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<http://sedp.cedram.org/item?id=SEDP_2004-2005_____A22_0>
Asymptotics for Bergman kernels for high powers of complex line bundles, based on joint works with B.Berndtsson and R.Berman

Johannes Sjöstrand CMLS, Ecole Polytechnique, FR-91128 Palaiseau Cédex

Résumé: Nous discutons l’asymptotique des noyaux de Bergman pour des puissances élevées de fibrés de droites, d’après deux travaux récents avec B.Berndtsson et R. Berman*

0. Introduction.

We present some new proofs and results around the so called Tian–Yau–Zelditch–Catlin asymptotics for the orthogonal projections onto the spaces of harmonic forms with coefficients in a high power of a complex line-bundle:

1) For (0,0)-forms: Here we give a new proof (joint work with B.Berndtsson and R.Berman [BeBeSj].)

2) For (0,q)-forms: New result and proof (joint work with R.Berman [BeSj]).

The subject has gained new interest recently through the work of geometers. M. Shubin suggested closely related problems to me 11 years ago, and later I got more stimulation mainly through the works of Shiffman, Zelditch and coworkers and from discussions with Berndtsson and Berman around the work [Be], as well as with X.Ma. The plan of the talk is:

1) Statement of the result.
2) Some historical remarks.
3) Quick outline of a new proof for (0,0)-forms.
4) Outline of the proof for (0,q)-forms.

1. The result

Let $L$ be a holomorphic line bundle over a complex compact manifold $X$ of dimension $n$. Assume the fibers $L_x$ and $\wedge^{1,0}T_x X$ carry Hermitian metrics that depend smoothly on $x \in X$.

If $s$ is a non-vanishing holomorphic section of $L$ on the open subset $\tilde{X} \subset X$, write $|s(x)| = e^{-\phi(x)}$ with $\phi(x)$ real and smooth. The curvature form of the line bundle is then determined by

$$\partial \bar{\partial} \phi = \sum \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} d z_j \wedge d \bar{z}_k,$$

where the right hand side is written in local holomorphic coordinates. Assume that $\partial \bar{\partial} \phi$ is non-degenerate of constant signature $(n_+, n_-)$ on $X$.

We shall replace $L$ by $L^k$ and consider the $\bar{\partial}$-complex:

$$C^\infty(X; L^k \otimes \wedge^{0,0} T^* X) \to C^\infty(X; L^k \otimes \wedge^{0,1} T^* X) \to \cdots \to C^\infty(X; L^k \otimes \wedge^{0,n} T^* X). \quad (1.1)$$

* Key words: complex, line, bundle, MSC 2000: 32L05, 35S30

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If we also fix a positive smooth integration density $m(dx)$, we have the adjoint $\overline{\partial}$-complex

$$C^\infty(X; L^k \otimes \wedge^{0,0} T^* X) \leftarrow C^\infty(X; L^k \otimes \wedge^{0,1} T^* X)$$

$$\leftarrow \ldots \leftarrow C^\infty(X; L^k \otimes \wedge^{0,n} T^* X).$$  (1.1*)

We introduce

$$h = 1/k \ll 1, \text{ for } k \gg 1$$  (1.2)

and the Hodge Laplacian for $(0,q)$-forms:

$$\Delta_q = \Delta_{q,k} = h \frac{\partial}{\partial h} h \overline{\partial} + h \overline{\partial}^* h \overline{\partial}.$$  (1.3)

$X$ being compact, $\Delta_q$ is essentially self-adjoint with discrete spectrum contained in $[0, +\infty[$. Let $\mathcal{N}(\Delta_q)$ be the kernel (i.e. the 0-eigenspace) and let

$$\Pi_q : L^2(X, L^k \otimes \wedge^{0,q} T^* X) \to \mathcal{N}(\Delta_q)$$

be the orthogonal (Bergman) projection. With $\widetilde{X}$, $s$, $\phi$ as above, we have the unitary identifications

$$L^2(\widetilde{X}; \wedge^{0,q} T^* X) \leftrightarrow L^2(\widetilde{X}; L^k \otimes \wedge^{0,q} T^* X)$$

$$u \leftrightarrow (e^\phi s)^k u$$

$$Z_\phi \leftrightarrow h \overline{\partial}$$

$$\Delta_{q,\text{loc}} \leftrightarrow \Delta_q$$

$$\Pi_{q,\text{loc}} \leftrightarrow \Pi_q,$$

with

$$Z_\phi = (e^\phi s)^{-k} \circ h \overline{\partial} \circ (e^\phi s)^k = h \overline{\partial} + (\overline{\partial} \phi)^\wedge,$$

$$\Delta_{q,\text{loc}} = Z_\phi^* Z_\phi + Z_\phi Z_\phi^*,$$

$$\Pi_{q,\text{loc}} = (e^\phi s)^{-k} \Pi_q (e^\phi s)^k.$$

For the proof in the case of $(0,0)$-forms we shall also use the unitary identification

$$L^2(\widetilde{X}; \wedge^{0,0} T^* X, e^{-\frac{2}{i} \phi} m) \leftrightarrow L^2(\widetilde{X}; L^k \otimes \wedge^{0,0} T^* X)$$

$$e^{\phi/h} u \leftrightarrow (e^\phi s)^k u.$$

$\Delta_{q,\text{loc}}$ has a scalar principal symbol $p \geq 0$ (times the identity matrix) vanishing precisely to the second order on the symplectic submanifold $\Sigma \subset T^* X$, given by

$$\zeta = \frac{2}{i} \frac{\partial \phi}{\partial z}, \quad z = x + iy, \quad \zeta = \xi - i\eta,$$

with $(x, y; \xi, \eta)$ as standard canonical coordinates on $T^* X$ (and $z = (z_1, ..., z_n)$ denoting local holomorphic coordinates).

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In [MeSj1] and later in [BoGu] it was established that there exist almost analytic manifolds (in the sense of [MelSj]) and we shall from now on use the term almost holomorphic

\[ J_+, J_- \subset T^* X, \quad J_- = \overline{J_+}, \]

such that \( J_+ \cap J_- = \Sigma^C \) with transversal intersection, such that locally

\[
J_+: f_1 = \ldots = f_n = 0, \quad \{f_j, f_k\}|_{J_+} = 0, \]

\[
(\frac{1}{i}\{f_j, \overline{f}_k\})_{j,k} > 0 \text{ on } \Sigma, \quad p|_{J_+} = 0.
\]

When \( n_- = 0 \), we can take \( f_j \) to be the semi-classical symbol of \( h \frac{\partial}{\partial x_j} + \frac{\partial \phi}{\partial x_j} \) that will be given more explicitly below. The following theorem is mainly due to S.Zelditch and D.Catlin when \( q = n_- = 0 \) and to R.Berman and Sjöstrand in the general case.

**Theorem 1.1.** For \( k = 1/h \) sufficiently large, we have \( \Pi_q = 0 \), \( q \neq n_- \) and for \( q = n_- \):

\[
\Pi_{q, loc} u(x) = h^{-n} \int e^{\frac{1}{i} \psi(x,y)} b(x, y; h) u(y) m(dy) + Ru,
\]

for \( x \in \tilde{X}, \ u \in L^2(\tilde{X}, L^k \otimes \wedge^{0,q} T^* X) \), where \( b \sim \sum_0^\infty b_j(x, y) h^j \) in \( C^\infty(\tilde{X} \times \tilde{X}; \mathcal{L}(\wedge^{0,q} T_y^* X, \wedge^{0,q} T_x^* X)) \).

\( Ru = \int r(x, y; h) u(y) m(dy), \quad \partial_{x,y}^\infty r = O(h^\infty) \). Further, \( \psi(x, x) = 0, \ Re \psi(x, y) \sim -dist(x, y)^2 \),

\[
\left\{ \begin{array}{l}
(x, d_x \frac{1}{i} \psi(x, y)) \in J_+ - \mathcal{O}(dist(x, y)^\infty) \\
(y, -d_y \frac{1}{i} \psi(x, y)) \in J_- - \mathcal{O}(dist(x, y)^\infty)
\end{array} \right.
\]

For \( x = y \):

\[
\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial x}, \quad \frac{\partial \psi}{\partial \overline{x}} = -\frac{\partial \phi}{\partial \overline{x}}, \quad \frac{\partial \psi}{\partial y} = -\frac{\partial \phi}{\partial \overline{y}}, \quad \frac{\partial \psi}{\partial \overline{y}} = \frac{\partial \phi}{\partial \overline{x}}.
\]

2. Historical remarks.

Most of the earlier results concern the positively curved case \( n_- = 0 \). G.Tian [Ti], followed by W.Ruan [Ru] and Z.Lu [Lu], computed increasingly many terms of the asymptotic expansion on the diagonal, using Tian’s method of peak solutions. T. Bouche [Bou] also got the leading term using heat kernels. S.Zelditch [Ze], D.Catlin [Ca] established the complete asymptotic expansion at \( x = y \) by using a result of Boutet de Monvel, Sjöstrand [BoSj] for the asymptotics of the Szegö kernel on a strictly pseudoconvex boundary (after the pioneering work of C.Fefferman [Fe]), here on the boundary of the unit disc bundle, and a reduction idea of Boutet de Monvel, Guillemin [BoSj]. Scaling asymptotics away from the diagonal was obtained later

* as follows from Hörmander’s \( L^2 - \partial \) estimates [Hö].
by P.Bleher, B.Shiffman, Zelditch [BlShZe] and the full asymptotics by L. Charles [Ch], using again the reduction method.

In more general situations, full asymptotic expansions on the diagonal and in some sense away from the diagonal were obtained by X.Dai, K.Liu, X.Ma [DaLiMa] (see also [MaMar] for related spectral results).

Without a positive curvature assumption there are fewer results. J.M.Bismut [Bi] used the heat kernel method in his approach to Demailly’s holomorphic Morse inequalities. X. Ma has pointed out to us that the method and results of [DaLiMa] can be extended to the case of non-positive holomorphic line bundles by using a spectral gap estimate from [MaMar].

3. Quick outline of a proof when $q = n_− = 0$ ([BeBeSj])

Locally, the problem is essentially to find the orthogonal projection from $L^2(\mathbb{C}^n, e^{-2\phi/h} m(dx))$ to its subspace of holomorphic functions. That projection was recently constructed in [MeSj3], and the method we present here is similar but differs on one essential point: A square root procedure is replaced by a simpler algorithm. Write for $u \in L^2_\phi \cap \text{Hol}$:

$$1 u(x) = \frac{1}{(2\pi h)^n} \int \int_{\Gamma(x)} e^{\frac{x-y}{h}\theta} u(y) dyd\theta$$

modulo an error $O(h^\infty)$, provided that the symbol $a \sim \sum_0^\infty a_j h^j$ (is almost holomorphic at a suitable set and) satisfies

$$\sum_{\alpha \in \mathbb{N}^n} \frac{h^{|\alpha|}}{\alpha!} (\partial^{\alpha}_\theta D^\alpha_y a(x, y, \theta; h))_{y=x} \sim 1. \quad (3.2)$$

Let $\Psi(x, y), M(x, y)$ be almost holomorphic with $\Psi(x, \overline{x}) = \phi(x), M(x, \overline{x}) = m(x)$. Recall that in the case $n_− = 0$, $\phi$ is strictly plurisubharmonic and we have

$$-\phi(x) + 2\text{Re} \Psi(x, \overline{y}) - \phi(y) \sim -|x - y|^2.$$ 

Consider

$$Ju(x) = \int \int e^{\frac{2}{h} \Psi(x, w) - \Psi(y, w)} c(x, w; h) M(y, w) u(y) \frac{dydw}{h^n}$$

$$= \int \int e^{\frac{2}{h} \Psi(x, \overline{y})} c(x, \overline{y}; h) u(y) e^{-\frac{2}{h} \phi(y)} m(y) \frac{dyd\overline{y}}{h^n}$$

where we integrate over $w = \overline{y}$ in the first integral.

Use the Kuranishi trick:

$$2(\Psi(x, w) - \Psi(y, w)) = i(x - y) \cdot \theta(x, y, w),$$

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\[ Ju(x) = \int \int e^{i\frac{1}{h}(x-y) \cdot \theta} a(x, y, \theta; h) u(y) \frac{dy d\theta}{(2\pi h)^n}, \]

\[ a(x, y, \theta; h) = (2\pi)^n c(x, w(x, y, \theta); h) M(y, w(x, y, \theta)) \frac{\det(\partial \theta)}{\det(\frac{\partial w}{\partial \theta})}. \]

Here the coefficients \( c_0, c_1, \ldots \) in the asymptotic expansion of \( c \) can be determined successively so that (3.2) holds.

4. Outline of the proof for general \( n \) (\([BESj]\))

We shall use the heat equation approach of \([MeSj]\) with a Witten complex trick. Work locally with

\[ \Delta_{q, \text{loc}} = Z_{\phi}^* Z_{\phi} + Z_{\phi} Z_{\phi}^*. \]

Let \( x_1, \ldots, x_{2n} \) be local coordinates. Construct a parametrix \( U_q(t; h) \) for

\[ (h \partial_t + \Delta_{q, \text{loc}}) U_q(t) = O(h^\infty), \quad U_q(0) = \text{id}, \quad (4.1) \]

\[ U_q(t) u(x) = \int \int e^{i\frac{1}{h}(\psi(t, x, \eta) - y \eta)} a(t, x, \eta; h) u(y) \frac{dy d\eta}{(2\pi h)^{2n}}. \quad (4.2) \]

Here we can solve

\[ i \partial_t \psi + p(x, \psi_x) = O(\text{Im} \psi)^\infty, \]

locally with \( \psi(0, x, \eta) = x \cdot \eta \) and with \( \text{Im} \psi \geq 0 \), and more precisely

\[ \text{Im} \psi \sim \text{dist}(x, \eta; \Sigma)^2, \quad t \geq t_0 > 0, \]

\[ \psi(t, x, \eta) = x \cdot \eta + O(\text{dist}(x, \eta; \Sigma)^2) \]

(See \([MelSj2]\) and references given there to work of Kucherenko and others.) In \([MeSj]\) a more detailed study was given, using that \( \Sigma \) is symplectic, and we showed that there exists a limiting function \( \psi(\infty, x, \eta) \) such that

\[ \partial_{t, x, \eta}^\alpha (\psi(t, x, \eta) - \psi(\infty, x, \eta)) = O(1) e^{-\frac{t}{C}}, \quad (4.3) \]

for \( t \geq 0, (x, \eta) \in \Sigma \). As used in \([MeSj1,2]\), \( J_\pm \) can be viewed as the stable outgoing and incoming manifolds for the \( i^{-1} H_p \) flow around the fixed point variety \( \Sigma^C \), and the canonical transformation \( \kappa_t \) generated by \( \psi(t, \cdot, \cdot) \) converges to the limiting canonical relation \( \kappa_\infty \) characterized by saying that \( (\rho, \mu) \in \text{graph}(\kappa_\infty) \) if \( \rho \in J_+, \mu \in J_- \) belong to bicharacteristics leaves of \( J_+, J_- \) respectively, containing the same point of \( \Sigma^C \).

The symbol

\[ a(t, x, \eta; h) \sim \sum_{j=0}^\infty a_j(t, x, \eta) h^j, \]

is determined by a sequence of transport equations, and adapting the approach of \([MeSj1]\) to the case of matrix-valued symbols, we get on \( \Sigma \):

\[ \partial_{t, x, \eta}^\alpha a_j = \begin{cases} O_{\alpha,j}(1) e^{-t/C}, & q \neq n_- \\ O_{\alpha,\alpha, j}(1) e^{\epsilon t}, & \forall \epsilon > 0, q = n_- \end{cases} \quad (4.4) \]

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Now, let $q = n_-$ and apply a Witten trick: From
\[
\Delta_{q+1,\text{loc}} Z_\phi = Z_\phi \Delta_{q,\text{loc}}, \quad \Delta_{q-1,\text{loc}} Z_\phi^* = Z_\phi^* \Delta_{q,\text{loc}},
\]
we get
\[
(h \partial_t + \Delta_{q-1,\text{loc}}) Z_\phi U_q(t) = O(h^\infty),
\]
\[
(h \partial_t + \Delta_{q+1,\text{loc}}) Z_\phi^* U_q(t) = O(h^\infty).
\]
Here $Z_\phi U_q$, $Z_\phi^* U_q$ have the general form (4.2) and since $q - 1 \neq n_- \neq q + 1$, one can show that the symbols satisfy the same decay estimate as in the first case in (4.4).

This also applies to
\[
\Delta_{q,\text{loc}} U_q = Z_\phi (Z_\phi^* U_q) + Z_\phi^* (Z_\phi U_q),
\]
and by (4.1) to
\[
h \frac{\partial U_q(t)}{\partial t} u = \int \int e^{i\psi(t, x, \eta) - y \eta} \left(i \frac{\partial \psi}{\partial t} a + h \frac{\partial a}{\partial t} u(y)\right) dyd\eta \frac{(2\pi h)^{2n}}{n}.
\]
This and (4.3) imply
\[
\partial_{t,x,\eta}^\alpha a_j(t, x, \eta) - a_j(\infty, x, \eta) = O(e^{-t/C} h^\infty),
\]
where the last equality follows from complex stationary phase ([MelSj]) and the last expression is as in the theorem.

The remaining part is more routine. We get:
\[
U_q(t) = \Pi_{q,\text{loc}}^\approx + V_q(t),
\]
\[
V_q(t) = O(e^{-t/C}) : H^-_{\text{comp}} \rightarrow H^\infty_{\text{loc}}, \quad t \geq t_0 > 0.
\]
Modulo $O(h^\infty)$:
\[
\Delta_{q,\text{loc}} \Pi_{q,\text{loc}}^\approx \equiv \Pi_{q,\text{loc}}^\approx \Delta_{q,\text{loc}} \equiv 0,
\]
\[
(\Pi_{q,\text{loc}}^\approx)^* \equiv \Pi_{q,\text{loc}}^\approx,
\]
\[
[\Pi_{q,\text{loc}}^\approx, V(t)] = O(e^{-t/C} h^\infty).
\]

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Approximate resolvent for $\text{Re} \ z < (2C)^{-1}$, $|z| \geq hN_0$:

$$R^\approx_{\text{loc}}(hz) = \frac{1}{hz} \Pi^\approx_{q, \text{loc}} - \frac{1}{h} \int_0^\infty e^{tz} V_q(t) dt.$$ 

(When $q \neq n_-$, we have the simpler formula for $\text{Re} \ z < (2C)^{-1}$:

$$R^\approx_{\text{loc}}(hz) = -\frac{1}{h} \int_0^\infty e^{tz} U_q(t) dt.$$ 

Notice that,

$$\Pi^\approx_{q, \text{loc}} = \frac{-1}{2\pi i} \int_{|z|=r} R^\approx_{\text{loc}}(z) dz, \quad h N_0 \leq r \leq \frac{h}{2C}.$$ 

Back to the global situation, we glue the different $R^\approx_{\text{loc}}$ together and get $R^\approx(z) : H^s(X) \to H^s(X)$ such that for $\text{Re} \ z < (2C)^{-1}$, $|z| \geq hN_0$:

$$(\Delta_q - hz) R^\approx(hz) \equiv R^\approx(hz)(\Delta_q - hz) \equiv \text{id}, \quad (4.5)$$

which implies that

$$(\Delta_q - hz)^{-1} \equiv R^\approx(hz),$$

$$\Pi_q = \frac{1}{2\pi i} \int_{|z|=r} (z - \Delta_q)^{-1} dz \equiv \frac{-1}{2\pi i} \int_{|z|=r} R^\approx(z) dz$$

Bibliography


