Centre de Mathématiques Laurent Schwartz
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SEMINAIRE

Equations aux Dérivées Partielles

1996-1997

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Séminaire É. D. P. (1996-1997), Exposé n° XX, 13 p.

<http://sedp.cedram.org/item?id=SEDP_1996-1997____A20_0>

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Smoothing effect for Schrödinger evolution equation via commutator algebra

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0. Introduction

This article concerns smoothing effects of dispersive evolution equations, especially Schrödinger evolution equations associated with complete Riemannian metrics. Our aim is to clarify the relationship between the global behavior of the Hamilton flow of the principal symbol and the smoothing effects specified later. More precisely we shall consider two problems.

Problem 1: Define for the complete Riemannian metric g the subset $S(g)$ of the unit cotangent bundle where a certain microlocal smoothing effect of the associated Schrödinger evolution group does not hold, and describe it as precisely as possible in terms of the geodesic flow.

Problem 2: Prove (higher order) smoothing effects not only for metrics with short range perturbation of the Euclidean metric but also for other types of metrics including (i) metrics with long range perturbation of the Euclidean metric, (ii) conformally compact metrics, (iii) metrics of warped product.

We shall explain the two problems by using examples.

Example 1. Let (M, g_0) be (i) the Euclidean space, or (ii) the hyperbolic space of constant curvature $-\rho^2\,(\rho > 0)$. Let g be a C^∞ Riemannian metric in M such that $g = g_0$ outside a compact set. Let $\Delta_g\,(\leq 0)$ be the associated Laplace-Beltrami operator. Then

$$L^2(M) \ni u \mapsto e^{it\Delta_g}u \in L^2_{loc}(\mathbf{R}; H^{1/2}_{loc}(M)) \tag{1}$$

is continuous if and only if there is no complete geodesic of g that is relatively compact. The continuity of the map (1) is an expression of smoothing effects. This example suggests that the existence of trapped geodesics prevents the smoothing effect. We shall explain this kind of relation more precisely from the microlocal viewpoint in Section 1 according to [Do1] (Problem 1).

Example 2. In Example 1, consider the case (i). If there is no complete geodesic of g that is relatively compact, then the following maps are continuous (see Craig-Kappeler-Strauss

[CKS]): for $k \in \mathbf{Z}_+ = \{0, 1, \dots\}$

$$L^2(\mathbf{R}^d, (1+|x|^k)dx) \ni u \mapsto t^{k/2} e^{it\Delta_g} u \in L^2_{loc}([0,\infty); H^{(1+k)/2}_{loc}(\mathbf{R}^d)) \tag{2}$$
$$L^2(\mathbf{R}^d, (1+|x|^k)dx) \ni u \mapsto t^{k/2} e^{it\Delta_g} u \in C^0([0,\infty); H^{k/2}_{loc}(\mathbf{R}^d)) \tag{3}$$

In comparison with Example 1, one might say that the continuity of maps (2) and (3) is an expression of smoothing effects of higher order. Craig-Kappeler-Strauss [CKS] consider the microlocal version of these mapping properties, and prove that if a point $z_0 \in T^*\mathbf{R}^d$ is not trapped backwards by the Hamilton flow, then the maps above suitably microlocalized near the backward orbit through z_0 are all continuous for every metric $g = \sum_{j,k=1}^d g_{jk}(x) dx^j \otimes dx^k$ satisfying the following conditions:

(i) with $C \geq 1$: $C^{-1}|dx|^2 \leq g \leq C|dx|^2$ in $\mathbf{R}^d$;

(ii) $|\partial^\alpha (g_{jk}(x) - \delta_{jk})| \leq C_\alpha (1+|x|)^{-\tau(|\alpha|)}$, $x \in \mathbf{R}^d$, for every $\alpha \in \mathbf{Z}^d_+, 1 \leq j,k \leq d$.

Here $\tau(m) > m+1$ $(m = 0, 1, \dots)$. Their method depends on the global calculus of pseudodifferential operators on $\mathbf{R}^d$, and on detailed analysis of the asymptotic behavior of the classical orbits. However, if we consider global smoothing effects not along each backward orbit as in [CKS] but in an "incoming region", then we shall find that they are reduced to a simple functional-analytic structure: a Mourre-type condition.

From this viewpoint, we shall first consider smoothing effects in an abstract settings. After preparing a scale of spaces associated with two operators A, B (Section 2), we shall introduce suitable classes of operators characterized by the mapping properties of the multiple commutators with A, B in the spirit of Gérard, Isozaki and Skibsted [GIS] (Section3) . Using the commutator calculus of these classes, we shall construct a series of conjugate operators under a Mourre-type condition, from which we shall deduce the desired smoothing effects in a suitable incoming region (Section 4). After that, we shall translate the results in our concrete settings; for this purpose, we shall need a kind of propagation of smoothing effects (Section 5).

We conclude the introduction by referring to the related works. In contrast to many works on smoothing effects for dispersive evolution equations of principal part of constant coefficient, there are few works that treat dispersive equations of principal part of variable coefficient: Craig, Kappeler and Strauss [CKS], Craig [Cr], Doi [Do1], Kapitanski and Rodianski [KR]. We remark here that the difficulty of the global behavior of the Hamilton flow appears only in the latter case (however, if we treat another category where the meaning of principal part differs from the normal case, for example, Schrödinger equations with quadratic potentials, the same type of problem occurs; see Zelditch [Ze], Yajima [Ya], Kapitanski and Safarov [KS]). In [KP], the link between the local energy decay of wave equations and the smoothing effect of

Schrödinger equations is investigated. In [CKS] and [Cr], they derive the detailed estimates of classical orbits, and prove the smoothing estimates by commutator methods.

The main part of this work is done during the stay of the author at Centre de Mathématiques de l'Ecole Polytechnique as a post-doc researcher with the aid of a grant of the French government. The author would like to express his gratitude to the staff members at CMAT for hospitality, especially to Professor J.-M. Bony for inviting him and giving him the opportunity to talk in this seminar, and to Professor C. Gérard for useful discussion.

1. Breaking of smoothing effects of the lowest order

Let (M, g) be a C^∞ complete Riemannian manifold, μ_g the associated density, and Δ_g the Laplace-Beltrami operator. Since M is complete, $\Delta|_{C_0^\infty(M)}$ is essentially self-adjoint in the Hilbert space $\mathcal{H} = L^2(M, \mu_g)$. Denote its self-adjoint extension by the same symbol Δ_g. Denote by $\Psi^m(M)$ ($\Psi^m_{cl}(M)$) the set of all (classical) pseudo-differntial operators of order m of type (1,0) on M. An operator in $\Psi^m(M)$ is called compactly supported if its distribution kernel has compact support.

Let h be the principal symbol of $-\Delta_g$, and Φ_t the flow of the Hamilton vector field $H_q$of $q = \sqrt{h}$. Put $S^*M = \{z \in T^*M; h(z) = 1\}$.

For $z \in S^*M$, the limit set $L(z)$ of z is the collection of all points $z' \in S^*M$ such that there exists a sequence of real numbers $\{t_n\}$ satistying $|t_n| \to \infty$, $\Phi_{t_n}(z) \to z'$ as $n \to \infty$.

Now we microlocalize the notion of trapping.

Definition 1.1. $S_0(g)$ is the set of all $z \in S^*M$ such that for every neighborhood of z in S^*M

$$\sup_{z' \in S^*M} m\{t \in \mathbf{R}; \Phi_t(z') \in U\} = \infty \quad (m: \;\; \text{1-dimensional Lebesgue measure}).$$

We can see that $S_0(g)$ is closed and Φ_t-invariant ($t \in \mathbf{R}$); that it cotains all limit points and is contained in the set of all non-wandering points.

Next we introduce the microlocal smoothing effect of the lowest order.

Definition 1.2. $S(g)$ is the subset of S^*M satisfying the following condition: $z \notin S(g)$ if there exists a compactly-supported operator $A \in \Psi^{1/2}_{cl}(M)$ with principal symbol $\sigma_{prin}(A) \neq 0$ at z such that the map

$$L^2(M) \ni u \mapsto A\, e^{it\Delta_g} u \in L^2(I; L^2(M))$$

is continuous for some nonempty (or equivalently, for all) open-bounded interval $I \subset \mathbf{R}$.

We are ready to state our main result obtained in [Do1].

Theorem 1.3. $S(g)$ is closed and Φ_t-invariant ($t \in \mathbf{R}$).

Theorem 1.4. $S(g)$ contains $S_0(g)$; so, if there exists a complete geodesic that remains in a compact set, then $S(g)$ is nonempty.

Corollary 1.5. If the volume of M is finite, then $S_0(g) = S(g) = S^*M$.

Corollary 1.6. If $S(g)$ is compact, then $S(g) \setminus S_0(g)$ is of measure 0 with respect to the Liouville measure on S^*M.

Corollary 1.7. If $S(g)$ is compact, then the following conditions are equivalent:

(i) $S(g) = \emptyset$; (ii) no complete geodesic is contained in a compact set.

Last, we give several applications.

I. Let $g = \sum_{j,k=1}^d g_{jk}(x)dx^j \otimes dx^k$ be a C^∞ Riemannian metric in $M = \mathbf{R}^d$. Assume that

(i) with $C \geq 1$: $C^{-1}|dx|^2 \leq g \leq C|dx|^2$ in $\mathbf{R}^d$;

(ii) $|\partial^\alpha g_{jk}(x)| \leq C_\alpha$, $x \in \mathbf{R}^d$ for all $\alpha \in \mathbf{Z}_+^d, 1 \leq j,k \leq d$;

(iii) $|\partial_i g_{jk}(x)| = o(|x|^{-1})$ as $|x| \to \infty$ for all $1 \leq i,j,k \leq d$.

Then $S(g)$ is compact

II. Let $\overline{M}$ be a C^∞ manifold with boundary ∂M, and let $\phi \in C^\infty(\overline{M}, \mathbf{R})$ be a defining function of ∂M; that is, $M := \overline{M} \setminus \partial M = \{\phi > 0\}$, $\partial M = \{\phi = 0\}, d\phi \neq 0$ on ∂M. Let g_0 be a C^∞ Riemannian metric in $\overline{M}$, and define the Riemannian metric in M by $g = \phi^{-2s} g_0$ $(s > 0)$. Then g is complete if and only if $s \geq 1$. Assume that $s \geq 1$. Then $S(g)$ is compact. In [Do1], conformal metrics $g = a(\phi)^{-2} g_0$ and their metric perturbations are considered.

III. Let (M, g) be a C^∞ Riemannian manifold. Assume that there exist a C^∞ compact Riemannian manifold (N, ω), and a C^∞ diffeomorphism χ from $(0, \infty) \times N$ to an open subset U of M satisfying

$$\chi^* g = dt \otimes dt + f(t)^2 \omega; \quad M \setminus \chi((1, \infty) \times N) \text{ is compact.}$$

Here $f \in C^\infty((0, \infty); \mathbf{R})$ is assumed to satisfy

$$|f^{(k)}(t)/f(t)| \leq C_k,\ t > 1, k = 1, 2, 3;\ f'(t) > 0\,(t >> 1).$$

Then $S(g)$ is compact. This is applicable, for example, to the case that $M = \mathbf{R}^d$; $g = |x|^{2s}|dx|^2$ for $|x| >> 1$ $(s > -1)$, $g = e^{2a|x|}|dx|^2$ for $|x| >> 1$ $(a > 0)$.

2. Scale of spaces associated with two self-adjoint operators

In this section, we shall briefly discuss the scale of spaces associated with two self-adjoint operators. Let $\mathcal{H}$ be a Hilbert space, and A, B positive definite self-adoint operators on $\mathcal{H}$. For simplicity, we assume $A \geq 1$, $B \geq 1$. Put $D^{(t,s)} = D(B^t A^s)$ $(t, s \geq 0)$, $\mathcal{S} = \mathcal{S}(A, B) = \cap_{t,s \geq 0} D^{(t,s)}$. We assume the following conditions.

(A1) For $z \notin \sigma(A)$, $(z-A)^{-1} \in L(D(B))$.

(A2) With some $0<\nu\le 1$: $D(A)\cap D(B)$ is dense in $D(B^{1-\nu})$; $\mathrm{ad}_A^N B$, firstly defined as a quadratic form on $D(A)\cap D(B)$, is extended to an operator in $L(D(B^{1-\nu}), D(B^0))$ inductively on $N \in \mathbf{N}$; further, it belongs to $L(D(B^{t+1-\nu}), D(B^t))$ for every $t \ge 0$.

Here $L(X,Y)$ denotes the set of all continuous linear operators from X to Y and $L(X) = L(X,X)$; $\mathrm{ad}_A^0 B = B$, $\mathrm{ad}_A B = [\,A, B\,]$. In application, $\mathcal{H} = L^2(M,\mu)$ where M is a C^∞ manifold with C^∞ positive density μ, A is a multiplication operator, and B is an elliptic differential operator of order $1/\nu$. A simple example is as follows: $\mathcal{H} = L^2(\mathbf{R}^d)$, $A = \langle x\rangle = (1+|x|^2)^{1/2}$ (multiplication operator), $B = 1-\Delta$ with domain $H^2(\mathbf{R}^d)$, $\nu = 1/2$.

Set $d_A(z) = \mathrm{dist}(z, \sigma(A))$, $d_B(z) = \mathrm{dist}(z,\sigma(B))$, $z \in \mathbf{C}$. $S^m(\mathbf{R})$ is the set of all $f \in C^\infty(\mathbf{R})$ such that for every $k \in \mathbf{Z}_+$

$$|f^{(k)}(t)| \le C_k(1+|t|)^{m-k}, \quad t \in \mathbf{R}.$$

Under the assumptions (A1) and (A2), $\mathcal{S}$ is dense in $D^{(t,s)}$ $(t,s\ge 0)$; $A^s, B^t \in L(\mathcal{S})$ $(s,t \in \mathbf{R})$. For the proof, we use the resolvent estimates following from (A1), (A2):

$\|B^t(z-A)^{-1}B^{-t}\| \le C_t d_A(z)^{-1}(1+d_A(z)^{-t/\nu}), \quad z\notin\sigma(A),\ t\ge 0;$

$\|A^s(z-B)^{-1}A^{-s}\| \le C_s d_B(z)^{-1}(1+(\langle z\rangle^{1-\nu}/d_B(z))^s), \quad z\notin\sigma(B),\ s\ge 0.$

Let $\mathcal{S}' = \mathcal{S}'(A,B)$ (respectively $\mathcal{H}'$) be the space of all anti-linear continuous functional on $\mathcal{S}$ (respectively $\mathcal{H}$). By Riesz's lemma, the mapping $\mathcal{H} \ni u \mapsto T_u = (u,\cdot)_{\mathcal{H}} \in \mathcal{H}'$ gives an identification, through which $\mathcal{H}$ can be regarded as a subset of $\mathcal{S}'$, since $\mathcal{S}$ is dense in $\mathcal{H}$.

For $t,s \in \mathbf{R}$, Put $D^{(t,s)} = \{u \in \mathcal{S}'; B^tA^su \in \mathcal{H}\}$. Here, in general, $T \in L(\mathcal{S})$ is identified with its unique extension in $L(\mathcal{S}')$ if $T^* \in L(\mathcal{S})$. We now summarize the basic properties.

(i) $D^{(t,s)} = \{u\in\mathcal{S}'; A^sB^tu \in \mathcal{H}\}$.

(ii) $\mathcal{S}$ is dense in $D^{(t,s)}$.

(iii) $f(A) \in L(D^{(t,s)}, D^{(t,s-\lambda)})$, $\quad f(B) \in L(D^{(t,s)}, D^{(t-\lambda,s)})$ if $f \in S^\lambda(\mathbf{R})$.

(iv) $(D^{(t,s)})' \cong D^{(-t,-s)}$ (with equivalent norms).

Finally we state the continuity property of the evolution group e^{-itB}.

Proposition 2.1. Put $D^{(k)} = \cap_{j=0}^k D^{(j(1-\nu),k-j)}$, $k=0,1,\ldots$. Then $e^{-itB} \in L(D^{(k)})$ and $\|\, e^{-itB}\,\|_{L(D^{(k)})} \le e^{C_k|t|}$ for every $t \in \mathbf{R}$. In particular, $e^{-itB} \in L(\mathcal{S})$, $t\in\mathbf{R}$.

3. Commutator algebras

In this section, we shall construct algebras of operators that admit certain asymptotic calculus and are characterized by the mapping properties of the multiple commutators with A and B. They are the basic tool in the following section. We shall preserve the notation in Section 2.

Definition 3.1. $P^{(b,a)}$ is the set of all $P \in L(\mathcal{S})$ such that P can be extented to an operator in $L(H^{(t+b,s+a)}, H^{(t,s)})$ for every $t, s \in \mathbf{R}$. Here $H^{(t,s)} = D^{(mt,s)}$, $m = 1/\nu \geq 1$.

Definition 3.2. $Q^{(b,a)}$ is the set of all $P \in P^{(b,a)}$ such that for every $N \in \{1, 2, \ldots\}, L_1, \cdots, L_N \in \{A, B\}$

$$\mathrm{ad}_{L_1} \cdots \mathrm{ad}_{L_N} P \in P^{(b+\beta m-N, a+\alpha-N)}.$$

Here $\alpha = \#\{1 \leq j \leq N; L_j = A\}$, $\beta = \#\{1 \leq j \leq N; L_j = B\}$.

Lemma 3.3. (1) If $P_j \in Q^{(b_j,a_j)}$ $(j = 1, 2)$, then $P_1 P_2 \in Q^{(b_1+b_2,a_1+a_2)}$.

(2) If $P \in Q^{(b,a)}$, then $P^* \in Q^{(b,a)}$ if P^* is identified with $P^*|_{\mathcal{S}} \in L(\mathcal{S})$ by convention.

Hereafter, we assume (A3) in addition to (A1) and (A2).

(A3) $A \in Q^{(0,1)}$, $B \in Q^{(m,0)}$; that is, for every $N \in \{0, 1, \ldots\}, L_0, \cdots, L_N \in \{A, B\}$

$$\mathrm{ad}_{L_N} \cdots \mathrm{ad}_{L_1} L_0 \in P^{(\beta m-N, \alpha-N)}.$$

Here $\alpha = \#\{0 \leq j \leq N; L_j = A\}$, $\beta = \#\{0 \leq j \leq N; L_j = B\}$.

Definition 3.4. $R^{(b,a)}$ is the set of all $P \in Q^{(b,a)}$ such that for every $d, c \in \mathbf{R}$, $Q \in Q^{(d,c)}$

$$\mathrm{ad}_P Q \in Q^{(b+d-1,a+c-1)}.$$

Example. $A \in R^{(0,1)}$, $B \in R^{(m,0)}$.

Lemma 3.5. (1) If $P_j \in R^{(b_j,a_j)}$ $(j = 1, 2)$, then $P_1 P_2 \in R^{(b_1+b_2,a_1+a_2)}$.

(2) If $P \in R^{(b,a)}$, then $P^* \in R^{(b,a)}$.

(3) If $P_j \in R^{(b_j,a_j)}$ $(j = 1, 2)$, then $[P_1, P_2] \in R^{(b_1+b_2-1,a_1+a_2-1)}$.

Definition 3.6. Let $P_j \in P^{(b_j,a_j)}, b_0 \geq b_1 \geq \cdots \to -\infty, a_0 \geq a_1 \geq \cdots \to -\infty$. For $P \in \cup P^{(b,a)}$, we write $P \sim \sum_{j=0}^{\infty} P_j$ if

$$P - \sum_{j=0}^{N-1} P_j \in P^{(b_N,a_N)}, \qquad N = 0, 1, \ldots.$$

Remark 3.7. If $P_j \in Q^{(b_j,a_j)}$ (respectively $R^{(b_j,a_j)}$), $b_0 \geq b_1 \geq \cdots \to -\infty, a_0 \geq a_1 \geq \cdots \to -\infty$, and if $P \sim \sum_{j=0}^{\infty} P_j$, then it follows easily that

$$P - \sum_{j=0}^{N-1} P_j \in Q^{(b_N,a_N)} \text{ (respectively } R^{(b_N,a_N)}), \qquad N = 0, 1, \ldots.$$

Lemma 3.8. (1) If $f \in S^{\lambda}(\mathbf{R}), P \in Q^{(b,a)}$, then

$$\begin{aligned} f(A)P &= \sum_{j=0}^{N-1} \frac{1}{j!} (\mathrm{ad}_A^j P) f^{(j)}(A) + L_N(f, A, P), \\ P f(A) &= \sum_{j=0}^{N-1} \frac{(-1)^j}{j!} f^{(j)}(A) \mathrm{ad}_A^j P + R_N(f, A, P), \end{aligned}$$

where $L_N(f,A,P) = R_N(\bar{f},A,P^*)^* \in P^{(b-N,a+\lambda-N)}$.

(2) If $f \in S^\lambda(\mathbf{R})$, then $f(A) \in R^{(0,\lambda)}$.

Lemma 3.9. (1) If $f \in S^\lambda(\mathbf{R}), P \in Q^{(b,a)}$, then

$$\begin{aligned} f(B)P &= \sum_{j=0}^{N-1} \frac{1}{j!}(\mathrm{ad}_B^j P) f^{(j)}(B) + L_N(f,B,P), \\ Pf(B) &= \sum_{j=0}^{N-1} \frac{(-1)^j}{j!} f^{(j)}(B)\mathrm{ad}_B^j P + R_N(f,B,P), \end{aligned}$$

where $L_N(f,B,P) = R_N(\bar{f},B,P^*)^* \in P^{(b+\lambda m-N,a-N)}$.

(2) If $f \in S^\lambda(\mathbf{R})$, then $f(B) \in R^{(\lambda m,0)}$.

Lemma 3.10. Let $E = E^* \in R^{(0,0)}$.

(1) $(z-E)^{-1} \in L(\mathcal{S}), z \notin \sigma(E)$; for $t,s \in \mathbf{R}$, $K > 0$, there exist $C = C(t,s,K) > 0, \lambda = \lambda(t,s,K) \geq 0$ such that

$$\|(z-E)^{-1}\|_{L(H^{(t,s)})} \leq C|\mathrm{Im}\, z|^{-1-\lambda},\ z \notin \sigma(E), |z| \leq K.$$

(2) If $f \in C_0^\infty(\mathbf{R}), P \in Q^{(b,a)}$, then

$$\begin{aligned} f(E)P &= \sum_{j=0}^{N-1} \frac{1}{j!}(\mathrm{ad}_E^j P) f^{(j)}(E) + L_N(f,E,P), \\ Pf(E) &= \sum_{j=0}^{N-1} \frac{(-1)^j}{j!} f^{(j)}(E)\mathrm{ad}_E^j P + R_N(f,E,P), \end{aligned}$$

where $L_N(f,E,P) = R_N(\bar{f},E,P^*)^* \in P^{(b-N,a-N)}$.

(3) If $f \in C^\infty$ near $\sigma(E)$, then $f(E) \in R^{(0,0)}$.

4. Commutator estimates

Let X and H be two self-adjoint operators in a Hilbert space $\mathcal{H}$ satisfying (A1), (A2) with "$m := 1/\nu > 1$" and (A3) if $A = X \geq 1$ and $B = H+1 \geq 1$. Put $\Lambda = (1+H)^{1/m}$.

In this section, we assume the Mourre-type condition:

(A4) there exist $R >> 1, \delta > 0, C_\delta > 0$ such that the following estimate holds for every $\alpha \in C^\infty(\mathbf{R})$ with $\alpha' \in C_0^\infty((R,\infty))$

$$\alpha(X)[\,iH,\,[\,iH,\,X^2\,]\,]\alpha(X) \geq 2\delta^2\alpha(X)\Lambda^{2(m-1)}\alpha(X) - 2C_\delta\alpha(X)\Lambda^{2m-3}\alpha(X).$$

Define the operator $E \in R^{(0,0)}$ by

$$E = \Lambda^{(1-m)/2} i[H,X]\Lambda^{(1-m)/2}.$$

Then it follows from (A1)-(A3) that

$$[iH,E]-\frac{1}{2}X^{-1}\Lambda^{1-m}[\,iH,[iH,X^2]\,]+X^{-1}\Lambda^{m-1}E^2\in Q^{(m-2,-2)}.$$

The assumption (A4) is used only in the Key Lemma 4.1 through this relation.

To state our main results, we need several notations.

Let $F(R)$ be the set of all $f\in C^\infty(\mathbf{R})$ such that $f'\geq 0$, $\sqrt{f},\sqrt{f'}\in C^\infty(\mathbf{R})$, $\operatorname{supp} f\subset(R,\infty)$, and that $f(t)=1$ if t is large enough. Let $G(\delta_1,\delta_2)$ $(0<\delta_1<\delta_2<\delta)$ be the set of all $g\in C^\infty(\mathbf{R})$ such that $g'\leq 0$, $\sqrt{g},\sqrt{-g'}\in C^\infty(\mathbf{R})$, $\operatorname{supp} g'\subset(-\delta_2,-\delta_1)$, and that $g(t)=1$ if $t\leq-\delta_2$. For $f,g\in C(\mathbf{R})$, write $f\subset\subset g$ if $g=1$ in a neighborhood of $\operatorname{supp} f$.

For $f,g\in C(\mathbf{R})$, denote by $Q^{(b,a)}(f,g)$ the set of all $P\in Q^{(b,a)}$ such that there exist $P_j\in Q^{(b_j,a_j)}$, $b=b_0\geq b_1\geq\cdots\to-\infty, a=a_0\geq a_1\geq\cdots\to-\infty$, $f_j,g_j\in S^0(\mathbf{R})$, satisfying

$$P\sim\sum_{j=0}^{\infty}P_jf_j(X)g_j(E);\ \operatorname{supp} f_j\subset\operatorname{supp} f,\ \operatorname{supp} g_j\subset\operatorname{supp} g.$$

This operator class is used to express error terms.

Put $L=(\frac{C_\delta}{\delta^2-\delta_2^2})^m$ (that is, a large constant depending only on $m,\delta,C_\delta,\delta_2$). Define $\chi(t)=0\,(t<L),=1\,(t\geq L)$.

Now we are ready to state the key lemma which gives the principal parts of a series of conjugate operators.

Key Lemma 4.1. Let $\phi\in S^a(\mathbf{R})\,(a\geq 0)$ such that $\phi\geq 0,\phi'\geq 0,\sqrt{\phi}\in S^{a/2}(\mathbf{R}),\sqrt{\phi'}\in S^{(a-1)/2}(\mathbf{R})$. Let $f\in F(R)$ and $g\in G(\delta_1,\delta_2)$. Then the following commutator estimate holds with $R\in Q^{(b+m-2,a-2)}(f,g)$:

$$\begin{aligned}&-\chi(H)[\,iH,\,|\,\Lambda^{b/2}(\phi f)(X)^{1/2}g(E)^{1/2}\,|^2\,]\chi(H)\\ &\geq\delta_1\chi(H)\,|\,\Lambda^{(b+m-1)/2}(\phi' f)(X)^{1/2}g(E)^{1/2}\,|^2\,\chi(H)\quad+\chi(H)R\chi(H).\end{aligned}$$

In application, we consider three cases:

(i) $\phi(t)=t^a$ with $R\in Q^{(b+m-2,a-2)}(f,g)\,(a>0)$;

(ii) $\phi(t)=1-t^{-\varepsilon}$ with $R\in Q^{(b+m-2,-2)}(f,g)\,(\varepsilon>0)$;

(iii) $\phi(t)=\log t$ with $R\in\cap_{s>0}Q^{(b+m-2,s-2)}(f,g)$.

To treat error terms, the following lemma is necessary.

Lemma 4.2. Let $f,\tilde f,g,\tilde g\in S^0(\mathbf{R})$ such that $f\subset\subset\tilde f$, $g\subset\subset\tilde g$. Then for every $P\in Q^{(b,a)}(f,g)$, there exist $C>0$ and $R=R^*\in Q^{(-\infty,-\infty)}:=\cap_{n\geq 0}Q^{(-n,-n)}$ satisfying

$$\operatorname{Re}P\leq C\,|\,\Lambda^{b/2}X^{a/2}\tilde f(X)\tilde g(E)\,|^{\,2}+R.$$

Using Lemmas 4.1 and 4.2, we are able to construct conjugate operators modulo sufficiently lower order terms.

Proposition 4.3. Let $a \geq 0, N \in \mathbf{N}, N > a, b \in \mathbf{R}$. For $j = 0, 1, \ldots, N$, define $\phi_{a,j}$ by

$\phi_{a,j}(t) = t^{a-j}\,(j < a)$;

$\phi_{a,j}(t) = 1 - t^{-\varepsilon}\,(j > a$ or $j = a = 0)$ with $0 < \varepsilon < 1 - a + [a]$;

$\phi_{a,j}(t) = \log t\,(j = a \in \mathbf{N})$.

Let $f_j \in F(R)$, $g_j \in G(\delta_1, \delta_2)\,(j = 0, \ldots, N,)$ such that $f_0 \subset\subset f_1 \subset\subset \cdots \subset\subset f_N$, $g_0 \subset\subset g_1 \subset\subset \cdots \subset\subset g_N$. Then there exist constants $C_j > 0$ satisfying

$$-\chi(H)\,[\,iH, P_{b,a,N}\,]\,\chi(H) \geq \chi(H)\,|\,\Lambda^{(b+m-1)/2}(\phi'_{a,0}f_0)(X)^{1/2}g_0(E)^{1/2}\,|^2\chi(H) + \chi(H)R_{b,a,N}\chi(H).$$

Here $R_{b,a,N} \in Q^{(b+m-2-N,-2)}(f_N, g_N)$ and

$$P_{b,a,N} = \sum_{j=0}^{N} C_j\,|\,\Lambda^{(b-j)/2}(\phi_{a,j}f_j)(X)^{1/2}g_j(E)^{1/2}\,|^2 \in Q^{(b,a)}(f_N, g_N).$$

According to Proposition 4.3, we arrive at obtaining a series of conjugate operators described below.

Theorem 4.4. Let $a \geq 0, b \in \mathbf{R}, \varepsilon > 0, K > 0$. Let $f_j \in F(R)$, $g_j \in G(\delta_1, \delta_2)\,(j = -1, 0, \ldots, [a])$ such that $f_j \subset\subset f_{j-1}$, $g_j \subset\subset g_{j-1}\,(j = 0, 1, \ldots, [a])$. Then there exist $Q_j = Q_j^* \in Q^{(b+(m-1)j, a-j)}\,(j = 0, 1, \ldots, [a])$ such that

$$\begin{aligned} Q_0 &\leq C\,|\,\Lambda^{b/2}X^{a/2}f_{-1}(X)^{1/2}g_{-1}(E)^{1/2}\,|^2 + R \\ Q_j &\geq \,|\,\Lambda^{(b+(m-1)j)/2}X^{(a-j)/2}f_j(X)^{1/2}g_j(E)^{1/2}\,|^2 \qquad (j = 0, 1, \ldots, [a]) \\ -\chi(H)\,[\,iH, Q_j\,]\,\chi(H) &\geq \chi(H)Q_{j+1}\chi(H) - \chi(H)R_j\chi(H) \quad (j = 0, 1, \ldots, [a]), \end{aligned}$$

where

$$\begin{aligned} Q_{[a]+1} &= \,|\,\Lambda^{(b+(m-1)([a]+1))/2}X^{(a-[a]-1)/2}f_{[a]}(X)^{1/2}g_{[a]}(E)^{1/2}\,|^2 \text{ if } a \text{ is a non-integer;} \\ Q_{a+1} &= \,|\,\Lambda^{(b+(m-1)(a+1))/2}X^{(-1-\varepsilon)/2}f_a(X)^{1/2}g_a(E)^{1/2}\,|^2 \text{ if } a \text{ is an integer;} \end{aligned}$$

and $R_j \in Q^{(b-K,-2)}$, $R \in Q^{(-\infty,-\infty)}$.

Lastly, we state our main theorem on regularizing effects.

Main Theorem 4.5. Let $a \geq 0, b \in \mathbf{R}, \varepsilon > 0, K > 0$. Let $f, \tilde{f} \in F(R)$, $g, \tilde{g} \in G(\delta_1, \delta_2)$ such that $f \subset\subset \tilde{f}$, $g \subset\subset \tilde{g}$. Then the following regularizing estimate holds: for $u \in \mathcal{S}, t \geq 0$

$$\begin{aligned} &\sum_{j=0}^{[a]} t^j\,\|\,P_j e^{-itH}u\,\|^2 + \sum_{j=0}^{[a]}\int_0^t \tau^j\,\|\,P_{j+1}e^{-i\tau H}u\,\|^2 d\tau \\ &\leq C\,\|\,\Lambda^{b/2}X^{a/2}\tilde{f}(X)^{1/2}\tilde{g}(E)^{1/2}\chi(H)u\,\|^2 + Ct(1 + t^{[a]})\|\,\Lambda^{(b-K)/2}\chi(H)u\,\|^2 + (Ru, u) \end{aligned}$$

Here $R = R^* \in Q^{(-\infty,-\infty)}$ and $C > 0$ are independent of u, t, and

$$\begin{aligned} P_j &= \Lambda^{(b+(m-1)j)/2} X^{(a-j)/2} f(X)^{1/2} g(E)^{1/2} \chi(H) \ (j = 0, \ldots, [a]) \\ P_{[a]+1} &= \Lambda^{(b+(m-1)([a]+1))/2} X^{(a-[a]-1)/2} f(X)^{1/2} g(E)^{1/2} \chi(H) \ (\text{if } a \notin Z_+) \\ P_{a+1} &= \Lambda^{(b+(m-1)(a+1))/2} X^{(-1-\varepsilon)/2} f(X)^{1/2} g(E)^{1/2} \chi(H) \ \ (\text{if } a \in \mathbf{Z}_+). \end{aligned}$$

Proof. By virtue of Theorem 4.4, it is sufficient to observe the following. In gereral, suppuose that $R_j, Q_j \in L(\mathcal{S})$ $(j = 0, 1, \ldots, n+1)$ satisfy $R_j = R_j^*, Q_j = Q_j^*$ $(j = 0, \ldots, n+1)$ and

$$-[\, iH, Q_j \,] \geq Q_{j+1} - R_{j+1}, \quad j = 0, 1, \ldots, n$$

on $\mathcal{S} \times \mathcal{S}$. Set $u(t) = e^{-itH}u$, $u \in \mathcal{S}$. Since

$$\frac{d}{dt}(Q_0 u(t), u(t)) = (\, [\, iH, Q_0 \,]\, u(t), u(t))$$

it follows that

$$(Q_0 u, u) \geq (Q_0 u(t), u(t)) + \int_0^t ((Q_1 - R_1) u(\tau), u(\tau)) d\tau.$$

By integration by parts, we obtain

$$\begin{aligned} \int_0^t (Q_1 u(\tau), u(\tau)) d\tau &= t(Q_1 u(t), u(t)) - \int_0^t \tau (\, [\, iH, Q_1 \,]\, u(\tau), u(\tau)) d\tau \\ &\geq t(Q_1 u(t), u(t)) + \int_0^t \tau((Q_2 - R_2) u(\tau), u(\tau)) d\tau. \end{aligned}$$

Repeating this argument, we obtain for $k = 0, 1, \ldots, n$

$$(Q_0 u, u) \geq \sum_{j=0}^{k} \frac{t^j}{j!} (Q_j u(t), u(t)) + \int_0^t \frac{\tau^k}{k!} (Q_{k+1} u(\tau), u(\tau)) d\tau - \sum_{j=0}^{k} \int_0^t \frac{\tau^j}{j!} (R_{j+1} u(\tau), u(\tau)) d\tau.$$

5. Propagation of regularizing estimates

In this section, we explain how to deduce the microlocal smoothing effects for dispersive evolution equations on manifolds with positive density, especially for Schrödinger evolution equations on Riemannian manifolds, from the results in the previous section.

Let M be a C^∞ manifold with C^∞ positive density μ, and put $\mathcal{H} = L^2(M, \mu) = L^2(M)$. Let $H \in \Psi_{cl}^m(M)$ $(m > 1)$ be a properly-supported formally self-adjoint operator with homogeneous principal symbol $\sigma_{prin}(H) = h > 0$ on $T^*M \setminus 0$, and assume that

(H0) $H|_{C_0^\infty(M)}$ is essentially self-adjoint; and the Hamiltom vector field of h, H_h, is complete.

Denote its self-adjoint extension by the same symbol H. Let Φ_t be the H_q-flow in $T^*M \setminus 0$, where $q = h^{1/m}$. Put $S^*M = \{z \in T^*M;\, h(z) = 1\}$.

The following proposition asserts that the smoothing property propagetes along the bicharacteristics in the forward direction.

Theorem 5.1. Assume (H0). Let $b \in \mathbf{R}, k \in \mathbf{Z}_+, K > 0, T > 0$. Let Γ and U be open subsets of S^*M such that $\overline{\Gamma}$ is compact in $V := \cup_{0\le t\le T}\Phi_t(U)$. Then for any compactly-supported operators $P_j \in \Psi_{cl}^{(b+(m-1)j)/2}(M)\,(j = 0,1,\ldots,k+1)$ with $S^*M \cap \text{ess-supp}\,P_j \subset \Gamma$, there exist compactly-supported operators $Q_j \in \Psi_{cl}^{b+(m-1)j}(M)$, $R_j, R'_j \in \Psi_{cl}^{b-K}(M)\,(j = 1,\ldots,k+1)$, and $S \in \Psi_{cl}^{b}(M)$, with $S^*M \cap \text{ess-supp}Q_j \subset U$, $S^*M \cap \text{ess-supp}R_j \subset V$, $S^*M \cap \text{ess-supp}R'_j \subset V$ and $S^*M \cap \text{ess-supp}\,S \subset V$, such that the following estimate holds:

$$\sum_{j=0}^{k}\left\{t^j\|P_je^{-itH}u\|^2 + \int_0^t \tau^j\|P_{j+1}e^{-i\tau H}u\|^2 d\tau\right\}$$
$$\le (Su,u) + \sum_{j=0}^{k}\int_0^t \tau^j((Q_{j+1}+R_{j+1})e^{-i\tau H}u, e^{-i\tau H}u)d\tau + \sum_{j=0}^{k} t^j(R'_je^{-itH}u, e^{-itH}u)$$

for every $t \ge 0,\ u \in C_0^\infty(M)$.

Hereafter assume $H \ge 0$ and set $\Lambda = (1+H)^{1/m}$. Let X be a multiplication operator by a function $r \in C^\infty(M,\mathbf{R})$ such that $r \ge 1$, $\lim_{x\to\infty} r(x) = \infty$. We shall assume that H, X satisfy (H1)-(H2).

(H1) For every $N \in \{0,1,\ldots\}, L_0,\cdots,L_N \in \{X,H\}$

$$\Lambda^{N-m\beta}X^a \mathrm{ad}_{L_N}\cdots\mathrm{ad}_{L_1}L_0X^{a'}|_{C_0^\infty(M)}$$

is extended to an operator in $L(\mathcal{H})$ for every $a, a' \in \mathbf{R}$ with $a + a' = N - \alpha$. Here $\alpha = \#\{0 \le j \le N; L_j = X\}$, $\beta = \#\{0 \le j \le N; L_j = H\}$.

(H2) There exist $R >> 1, \delta > 0, C_\delta > 0$ such that the following estimate holds for every $\alpha \in C^\infty(\mathbf{R})$ with $\alpha' \in C_0^\infty((R,\infty))$

$$\alpha(X)[\,iH,\,[\,iH,\,X^2\,]\,]\alpha(X) \ge 2\delta^2\alpha(X)\Lambda^{2(m-1)}\alpha(X) - 2C_\delta\alpha(X)\Lambda^{2m-3}\alpha(X)$$

on $C_0^\infty(M)\times C_0^\infty(M)$.

The assumption (H1) corresponds to (A1)-(A3), and (H2) to (A4). Thus, the corresponding spaces $\mathcal{S}$, $\mathcal{S}'$, $H^{(b,a)}$ are well-defined. The assumtion (H2) implies the classical Mourre-type condition:

(H2)' $H_h^2(r^2) \ge 2\delta^2$ if $r \ge R$.

The following lemma describes the "incoming" region that absorbs every backwardly non-trapped bicharacteristics as $t \to -\infty$.

Lemma 5.2. Assume (H2)'. If an integral curve of H_{h_m}, $\gamma \in C^\infty(\mathbf{R}, S^*M)$, verifies that $\{\gamma(t); t \le 0\}$ is not relatively compact, then for every $0 < \delta_1 < \delta$ there exists $T > 0$ such that

$$r(\gamma(t)) \ge \delta_1|t|;\ \ \&\ \ (H_hr)(\gamma(t)) \le -\delta_1 \quad \text{for } t \le -T.$$

So, if the regularizing estimate holds in the region

$$\{z = (x, \xi) \in S^*M; H_h(z) < -\delta_1, r(x) > R\},$$

then by virtue of the propagation of regularity in the forward direction, it is also valid at every backwardly non-trapped point.

By Theorem 4.5, Lemma 5.2, and Theorem 5.1, we are able to obtain the microlocal smoothing effect in the following form.

Theorem 5.3. Assume (H0)-(H2). Let $z_0 \in S^*M$ such that $\gamma := \{\Psi_t(z_0); t \leq 0\}$ is not relatively compact. Let $b, b' \in \mathbf{R}, b \geq b'$; $a \geq 0$. Let $f \in F(R)$, $g \in G(\delta_1, \delta_2)$. For $u \in \mathcal{S}' = \mathcal{S}'(X, H)$, assume that $\Lambda^{b'/2}u \in \mathcal{H}$, $\Lambda^{b/2}X^{a/2}f(X)^{1/2}g(E)^{1/2}u \in \mathcal{H}$ (for example, $\Lambda^{b/2}X^{a/2}u \in \mathcal{H}$), and that u is microlocally $H^{b/2}_{loc}$ at every point of γ. Then there exist a neighborhood of z_0, U, in S^*M such that the following assertion holds: for any compactly-supported operators $P_j \in \Psi_{cl}^{(b+(m-1)j)/2}(M)$ $(j = 0, 1, \ldots, [a] + 1)$ with $S^*M \cap \text{ess-supp}\, P_j \subset U$,

$$\sum_{j=0}^{[a]} \left\{ \sup_{0 \leq \tau \leq t} \tau^j \|P_j e^{-i\tau H} u\|^2 + \int_0^t \tau^j \|P_{j+1} e^{-i\tau H} u\|^2 d\tau \right\} < \infty \quad \text{for } t \geq 0.$$

Corollary 5.4. Assume (H0)-(H2). Let $z_0 \in S^*M$ such that $\gamma := \{\Phi_t(z_0); t \leq 0\}$ is not relatively compact. If $u \in \mathcal{S}' = \mathcal{S}'(X, H)$ satisfies that $\Lambda^b X^a u \in \mathcal{H}$ for every $a \geq 0$ with a fixed $b \in \mathbf{R}$, then $z_0 \notin WF(e^{-itH}u)$ for every $t > 0$.

Corollary 5.5. Assume (H0)-(H2). If there is no complete bicharacteristics of h in S^*M that is relatively compact, then e^{-itH} is continuous from $H^{-s}_{cpt}(M)$ to $H^s_{loc}(M)$ for every $s \neq 0, t \neq 0$.

6. Application

We shall return to the three cases considered in the second half part of Section 1.

I. Let $g = \sum_{j,k=1}^d g_{jk}(x)dx^j \otimes dx^k$ be a C^∞ Riemannian metric in $\mathbf{R}^d$. Assume (i), (iii) in Section 1 and

(ii)' $|\partial^\alpha g_{jk}(x)| \leq C_\alpha(1 + |x|)^{-|\alpha|}$, $x \in \mathbf{R}^d$ for all $\alpha \in \mathbf{Z}^d_+, 1 \leq j, k \leq d$;
Then $\mathcal{H} = L^2(\mathbf{R}^d, \mu_g)$, $H = -\Delta_g$, $X = \sqrt{1 + |x|^2}$ satisfy (H0)-(H2) with $m = 1/\nu = 2$.

II. Under the same setting of Section 1, Assume that $s > 1$. Then $\mathcal{H} = L^2(M, \mu_g)$, $H = -\Delta_g$, $X = \phi^{1-s}$ satisfy (H0)-(H2) with $m = 1/\nu = 2$. For $s = 1$, our method does not work.

III. Under the same setting of Section 1 assume that

$$|f^{(k)}(t)/f(t)| \leq C_k t^{-k},\ t > 0\,(k = 0, 1, \ldots);\ \ \text{with } \delta > 0,\ tf'(t)/f(t) \geq \delta\,(t >> 1).$$

Then $\mathcal{H} = L^2(M, \mu_g)$, $H = -\Delta_g$, $X = r$ satisfy (H0)-(H2) with $m = 1/\nu = 2$. Here $r \in C^\infty(M, \mathbf{R})$ is a function satisfying $r \geq 1$ and $\chi^* r = t \, (t > 2)$.

This is applicable to the case that $M = \mathbf{R}^d$; $g = |x|^{2s}|dx|^2$ for $|x| >> 1$ $(s > -1)$, or $g = e^{2a|x|}|dx|^2$ for $|x| >> 1$ $(a > 0)$.

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