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Improved Sobolev embedding theorem

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1 Introduction.

Let ∇ denote the gradient operator. It is well known that we have in any dimension $n \geq 1$

$$(1.1) \qquad \|f\|_{n^*} \leq C_n \|\nabla f\|_1 \ , \ n^* = \frac{n}{n-1}$$

whenever f vanishes at infinity in some mild sense. Here $\|f\|_p = (\int_{\mathbb{R}^n} |f(x)|^p dx)^{1/p}$ is the usual L^p-norm.

Estimate (1.1) is invariant under the $ax+b$ group action ($a > 0, b \in \mathbb{R}^n$). However (1.1) is not invariant under the Weyl-Heisenberg group action. Indeed let $\varphi(x)$ be any function in the Schwartz class and let $f_\omega(x)$ be $\exp(i\omega.x)\varphi(x)$ where $|\omega| \to +\infty$. Then $\|\nabla f_\omega\|_1 = |\omega| \ \|\varphi\|_1 + 0(1)$ when $|\omega| \to +\infty$. It implies that (1.1) is unaccurate for such modulated functions.

We want to improve (1.1) into

$$(1.2) \qquad \|f\|_{n^*} \leq C_n \|\nabla f\|_1^{(n-1)/n} \|f\|_B^{1/n}$$

where $B = \dot{B}_\infty^{-(n-1),\infty}$ is the homogeneous Besov space which will be defined in the next section.

This improvement will appear as a by-product of a result in the paper [1], which proof is rewritten here.

This sharp estimate obviously implies (1.1) since

$$\|f\|_B \leq C\|f\|_{n^*} \ . \tag{1.3}$$

Moreover if $f = f_\omega = e^{i\omega . x}\varphi(x)$ as above, $\|f_\omega\|_B = |\omega|^{(-n-1)}\|\varphi\|_\infty + 0(|\omega|^{-n})$, $\|\nabla f_\omega\|, = |\omega| \ \|\varphi\|_1 + 0(1)$ and the asymptotics of (1.2) yield the trivial estimate

$$\|f\|_{n^*} \leq \|\varphi\|_1^{(n-1)/n}\|\varphi\|_\infty^{1/n} \ . \tag{1.4}$$

2 A special Besov space.

The special Besov space B which plays a fundamental role in this paper can be defined through several approaches.

The first one starts with the celebrated *Zygmund class*. A function $f : \mathbb{R}^n \to \mathbb{C}$ belongs to the Zygmund class if and only if $f(x)$ is continuous on $\mathbb{R}^n$ and there exists a constant C such that

$$|f(x+y) + f(x-y) - 2f(x)| \leq C|y| \tag{2.1}$$

for $x \in \mathbb{R}^n, y \in \mathbb{R}^n$.

For instance $f(x) = x \log |x|$ belongs to the Zygmund class while $f(x) = |x| \log |x|$ does not belong.

The Zygmund class, as defined by (2.1), is a quotient space modulo affine functions.

We now concentrate on $n \geq 2$ since (1.2) is obviously wrong when $n = 1$. Our first definition is the following one.

Définition 1 *Let S be a tempered distribution. We write $S \in B = \dot{B}_\infty^{-(n-1),\infty}$ if and only if $S = \sum_{|\alpha|=n} \partial^\alpha f_\alpha$ where $f_\alpha, \alpha \in \mathbb{N}^n$, belong to the Zygmund class.*

Let us provide the reader with a few equivalent definitions of the Banach space B. This new definition is using the celebrated Littlewood-Paley analysis.

We start with a function φ belonging to the Schwartz class with the following properties

$$\hat{\varphi}(\xi) = 1 \text{ for } |\xi| \leq 2/3 \tag{2.2}$$

$$\hat{\varphi}(\xi) = 0 \text{ for } |\xi| \geq 4/3 \tag{2.3}$$

where $\hat{\varphi}(\xi)$ is the Fourier transform of φ.

We then define

$$\varphi_j(x) = 2^{nj}\varphi(2^j x), j \in \mathbb{Z} \tag{2.4}$$

and, for any tempered distribution f,

$$S_j f = f * \varphi_j \ . \tag{2.5}$$

Then we have

Lemme 1 *A tempered distribution f belongs to B if and only if there exists a constant C such that*

$$\|S_j(f)\|_\infty \leq C 2^{j(n-1)}, j \in \mathbb{Z} \ . \tag{2.6}$$

The norm of f in B being $\sup\{2^{-j}\|S_j(f)\|_\infty \ ; j \in \mathbb{Z}\}$. Distinct choices of φ lead to equivalent norms.

Let us observe that (2.6) is equivalent to

$$\|\Delta_j(f)\|_\infty C' 2^{j(n-1)}, j \in \mathbb{Z}, \tag{2.7}$$

together with

$$\|S_j(f)\|_\infty \to 0 \ , \ j \to -\infty \tag{2.8}$$

where $\Delta_j = S_{j+1} - S_j$.

Then one can write

$$f = \sum_{-\infty}^{\infty} \Delta_j(f) \tag{2.9}$$

where $\Delta_j(f)$ are the celebrated dyadic blocks of a Littlewood-Paley analysis.

It is now a simple exercise to check that B can be given the following equivalent definition

Lemme 2 *A tempered distribution f belongs to B if and only if there exists a constant C such that*

$$|\langle S, g_{a,b}\rangle| \leq C \ , \ 0 < a < \infty, b \in \mathbb{R}^n, \tag{2.10}$$

where

$$g_{(a,b)}(x) = \frac{1}{a}\, g(\frac{x-k}{a}) \tag{2.11}$$

and

$$g(x) = \exp(-|x|^2/2) \ . \tag{2.12}$$

This new definition is intersting for the following observation. If S is a non-negative Radon measure μ, then $S \in B$ if and only if μ satisfies the following familiar property

(2.13)
there exists a constant Csuch that for $r \in (0,\infty), x_0 \in \mathbb{R}^n$and $B(x_0,r) =$

$$\{x \in \mathbb{R}^n \ ; |x-x_0| \le r\} \ ,$$

$$\mu(B(x_0,r)) \le Cr \ . \tag{2.14}$$

From (2.14) we immediately observe that in dimension $n \ge 2$, $|x|^{-(n-1)}$ belongs to $\dot{B}_\infty^{-(n-1),\infty}(\mathbb{R}^n)$. From (2.6) we deduce that a function $f \in \dot{B}_\infty^{-1,\infty}$ should have a zero mean in the distributional sense. Let us be more specific

Définition 2 *Let f be a tempered distribution. We say that f has a zero mean in the distributional sense if for any testing function $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\lim_{R\uparrow+\infty} \langle f, \varphi_R\rangle = 0$ where $\varphi_R(x) = R^{-n}\varphi(\frac{x}{R})$.*

We now describe a wavelet based characterization of $B = \dot{B}_\infty^{-1,\infty}$.

In this paper an orthonormal wavelet basis will always be defined as $2^{nj/2}\psi(2^jx-k), j \in \mathbb{Z}\ ,\ k \in \mathbb{Z}^n$ where ψ belongs to a finite set of 2^n-1 mother wavelets. These mother wavelets are compactly supported and smooth. For our purpose, C^1 is enough while continuity is not sufficient. We then write $\psi_{j,k}(x) = 2^{nj/2}\psi(2^jx-k)$ and have

Lemme 3 *There exist two positive constants C_2, C_1 such that*

$$C_1\|f\|_B \le \sup_{\{j,k,\psi\}} 2^{j(1-n/2)}|\langle f, \psi_{j,k}\rangle| \le C_2\|f\|_B \ . \tag{2.15}$$

Two remarks might be useful. First the supremum in (2.15) should be computed over the 2^{n-1} mother wavelets ψ. Secondly, it is not true that the Banach space B consists of all tempered distribution f for which (2.15) is finite.

The reader is referred to [2] where a detailed proof of lemma 3 is given.

3 The main facts.

Our first theorem is a rephrasing of a result by A. Cohen et al. [1].

Let us remind the reader with the notations which were previously introduced. An orthonormal wavelet basis of $L^2(\mathbb{R}^n)$ is defined as

$$\psi_{j,k}(x) = 2^{nj/2}\psi(2^j x - k), j \in \mathbb{Z}, k \in \mathbb{Z}^n \tag{3.1}$$

where ψ belongs to a finite collection of $2^n - 1$ mother wavelets. These mother wavelets are assumed to belong to $L^\infty(\mathbb{R}^n)$ and to be compactly supported. We then have

Theorem 1 *If $n \geq 2$, there exists a constant $C = C(\psi, n)$ with the following property. Whenever the gradient ∇f of a function f belongs to $L^1(\mathbb{R}^n)$, then $\beta(j,k) = 2^{j(1-n/2)}\langle f, \psi_{j,k}\rangle$ belongs to weak $-\ell^1$ of $\mathbb{Z} \times \mathbb{Z}^n$. In other words, for $\lambda \in (0,\infty)$ we denote by $E_\lambda \subset \mathbb{Z} \times \mathbb{Z}^n$ the collection of (j,k) for which $|\beta(j,k)| > \lambda$ and we have*

$$\sharp E_\lambda \leq C\|\nabla f\|_1 \lambda^{-1} \ . \tag{3.2}$$

This is not the case in one dimension. Indeed a simple limiting argument shows the following : if (3.2) is true whenever $\nabla f \in L^1(\mathbb{R}^n)$, then (3.2) should remain true whenever $f \in BV$ which means that ∇f is a finite Radon measure.

When $n = 1$, the Heaviside function is a counter-example since $\beta(j,k) = \beta(k)$ does not depend on j. If ψ is not the Haar wavelet, $\beta(k)$ does not vanish identically. There exists $ak_0 \in \mathbb{Z}$ such that $\beta(k_0) \neq 0$ and $\beta(j,k)$ cannot belong to weak $-\ell^1$.

We now return to the improved Sobolev embedding

Theorem 2 *There exists a constant C_n such that*

$$\|f\|_{n^*} \leq C_n \|\nabla f\|_1^{(n-1)/n} \|f\|_B^{1/n} \tag{3.3}$$

where $B = \dot{B}_\infty^{-(n-1),\infty}$.

This estimate can be improved one step further since the assumption $\nabla f \in L^1(\mathbb{R}^n)$ can be replaced by $f \in BV$ (in other terms ∇f is a finite Radon measure).

In order to deduce theorem 2 from theorem 1, it suffices to apply the following lemmata

Lemme 4 *Let $\alpha(j,k) = \langle f, \psi_{j,k}\rangle$ be the wavelet coefficients of f and $\beta(j,k) = 2^{j(1-n/2)}\alpha(j,k)$. If $1 < p \leq 2$, there exists a constant $C_{p,n}$ such that*

$$\|f\|_p \leq C_{p,n} (\sum_{j,k}\sum |\beta(j,k)|^p)^{1/p} \ . \tag{3.4}$$

Indeed the standard Littlewood-Paley theory yields the following. If $S(f)(x) = \left(\sum_j \sum_k |\alpha(j,k)|^2 2^{nj} |\psi(2^j x - k)|^2\right)^{1/2}$, then $\|f\|_p$ and $\|S(f)\|_p$ are equivalent norms for $1 < p < \infty$. It then suffices to observe that $S(f)(x)$ is an ℓ^2 norm which is smaller than an ℓ^p norm when $1 \leq p \leq 2$. Then the L^p-norm of this ℓ^p-norm is a trivial computation which is left to the reader.

Lemme 5 *Let $\alpha > 0$ and $\beta > 0$ be two positive numbers. Let $x_n, n \in \mathbb{N}$, be a sequence of real or complex numbers such that $|x_n| \leq \alpha (n \in \mathbb{N})$ and*

$$\sharp\{n \in \mathbb{N} \ ; \ |x_n| > \lambda\} \leq \beta\lambda^{-1} \ , 0 < \lambda < \alpha \ . \tag{3.5}$$

Then if $1 < p \leq \infty$

$$\left(\sum_0^\infty |x_n|^p\right)^{1/p} \leq C_p \ \alpha^{1-1/p}\beta^{1/p} \ . \tag{3.6}$$

Then lemma 5, lemma 6 and theorem 1 alltogether imply theorem 2.

4 The first part of the proof of theorem 1.

We will forget for a while the Daubechies wavelet $\psi(x)$ and instead use an other function $w(x)$ which is supported by the unit cube $[0,1)^n$ and moreover satisfies the following two conditions

$$|w(x)| \leq 1 \text{ and } \int w(x)dx = 0 \ . \tag{4.1}$$

If $I = I(j,k) = \{x \in \mathbb{R}^n \ ; 2^j x - k \in [0,1)^n\}$ is a dyadic cube, we consider

$$w_I(x) = 2^{nj/2} w(2^j x - k) \tag{4.2}$$

which is supported by I.

We then consider the corresponding “wavelet coefficients”

$$\alpha(I) = \langle f, w_I \rangle \ . \tag{4.3}$$

The collection of all dyadic cubes will be denoted by $\mathcal{I}$

If S is any discrete set, we will equip it with the obvious counting measure and weak $-\ell^1(S)$ will have the obvious meaning : for $\lambda > 0$, we count the number N, of $s \in S$ for which $|x(s)| > \lambda$ and $(x(s))_{s\in S}$ belongs to weak $-\ell^1(S)$ if $N_\lambda \leq C\lambda^{-1}$ for some constant C.

With this in mind we have

Theorem 3 *In any dimension $n \geq 2$, there exists a constant $C(n)$ with the following property. For any function $f(x)$ such that $\nabla f(x) \in L^1(\mathbb{R}^n)$, the renormalized wavelet coefficients*

$$\beta(I) = 2^{j(1-n/2)}\langle f, w_I \rangle \tag{4.4}$$

belong to weak $\ell^1(\mathcal{I})$ and

$$\sharp\{I \in \mathcal{I}\ ;\ |\beta(I)| > \lambda\} \leq C(n)\lambda^{-1}\|\nabla f\|_1\ . \tag{4.5}$$

Before entering the detailed proof, a few trivial remarks are needed. First we have

Lemme 6 *With the same notations as in theorem 3, we have*

$$|\beta(I)| \leq n \int_I |\nabla f| dx\ . \tag{4.6}$$

Indeed we first write

$$w(r) = \partial_1\theta_1(x) + \cdots + \partial_n\theta_n(x) \tag{4.7}$$

where $\theta_1, \cdots, \theta_n$ are supported by the unit cube $[0,1]^n$ and $\partial_j = \partial/\partial x_k$. Moreover $\theta_1, \cdots, \theta_n$ are lipschitzian and satisfy $\|\theta_j\|_\infty \leq 1$.

We then obtain

$$\beta(I) = \int 2^j w(2^j x - k) f(x) dx = \tag{4.8}$$

$$- \int \theta_1(2^j x - k)\partial_1 f(x) dx - \cdots - \int \theta_n(2^j x - k)\partial_n f(x) dx\ .$$

If lemma 6 was the only estimate at our disposal, theorem 3 would be out of reach. Indeed $|\beta(I)| > 2^{-q}$ would lead to $\int_I |\nabla f| dx > n^{-1}2^{-q}$. We will systematically study this collection of dyadic cubes. Indeed n^{-1} will be forgotten and one writes $I \in A_q$ whenever $\int_I |\nabla f| dx > 2^{-q}$. Unfortunately A_q is infinite since $I \in A_q$ and $J \supset I$ implies $J \in A_q$. That is why lemma 6 is not sharp and A_q is not the finite collection of dyadic cubes we are looking for.

A sharpening of lemma 6 is definitely needed and this improvement is provided by (6.4) in Proposition 2. For the time being, a digression is needed since we will systematically use Poincaré's inequalities. This digression will provide some information about the size of some constant which appears in this estimate.

A domain $\Omega \subset \mathbb{R}^n$ is defined as a bounded connected open set.

A domain Ω is lipschitzian if its boundary $\partial\Omega$ is locally (in a suitable coordinate frame) the graph of a lipschitzian function.

Poincaré's inequality reads the following

Lemme 7 *If Ω is a lipschitzian domain, there exists a constant $C(\Omega)$ such that*

$$\|f - m_\Omega f\|_{L^{n/n-1}(\Omega)} \leq C(\Omega) \int_\Omega |\nabla f| dx \tag{4.9}$$

for any function $f : \Omega \to \mathbb{R}$ (or $\mathbb{C}$).

Here and in what follows $m_\Omega f$ denotes the mean value of f over Ω.

The constant $C(\Omega)$ heavily depends on the global geometrical property of Ω. However if Ω_1, Ω_2 are two lipschitzian domains such that

$$\Omega_2 = a\Omega_1 + b \ , \ a > 0, b \in \mathbb{R}^n, \text{ then } C(\Omega_2) = C(\Omega_1)$$

and this observation will play a key role in the proof. An other crucial point is the following observation : (4.9) is definitely wrong if Ω is not connected.

We now return to the proof of theorem 3. Once for all we assume $\int_{\mathbb{R}^n} |\nabla f| dx \leq 1/n$. Then we are interested in counting these dyadic cubes I such that $\beta(I) > 2^{-q}$. Since $\beta(I) \leq 1$ (lemma 6) we can restrict our attention to $q \in \mathbb{N}$. Using lemma 6 once more, we consider the following collection of dyadic cubes

Définition 3 *If I is a dyadic cube and $q \in \mathbb{N}$, we write $I \in A_q$ if (and only if)*

$$\int_I |\nabla f| dx > 2^{-q} \ . \tag{4.10}$$

From lemma 6, we know that $\beta(I) \leq n2^{-q}$ if $I \notin A_q$. However A_q is not the collection of cubes we are looking for. Indeed $I \in A_q$ and $J \supset I$ imply $J \in A_q$.

Therefore A_q is an infinite collection of dyadic cubes and we are instead looking for a finite collection Λ_q such that

$$\sharp \Lambda_q \leq C 2^q \tag{4.11}$$

$$I \notin \Lambda_q \Rightarrow \beta(I) \leq C' 2^{-q} \tag{4.12}$$

where C and C' are two constant.

For constructing this finite set Λ_q which is contained in A_q we begin with two simpler finite subsets F_q and B_q of A_q.

Définition 4 *A leaf $I \in A_q$ is a minimal dyadic cube in A_q : $J \subset I$ and $J \in A_q \Rightarrow J = I$. The collection of all leaves is denoted by F_q.*

Lemme 8 $\sharp F_q \leq 2^q$.

Indeed if we are given N distinct leaves, they are pairwise disjoint. Therefore

$$N2^{-q} \leq \int_{I_1} |\nabla f| dx + \cdots + \int_{I_N} |\nabla f| dx \leq \int |\nabla f| dx \leq 1 \ .$$

This yields $N \leq 2^q$ as announced.

Before moving to B_q we need to define "sons" and "parents".

A "son" I' of the dyadic cube $I = I(j,k)$ is one of the 2^n dyadic cubes $I(j+1,k')$ which are contained in $I(j,k)$.

Conversely I is the (only) parent of I'. Let us stress again that each dyadic cube I has 2^n sons but one (and only one) parent.

These 2^n sons are ordered the following way. The "first son" I' is, among all sons $I(j+1,k')$ of $I(j,k)$ the one for which $\int_{I(j+1,k')} |\nabla f| dx$ attains the largest value (among all other sons).

The "second son" is the one for which $\int_{I(j+1,k')} |\nabla f| dx$ attains the second largest value and so on... If there are several k' for which $\int_{I(j+1,k')} |\nabla f| dx$ attains the largest value, I' will denote one among these several k' and I'' an other one.

We now arrive to the definition of the second finite subset $B_q \subset A_q$.

Définition 5 *If $I \in A_q$, we write $I \in B_q$ if both I' and I'' belong to A_q.*

A crucial remark is given by the following lemme.

Lemme 9 *The cardinality of B_q does not exceed 2^q.*

Indeed, if N_q denotes this cardinality, the number of leaves exceeds N_q. It then suffices to apply our remark concerning the number of leaves.

The set Λ_q is now defined by the following rule

(4.13)
$$\text{if } J \in B_p \ , \ 0 \leq p \leq q \ , \ I \supset J \text{ and } d(I,J) \leq 2(q-p) \ , \text{ then } I \in \Lambda_q \ .$$

The distance between I and I' is $j'-j$ whenever $I = I(j,k), I' = I(j',k')$ and $I' \subset I$. Moreover Λ_q is the smallest collection of dyadic cubes satisfying (4.13).

In other words Λ_q contains B_q. Next Λ_q contains the parents and grand parents of all dyadic cubes J in B_{q-1} and so on.

¿From this definition, it is trivial to estimate $\sharp\Lambda_q$. But this computation will be postponed and we will instead give a more algorithmic approach to Λ_q.

Définition 6 *If I is a dyadic cube, $m(I)$ is defined as* $\inf\{q \in \mathbb{N} \ ; \ I \in B_q\}$. *If this set is empty, $m(I) = +\infty$.*

In other words $m(I) = q$ implies $I' \in A_q$, $I'' \in A_q$ and q is minimal with this property. In other words, $I \notin B_{q-1}$ which implies the following property. If the first son I' is kept apart, we have $\int_J |\nabla f| dx \leq 2^{-m(I)+1}$ for all the other sons, This will be rewritten as

$$\int_{I \setminus I'} |\nabla f| dx \leq 2(2^n - 1) 2^{-m(I)} = C 2^{-m(I)} .$$

Définition 7 *If $0 \leq p \leq q$, then $\Lambda_{p,q}$ is the collection of all dyadic cubes $I \in A_q$ for which there exists an other dyadic cube $J \subset I$ such that*

$$m(J) = p \text{ and } d(I, J) \leq 2(q - p) . \tag{4.14}$$

Finally $\Lambda_q = \bigcup_{0 \leq p \leq q} \Lambda_{p,q}$.

We then want to prove the following crucial fact

Proposition 4 *There exist two (absolute) constants C and C' such that*

$$\sharp \Lambda_q \leq C' 2^q \tag{4.15}$$

$$\text{if } I \notin \Lambda_q, \text{ then } |\beta(I)| \leq C'' 2^{-q} . \tag{4.16}$$

As was already observed, this proposition implies theorem 3.

5 Proof of theorem 3 : the cardinality of Λ_q.

We begin with the following remark

$$\sharp \Lambda_{p,q} \leq (1 + 2(q - p)) 2^p . \tag{5.1}$$

Indeed $m(J) = p$ implies $J \in B_p$. The cardinality of B_p does not exceed 2^p. It then suffices for each frozen J to count the number of I containing J with $d(I, J) \leq 2(q - p)$. This number is $1 + 2(q - p)$. Observe that a child has one parent only.

Finally $\Lambda_q = \bigcup_{0 \leq p \leq q} \Lambda_{p,q}$ implies $\sharp \Lambda_q \leq \sum_{0 \leq p \leq q} \sharp \Lambda_{p,q} \leq \sum_{0 \leq p \leq q} (1 + 2(q - p)) 2^p = 2^q \sum_{0 \leq p \leq q} (1 + 2(q - p)) 2^{p-q}$. It then suffices to observe that $\sum_0^\infty (1 + 2j) 2^{-j}$ is finite.

6 The end of the proof of theorem 3.

We need to estimate the wavelet coefficients of f when $I \notin \Lambda_q$.

We separately treat two cases.

The first one is $I \notin A_q$. Then $\int_I |\nabla f| dx < 2^{-q}$ and $|\langle f, w_I \rangle|$ is trivially estimated by an integration by parts. We obtain $|\beta(I)| \leq C' 2^{-q}$. The second (and last case) is $I \in A_q$ and $I \notin \Lambda_q$. If I belongs to A_q, we construct a decreasing sequence I_j , $0 \leq j \leq r$, defined by the three rules

$$I_0 = I \tag{6.1}$$

$$I_{j+1} = I'_j \text{ if the “first son” } I'_j \in A_q \tag{6.2}$$

$$\text{if } I'_j \notin A_q \text{ , we set } j = r \text{ and the chain stops here.} \tag{6.3}$$

We then want to prove the following estimate

Proposition 5 *If $I \in A_q$, then*

$$|\beta(I)| \leq C \sum_{j=0}^{r-1} 2^{-j} 2^{-m(I_j)} + C 2^{-r} 2^{-q} \ . \tag{6.4}$$

Once (6.4) will be proved, the hypothesis $I \notin \Lambda_q$ will be used in estimating the right-hand side of (6.4).

The proof of (6.4) is based on a few simple remarks. Let us denote by $K_j = I_j \setminus I_{j+1}$ the complement of I_{j+1} inside I_j and write $K_r = I_r$.

We then have

Lemme 10 *With the preceding notations, we obtain*

$$\int_{K_j} |\nabla f| dx \leq C 2^{-m(I_j)} \ , 0 \leq j \leq r-1 \ , \tag{6.5}$$

$$\int_{K_r} |\nabla f| dx \leq C 2^{-q} \ . \tag{6.6}$$

Indeed, if $0 \leq j < r$, I_j does not belong to B_{q_j} for $q_j = m(I_j) - 1$. It means that $\int_I |\nabla f| dx < 2^{-q_j}$ if $I = I'_j$ is excepted and I is one of the $2^n - 1$ other sons of I_j.

In other words $\int_{K_j} |\nabla f| dx \leq C 2^{-q_j}$ as announced. When $j = r$, we have $\int_I |\nabla f| dx < 2^{-q}$ whenever I is a son of I_r which implies (6.6).

We now want to prove the following estimate

Lemme 11 *With the preceding notations, we have*

$$|\beta(I)| \leq C\sum_{j=0}^{r} 2^{-j}\int_{K_j}|\nabla f|dx\ . \tag{6.7}$$

Indeed we first write

$$\alpha(I)=\int_I f w_I dx=\sum_{j=0}^{r}\int_{K_j} f w_I dx$$

since $K_j, 0\leq j\leq r$, is a partitioning of $I=I_0$.

Next $\alpha_j=\int_{K_j} f w_I dx=\beta_j+\gamma_j$ where

$$\beta_j=\int_{K_j}(f-m_{K_j}f)w_I dx\ ,\ \gamma_j=(m_{K_j}.f)\int_{K_j} w_I dx$$

and $m_{K_j}f$ is the mean value of f over K_j.

We first estimate β_j. We have

$$|\beta_j|\leq |I|^{-1/2}\int_{K_j}|f-m_{K_j}f|dx\leq$$

$$|I|^{-1/2}|K_j|^{1/n}\|f-m_{K_j}f\|_{L^{n/n-1}(K_j)}\ .$$

We observe that K_j is a dilated copy of a set E which belongs to a finite collection of Lipschitz domains. Therefore the Poincaré's inequality holds with a constant which does not depend on j. We then have

$$\|f-m_{K_j}f\|_{L^{n/n-1}(K_j)}\leq C\int_{K_j}|\nabla f|dx \tag{6.8}$$

and

$$|\beta_j|\leq C'|I|^{(1/n-1/2)}2^{-j}\int_{K_j}|\nabla f|dx\ . \tag{6.9}$$

We now treat the sum $\sum_0^r \gamma_j$.

We first write $\sum_0^r \gamma_j=\sum_{j=0}^r \eta_j(\theta_j-\theta_{j+1})$ with

$$\eta_j=m_{K_j}f\ ,\theta_j=\int_{I_j} w_I(x)dx \text{ and } \ \theta_{r+1}=0$$

(by definition). We then observe that $\theta_0 = 0$ since $\int w(x)dx = 0$. We write (discrete integration by parts)

$$(6.10) \qquad \eta_0(\theta_0 - \theta_1) + \cdots + \eta_r(\theta_r - \theta_{r+1}) =$$

$$\theta_1(\eta_1 - \eta_0) + \theta_2(\eta_2 - \eta_1) + \cdots + \theta_r(\eta_r - \eta_{r-1}) \ .$$

We then make a crucial observation. In any dimension larger than 1, $K^j = K_j \cup K_{j-1}$ is a connected lipschitz domain. Moreover this domain is a dilated copy of a domain belonging to a finite collection. These two properties imply that Poincaré's inequality is valid for K^j.

We then write

$$\eta_j - \eta_{j-1} = \frac{1}{|K_j|}\int_{K_j}(f(x) - m_{K^j}f)dx -$$

$$\frac{1}{|K_{j-1}|}\int_{K_{j-1}}(f(x) - m_{K^j}f)dx \ .$$

Moreover $|K_j| \geq c|K^j|$ and $|K_{j-1}| \geq c|K^j|$ where $c > 0$ is a constant. This implies

$$|\eta_j - \eta_{j-1}| \leq \frac{1}{c|K^j|}\int_{K^j}|f - m_{K^j}|dx \leq$$

$$C|K^j|^{1/n-1}\|f - m_{K^j}f\|_{L^{n/n-1}(K^j)} \leq C|K^j|^{1/n-1}\int_{K^j}|\nabla f|dx$$

$$\leq C'|I|^{1/n-1}2^{-jn(1/n-1)}\left(\int_{K_j}|\nabla f|dx + \int_{K_{j-1}}|\nabla f|dx\right) \ .$$

Since $|\int_{I_j} w_I(x)dx| \leq |I|^{-1/2}|I_j|$, we end up with

$$(6.11) \qquad |\sum_0^r \gamma_j| \leq C|I|^{(1/n-1/2)}\sum_{j=1}^r 2^{-j}\left(\int_{K_j}|\nabla f|dx + \int_{K_{j-1}}|\nabla f|dx\right)$$

Lemme 11 follows from (6.9) and (6.11).

In order to complete the proof of theorem 3, lemma 10 and lemma 11 are being put together which yields

$$(6.12) \qquad |\beta(I)| \leq C\sum_0^{r-1} 2^{-j}2^{-m(I_j)} + C2^{-r}2^{-q} \ .$$

We need to estimate the right-hand side of (6.12) by $C'2^{-q}$. The last term can be forgotten and we concentrate on the sum. We split this series into

$$\sum_{\{m(I_j)\leq q\ ,\ 0\leq j\leq r-1\}} 2^{-j}2^{-m(I_j)} = S_1$$

and

$$\sum_{\{m(I_j)>q\ ,\ 0\leq j\leq r-1\}} 2^{-j}2^{-m(I_j)} = S_2$$

Concerning S_2 everything is trivial since $2^{-m(I_j)} < 2^{-q}$ and $\sum_{j\geq 0} 2^{-j} = 2$.

We then concentrate on S_1 and use the assumption $I \notin \Lambda_q$. Since $I_j \in A_q$ and $0 \leq m(I_j) \leq q$, we certainly have $j = d(I, I_j) > 2(q - m(I_j))$.

We then forget the dependance of $m(I_j)$ in j and treat $k = m(I_j)$ as an independant variable.

This treatment yields

$$S_1 \leq \sum\sum_{\{k\leq q\ ,\ j>2(q-k)\}} 2^{-j}2^{-k} = \sum_{k\leq q} 2^{-2(q-k)}2^{-k} = 2^{-q}\ .$$

Theorem 3 is proved.

7 From the "fake wavelets" w_I to the Daubechies wavelets.

For the sake of simplicity, we concentrate on the two-dimensional case.

We assume that $2^j\psi(2^jx - k), j \in \mathbb{Z}, k \in \mathbb{Z}^2\ , \psi \in E$ is an orthonormal basis of $L^2(\mathbb{R}^2)$ where ψ belongs to a finite set of mother wavelets. Moreover ψ is C^1 and the support of ψ is contained in $[0,p] \times [0,p]$ where p is a (large) prime number.

Then $\alpha(j,k) = \int \psi_{j,k}(x)f(x)dx\ ,\ \psi_{j,k}(x) = 2^j\psi(2^jx-k)$ and we want to prove the following lemma

Lemme 12 *If $\nabla f \in L^1(\mathbb{R}^2)$, we have*

$$\sharp\{(j,k) \in \mathbb{N}\times\mathbb{Z}^2\ ; |\alpha(j,k)| > \lambda\}$$
$$\leq C\lambda^{-1}\|\nabla f\|_1\ .$$

Observe that we restricted j to belong to $\mathbb{N}$.

Once this lemma is proved, theorem immediately follows from a rescaling argument.

Indeed lemma 12 will be applied to $f_q(x) = 2^q f(2^q x)$ where q is a *large* integer. We than obtain

$$\sharp\{(j,k)\ ; j \geq 0 \text{ and } |\beta(j,k)| > \lambda\} \leq C\lambda^{-1}\|\nabla f\|_1 \ ,$$

with

$$\beta(j,k) = \int f_q(x)\psi_{j,k}(x)dx = \alpha(j-q,k) \ .$$

Therefore

$$\sharp\{(j,k)\ ; j \geq 0 \text{ and } |\alpha(j-q,k)| > \lambda\} \\ \leq C\lambda^{-1}\|\nabla f\|_1 \ .$$

It now suffices to let q tend to infinity to obtain theorem 1.

We now return to lemma 12 and to the n-dimensional case. Let $E \subset \mathbb{Z}^n$ be $\{0, 1, 2, \cdots, p-1\}^n$ which is identified to F_p^n with $F_p = \mathbb{Z}/p\mathbb{Z}$.

We then have

Lemme 13 *For any $\ell \in \mathbb{Z}^n$ and $j \in \mathbb{N}$, there exists a unique pair $(k,r) \in \mathbb{Z}^n \times E$ such that*

$$\ell = pk + 2^j r \ . \tag{7.1}$$

This becomes obvious if we observe that $x \to 2x$ is an isomorphism of F_p. The same observation applies to the mapping $y \to 2^j y \ , \ y \in F_p^n$ which is $1-1$ for each $j \in \mathbb{N}$.

Finally one writes $\ell = pk + s \ , \ s \in E$. Next $s = 2^j r + pm$ with $m \in \mathbb{Z}^n \ , \ r \in E$. Alltogether it yields (7.1).

We now apply theorem 3 to $w(x) = \psi(px)$ and $g(x) = f(px+r), r \in E$.

Then $\alpha(j,k) = 2^{nj/2}\int w(2^j x - k)g(x)dx =$

$$2^{nj/2}\int \psi(2^j px - pk)f(px+r)dx =$$

$$2^{nj/2}p^{-n}\int \psi(2^j x - pk)f(x+r)dx =$$

$$2^{nj/2}p^{-n}\int \psi(2^j x - (2^j r + pk))f(x)dx$$

$$= p^{-n}2^{nj/2}\int \psi(2^j x - \ell)f(x)dx \ .$$

When $j \in \mathbb{N}$ is frozen, lemma shows that the mapping $(k,r) \in \mathbb{Z}^n \times E \to 2^j r + pk \in \mathbb{Z}^n$ is onto. It implies that all the wavelet coefficients of f can be writtent that way. Therefore theorem 3 implies theorem 1.

Référence

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