in \mathcal{A}$, the operator a^w is semi-bounded from below and, more precisely, satisfies*

$$a^w + C\|a^{(4)}\|_{\mathcal{A}} \geq 0.$$

The constant C depends only on the dimension n.

Note that this answers positively to our second question (about relaxing the assumption on $a^{(4)}$), and as a byproduct gives the answer $4+2n+\epsilon$ for the number of derivatives needed to control C in the Fefferman-Phong inequality. Some results of this type were proven by Sjöstrand in [S2], namely the standard Gårding inequality with gain of one derivative for his class, $a \geq 0, a'' \in \mathcal{A} \Longrightarrow a(x, h\xi)^w + Ch \geq 0$. A version of the Hörmander-Melin inequality with gain of 6/5 of derivatives (see [H1]) was given by Hérau ([Hé]) who used a limited regularity on the symbol a, only such that $a^{(3)} \in \mathcal{A}$.

2. The Wick calculus

Some basic facts on this calculus can be found in section 2 of [LM] and in [Le].

2.1. Definitions.

Definition. Let $Y = (y, \eta)$ be a point in $\mathbb{R}^n \times \mathbb{R}^n$.

(i) The operator Σ_Y is defined as $\left[2^n e^{-2\pi|\cdot - Y|^2}\right]^w$. This is a rank-one orthogonal projection: $\Sigma_Y u = (Wu)(Y)\tau_Y\varphi$ with $(Wu)(Y) = \langle u, \tau_Y\varphi\rangle_{L^2}$, where $\varphi(x) = 2^{n/4}e^{-\pi|x|^2}$ and $(\tau_{y,\eta}\varphi)(x) = \varphi(x-y)e^{2i\pi\langle x-\frac{y}{2},\eta\rangle}$.

(ii) Let a be in $L^\infty(\mathbb{R}^{2n})$. The Wick quantization of a is defined as $a^{\mathrm{Wick}} = \int_{\mathbb{R}^{2n}} a(Y)\Sigma_Y dY$.

Lemma 4.

(i) Let a be in $L^\infty(\mathbb{R}^{2n})$. Then $a^{\mathrm{Wick}} = W^* a^\mu W$ and $1^{\mathrm{Wick}} = Id_{L^2}$ where W is the isometric mapping from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$ given above, and a^μ the operator of multiplication by a in $L^2(\mathbb{R}^{2n})$. The operator $\pi_H = WW^*$ is the orthogonal projection on a closed proper subspace H of $L^2(\mathbb{R}^{2n})$. We have also $\left\|a^{\mathrm{Wick}}\right\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq \|a\|_{L^\infty(\mathbb{R}^{2n})}$, and $a(X) \geq 0$ for all X implies $a^{\mathrm{Wick}} \geq 0$.

(ii) Moreover $a^{\mathrm{Wick}} = a^w + r(a)^w$ with

$$r(a)(X) = \int_0^1 \int_{\mathbb{R}^{2n}} (1-\theta)a''(X+\theta Y)Y^2 e^{-2\pi|Y|^2} 2^n dY d\theta.$$

That lemma implies readily the improvement of the Gårding inequality with gain of one derivative. Take $a \geq 0$ such that $a'' \in \mathcal{A}$: then $a^w = a^{\text{Wick}} - r(a)^w \geq -r(a)^w$, with $r(a)(X) = \int_0^1 \int_{\mathbb{R}^{2n}} (1-\theta) a''(X + \theta Y) Y^2 e^{-2\pi |Y|^2} 2^n dY d\theta$. Since $\mathcal{A}$ is stable by translation (see the lemma 3), we see that $r(a) \in \mathcal{A}$ and thus $r(a)^w$ is bounded on $L^2(\mathbb{R}^n)$ from the lemma 2.

2.2. Sharp estimates for the remainders. The next three lemmas are lemmas 2.2.1, 2.3.1, 2.3.3 in [LM].

Lemma 5. *Let a be a function defined on $\mathbb{R}^{2n}$ such that the fourth derivatives $a^{(4)}$ belong to $\mathcal{A}$. Then we have*

$$a^w = \Big(a - \frac{1}{8\pi} \operatorname{Tr} a''\Big)^{\text{Wick}} + \rho_0(a^{(4)})^w,$$

with $\rho_0(a^{(4)}) \in \mathcal{A}$: more precisely $\|\rho_0(a^{(4)})\|_{\mathcal{A}} \leq C_n \|a^{(4)}\|_{\mathcal{A}}$.

One should not expect the quantity $a - \frac{1}{8\pi} \operatorname{Tr} a''$ to be nonnegative: this quantity will take negative values even in the simplest case $a(x,\xi) = x^2 + \xi^2$, so that the positivity of the quantization expressed by the lemma 4 is far from enough to get our result.

Remark. We note that, from the lemma 5 and the L^2 boundedness of operators with symbols in $\mathcal{A}$, the theorem is reduced to proving

$$a \geq 0, a^{(4)} \in \mathcal{A} \Longrightarrow \Big(a - \frac{1}{8\pi} \operatorname{Tr} a''\Big)^{\text{Wick}} + C \geq 0.$$

2.3. Composition formula for the Wick quantization.

Lemma 6. *For $p, q \in L^\infty(\mathbb{R}^{2n})$ real-valued with $p'' \in L^\infty(\mathbb{R}^{2n})$, we have*

$$\operatorname{Re}\big(p^{\text{Wick}} q^{\text{Wick}}\big) = \Big(pq - \frac{1}{4\pi} \nabla p \cdot \nabla q\Big)^{\text{Wick}} + R, \ \textit{with}\ \|R\|_{\mathcal{L}(L^2)} \leq C(n) \left\|p''\right\|_{L^\infty} \left\|q\right\|_{L^\infty}.$$

Lemma 7. *For p measurable real-valued function such that $p'', (p'p'')', (pp'')'' \in L^\infty$, we have*

$$p^{\text{Wick}} p^{\text{Wick}} = \int \Big[p(Z)^2 - \frac{1}{4\pi} |\nabla p(Z)|^2\Big] \Sigma_Z dZ + S,$$
$$\|S\|_{\mathcal{L}(L^2)} \leq C(n) \left(\|p''\|_{L^\infty}^2 + \|(p''p')'\|_{L^\infty} + \|(pp'')''\|_{L^\infty}\right).$$

Further reduction. To get our theorem, we shall prove

$$a \geq 0, a^{(4)} \in L^\infty(\mathbb{R}^{2n}) \Longrightarrow \Big(a - \frac{1}{8\pi} \operatorname{Tr} a''\Big)^{\text{Wick}} + C \geq 0.$$

We leave now the arguments of harmonic analysis and we will use a structure theorem on nonnegative $C^{3,1}$ functions as sum of squares of $C^{1,1}$ functions to write the operator $\Big(a - \frac{1}{8\pi} \operatorname{Tr} a''\Big)^{\text{Wick}}$ as a sum of squares of operators, up to L^2-bounded operators, thanks to the lemmas 6,7.

3. Sketching the proof of our reduction

3.1. Nonnegative functions as sum of squares.

Theorem II. *Let $m \in \mathbb{N}$. There exists an integer N and a positive constant C such that the following property holds. Let a be a nonnegative $C^{3,1}$ function defined on $\mathbb{R}^m$ such that $a^{(4)} \in L^\infty$; then we can write*

$$a = \sum_{1 \le j \le N} b_j^2$$

where the b_j are $C^{1,1}$ functions such that $b_j'', (b_j' b_j'')', (b_j b_j'')'' \in L^\infty$. More precisely, we have

$$\|b_j''\|_{L^\infty}^2 + \|(b_j' b_j'')'\|_{L^\infty} + \|(b_j b_j'')''\|_{L^\infty} \le C \|a^{(4)}\|_{L^\infty}.$$

Note that this implies that each function b_j is such that b_j^2 is $C^{3,1}$ and that N and C depend only on the dimension m.

Part of this theorem is a consequence of the classical proof of the Fefferman-Phong inequality in [FP] and of the more refined analysis of Bony ([Bo1]) (see also the papers by Guan [Gu] and Tataru [Ta]). However the control of the L^∞ norm of the quantities $(b_j' b_j'')', (b_j b_j'')''$ seems to be new and is important for us.

Sketching the proof. We use a Calderón-Zygmund method and define

$$\rho(x) = \bigl(|a(x)| + |a''(x)|^2\bigr)^{1/4}, \quad \Omega = \{x, \rho(x) > 0\},$$

assuming as we may $\|a^{(4)}\|_{L^\infty} \le 1$. Note that, since ρ is continuous, the set Ω is open. The metric $|dx|^2/\rho(x)^2$ is slowly varying in Ω: $\exists r_0 > 0, C_0 \ge 1$ such that

$$x \in \Omega, |y - x| \le r_0 \rho(x) \Longrightarrow y \in \Omega, C_0^{-1} \le \frac{\rho(x)}{\rho(y)} \le C_0.$$

The constants r_0, C_0 can be chosen as "universal" constants, thanks to the normalization on $a^{(4)}$ above. Moreover the nonnegativity of a implies with $\gamma_j = 1$ for $j = 0, 2, 4$, $\gamma_1 = 3, \gamma_3 = 4$,

$$|a^{(j)}(x)| \le \gamma_j \rho(x)^{4-j}, \quad 1 \le j \le 4.$$

Remark. We shall use the following notation: let A be a symmetric k-linear form on real normed vector space V. We define the norm of A by

$$\|A\| = \sup_{\|T\|=1} |AT^k|.$$

Since the symmetrized products of $T_1 \otimes \cdots \otimes T_k$ can be written as a linear combination of k-th powers, that norm is equivalent to the natural norm

$$|||A||| = \sup_{\substack{\|T_j\|=1,\\ 1 \le j \le k}} |AT_1 \dots T_k|$$

and in fact, when V is Euclidean, we have the equality $\|A\| = |||A|||$ (see [Ke]). For an arbitrary normed space, the best estimate is $|||A||| \le \frac{k^k}{k!} \|A\|$ (see the remark 3.1.2 in [LM]).

The basic properties of slowly varying metrics are summarized in the following lemma (see e.g. section 1.4 in [H2]).

Lemma 8. *Let a, ρ, Ω, r_0 be as above. There exists a positive number $r_0' \le r_0$, such that for all $r \in]0, r_0']$, there exists a sequence $(x_\nu)_{\nu \in \mathbb{N}}$ of points in Ω and a positive number M_r, such that the following properties are satisfied. We define $U_\nu, U_\nu^*, U_\nu^{**}$ as the closed Euclidean balls with center x_ν and radius $r\rho_\nu, 2r\rho_\nu, 4r\rho_\nu$ with $\rho_\nu = \rho(x_\nu)$. There exist two families of nonnegative smooth functions on $\mathbb{R}^m$, $(\varphi_\nu)_{\nu \in \mathbb{N}}$, $(\psi_\nu)_{\nu \in \mathbb{N}}$ such that*

$$\sum_\nu \varphi_\nu^2(x) = \mathbf{1}_\Omega(x), \ \operatorname{supp} \varphi_\nu \subset U_\nu, \psi_\nu \equiv 1 \quad \textit{on } U_\nu^*,$$

$\operatorname{supp} \psi_\nu \subset U_\nu^{**} \subset \Omega$. *Moreover, for all integers l, we have* $\sup_{x \in \Omega, \nu \in \mathbb{N}} \|\varphi_\nu^{(l)}(x)\| \rho_\nu^l + \sup_{x \in \Omega, \nu \in \mathbb{N}} \|\psi_\nu^{(l)}(x)\| \rho_\nu^l < \infty$. *The overlap of the balls U_ν^{**} is bounded, i.e.*

$$\bigcap_{\nu \in \mathcal{N}} U_\nu^{**} \neq \emptyset \qquad \Longrightarrow \qquad \#\mathcal{N} \le M_r.$$

Moreover, $\rho(x) \sim \rho_\nu$ all over U_ν^{**} (i.e. the ratios $\rho(x)/\rho_\nu$ are bounded above and below by a fixed constant, provided that $x \in U_\nu^{**}$).

Since a is vanishing on Ω^c, we obtain

$$a(x) = \sum_{\nu \in \mathbb{N}} a(x) \varphi_\nu^2(x).$$

Definition. Let a, ρ, Ω be as above. Let θ be a positive number $\le \theta_0$, where $\theta_0 < 1/2$ is a fixed constant. A point $x \in \Omega$ is said to be

(i) θ-elliptic whenever $a(x) \ge \theta \rho(x)^4$,

(ii) θ-nondegenerate whenever $a(x) < \theta \rho(x)^4$: we have then $\|a''(x)\|^2 \ge \rho(x)^4/2$.

<u>Let us first consider the "elliptic" indices</u> ν such that x_ν is θ-elliptic. For $x \in U_\nu^{**}$, we have $a(x) \sim \rho_\nu^4$, so that with

$$b_\nu(x) = a(x)^{1/2} \psi_\nu(x), \quad b_\nu^2 = a\psi_\nu^2, \quad \varphi_\nu^2 b_\nu^2 = a\varphi_\nu^2$$

and on $\operatorname{supp} \varphi_\nu$ (where $\psi_\nu \equiv 1$),

$$\begin{cases} b_\nu' &= 2^{-1} a^{-1/2} a', \\ b_\nu'' &= -2^{-2} a^{-3/2} a'^2 + 2^{-1} a^{-1/2} a'', \\ b_\nu''' &= 3 \times 2^{-3} a^{-5/2} a'^3 - \frac{3}{4} a^{-3/2} a' a'' + 2^{-1} a^{-1/2} a''', \\ b_\nu^{(4)} &= -\frac{15}{16} a^{-7/2} a'^4 + \frac{9}{4} a^{-5/2} a'^2 a'' - \frac{3}{4} a^{-3/2} a''^2 \\ & \qquad\qquad -a^{-3/2} a' a''' + \frac{1}{2} a^{-1/2} a^{(4)}, \end{cases}$$

yielding easily the result. The whole difficulty is concentrated on the next case.

<u>The nondegenerate indices</u> ν are those for which x_ν is θ-nondegenerate. Since a'' is large, according to our scaling, we may choose the coordinates on U_ν such that

$$\partial_1^2 a(x) \ge \rho_\nu^2/2 \text{ for } |x - x_\nu| \lesssim \rho_\nu.$$

Since we know also that a is small at some point in U_ν (if the constant θ_0 is suitably chosen, cf. the lemma A.1.5 in [LM]), we get that $\partial_1 a$ vanishes somewhere in U_ν. ¿From the implicit function theorem, there exists α such that $\partial_1 a(\alpha(x'), x') = 0$ and thus, with $\beta = x_1 - \alpha(x'), R = \Big(\int_0^1 (1-t)\partial_1^2 a\big(\alpha(x') + t(x_1 - \alpha(x')), x'\big)dt\Big)^{1/2}$,

$$
\begin{aligned}
a(x) = a(x_1, x') &= R^2\beta^2 + a(\alpha(x'), x') \\
&= \int_0^1 (1-t)\partial_1^2 a\big(\alpha(x') + t(x_1 - \alpha(x')), x'\big)dt\big(x_1 - \alpha(x')\big)^2 + a(\alpha(x'), x').
\end{aligned}
$$

We find easily $|\alpha(x') - x_{\nu 1}| \lesssim \rho_\nu, \quad |\alpha'(x')| \lesssim 1,\ |\alpha''(x')| \lesssim \rho_\nu^{-1},\ |\alpha'''(x')| \lesssim \rho_\nu^{-2}$. Following Bony's argument, we compute the derivatives of

$$x' \mapsto a(\alpha(x'), x') = c(x').$$

We have, denoting by ∂_2 the x'-partial derivative,

$$
\begin{aligned}
c' &= \alpha'\partial_1 a + \partial_2 a = \partial_2 a, \\
&\quad \textbf{(here we have used the identity } \boldsymbol{\partial_1 a(\alpha(x'), x') \equiv 0}\textbf{)}, \\
c'' &= \alpha'\partial_1\partial_2 a + \partial_2^2 a, \\
c''' &= \alpha''\partial_1\partial_2 a + {\alpha'}^2\partial_1^2\partial_2 a + 2\alpha'\partial_1\partial_2^2 a + \partial_2^3 a, \\
c'''' &= \alpha'''\partial_1\partial_2 a + 3\alpha''\alpha'\partial_1^2\partial_2 a + 3\alpha''\partial_1\partial_2^2 a \\
&\qquad + {\alpha'}^3\partial_1^3\partial_2 a + 3{\alpha'}^2\partial_1^2\partial_2^2 a + 3\alpha'\partial_1\partial_2^3 a + \partial_2^4 a,
\end{aligned}
$$

$$\text{so that} \quad |c'| \lesssim \rho^3, \quad |c''| \lesssim \rho^2, \quad |c'''| \lesssim \rho, \quad |c''''| \lesssim 1.$$

This forces the function $B(x) = R(x)^2(x_1 - \alpha)^2$ to be $C^{3,1}$ with a j-th derivative bounded above by ρ_ν^{4-j} $(0 \le j \le 4)$, since it is the case for a and c. Defining $b(x) = R(x)\big(x_1 - \alpha(x')\big)$ we see that

$$a = b^2 + c, \quad |(b^2)^{(j)}| = |B^{(j)}| \lesssim \rho_\nu^{4-j},\ 0 \le j \le 4.$$

As a consequence, we have

$$
\begin{gathered}
R^2\beta^2 = \overbrace{B(\alpha(x'), x')}^{=0} + \overbrace{\int_0^1 \partial_1 B(\alpha(x') + \theta(x_1 - \alpha(x')), x')d\theta}^{\in C^{2,1}}\, \beta, \\
|\beta^{(j)}| \lesssim \rho^{1-j}, 0 \le j \le 3,
\end{gathered}
$$

and since the open set $\{\beta \ne 0\}$ is dense,

$$
\begin{gathered}
R^2\beta = \int_0^1 \partial_1 B(\alpha(x') + \theta(x_1 - \alpha(x')), x')d\theta \in C^{2,1}, \\
|(R^2\beta)^{(j)}| \lesssim \rho_\nu^{3-j},\ 0 \le j \le 3.
\end{gathered}
$$

Also we have $0 < R^2 = \omega \in C^{1,1}, \omega \sim \rho_\nu^2$ and

$$|\omega^{(j)}| \lesssim \rho_\nu^{2-j}, 0 \le j \le 2,$$

entailing that with $R = \omega^{1/2}$,

$$|R' = \frac{1}{2}\omega^{-1/2}\omega'| \lesssim 1,\ |R'' = -\frac{1}{4}\omega^{-3/2}\omega'^2 + \frac{1}{2}\omega^{-1/2}\omega''| \lesssim \rho_\nu^{-1}.$$

Using Leibniz' formula, we get

$$(R^2\beta)''' = (\omega\beta)''' = \omega'''\beta + 3\omega''\beta' + 3\omega'\beta'' + \omega\beta''',$$

which makes sense since ω''' is a distribution of order 1 and β is $C^{2,1}$. We know that $(\omega\beta)'''$ is L^∞, and since it is also the case of $\omega''\beta', \omega'\beta'', \omega\beta'''$, we get that $\omega'''\beta$ is bounded. On the other hand we have, since $\omega = R^2$,

$$\omega''' = 6R'R'' + 2\underbrace{R}_{C^{1,1}}\ \underbrace{R'''}_{\substack{\text{distribution}\\ \text{of order 1}}}$$

entailing that $\beta(6R'R'' + 2RR''')$ is L^∞ and since it is the case of $\beta R'R''$, we get that $\beta RR'''$ is L^∞ . With $b = R\beta$, we get $b'b'' = (R'\beta + R\beta')(R''\beta + 2R'\beta' + R\beta'')$ and to check that $(b'b'')'$ is in L^∞, it is enough to check the derivatives of $R''\beta R'\beta,\ R''\beta R\beta'$ which are, up to bounded terms,

$$R'''\beta R'\beta = R'''\beta RR'\frac{\beta}{R}, \quad R'''\beta R\beta'$$

which are bounded according to the estimates above. Note that b'' is bounded. We want also to verify that $(bb'')''$ is bounded. We use that $(b^2)^{(4)}$ is bounded and since we have

$$\underbrace{(b^2)''''}_{\text{bounded}} = 2(b' \otimes b' + bb'')'' = 2\underbrace{(b' \otimes b'' + b'' \otimes b')'}_{\text{bounded}} + 2(bb'')'',$$

we obtain the boundedness of $(bb'')''$. We can conclude by using an induction on the dimension (c is defined on $\mathbb{R}^{m-1}$) and a standard argument due to Guan ([Gu]) on slowly varying metrics.

3.2. End of the proof.

Lemma 9. *Let a be a nonnegative function defined on $\mathbb{R}^{2n}$ such that $a^{(4)}$ belongs to $L^\infty(\mathbb{R}^{2n})$. We have from the theorem II the identity $a = \sum_{1\le j\le N} b_j^2$ along with some estimates on each b_j and its derivatives. Then we have*

$$\Big(a - \frac{1}{8\pi}\operatorname{Tr} a''\Big)^{\mathrm{Wick}} = \sum_{1\le j\le N}\left[\Big(b_j - \frac{1}{8\pi}\operatorname{Tr} b_j''\Big)^{\mathrm{Wick}}\right]^2 + R$$

where R is a L^2-bounded operator such that $\|R\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C\|a^{(4)}\|_{L^\infty(\mathbb{R}^{2n})}$, C depending only on the dimension n.

3.3. A final comment. One may ask the following question: why did we not apply the induction argument on the Sjöstrand algebra $\mathcal{A}$ directly, and avoid that complicated detour with the Wick calculus? The answer to that (self-raised) interrogation is simple: as seen above the Fefferman-Phong induction procedure requires a cutting process (this is the metric $dX^2/\rho(X)^2$) and also a bending of the phase space (the function α is not linear). Although the cutting part may respect $\mathcal{A}$, it is not very likely that the rigid affine structure of $\mathcal{A}$ would survive the bending. We were somehow forced to push the induction procedure in some other corner, far away from the quantization business, and our theorem on nonnegative functions, although proven by induction on the dimension, is collecting all the information on lower dimensions.

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