Séminaire Équations aux dérivées partielles – École Polytechnique

V. Ivrii

Non weylian spectral asymptotics with accurate remainder estimate

Séminaire Équations aux dérivées partielles (Polytechnique) (1990-1991), exp. nº 5, p. 1-10

<http://www.numdam.org/item?id=SEDP_1990-1991____A5_0>

CENTRE
DE
MATHEMATIQUES
Unité de Recherche Associée D 0169

ECOLE POLYTECHNIQUE
F-91128 PALAISEAU Cedex (France)
Tél. (1) 69.41.82.00
Télex ECOLEX 601.596 F

Séminaire 1990-1991

ÉQUATIONS AUX DÉRIVÉES PARTIELLES

NON WEYLIAN SPECTRAL ASYMPTOTICS WITH ACCURATE REMAINDER ESTIMATE

par V. IVRII

Exposé n°V

20 Novembre 1990

In this lecture we discuss three spectral problems in which non-Weylian asymptotics of eigenvalue counting function arise. Namely we consider detaily operators in the domains with thick cusps$^{\dagger)}$ and we consider briefly operators with potentials degenerating at infinity and operators degenerating at symplectic submanifolds. There following common features of these problems: if problem isn't too bad then standard Weylian asymptotics is valid, if the badness is temperate then the same asymptotics remains true but with worth remainder estimate and if the problem is very bad then Weylian formula is completely wrong (it even predicts wrongly that spectrum is no more discrete); in the second case correct asymptotics include non-Weylian term and in the third case this non-Weylian term is principal; this term in fact is obtained by the Weylian way for certain operator with operator-valued symbol in some auxiliary Hilbert space.

1. Let us consider operator A in the domain $X \subset \mathbb{R}^d$ with the cusp. That means that

$$
\begin{aligned}
(1) \quad & X = X_0 \cup X_1, \partial X \in C^K, X_0 \Subset \mathbb{R}^d, \\
& X_1 = \{x = (x', x''), x'' \in \rho(x')Y, x' \in \mathbb{R}^{d'}\}, \quad Y \Subset \mathbb{R}^{d''}, \partial Y \in C^K
\end{aligned}
$$

where K is a large enough exponent and

$$
\begin{aligned}
(2) \quad & \rho \in C^K, \quad \rho(x') \asymp |x'|^{-\mu}, \\
& \qquad D^\alpha_{x'}\rho(x') = O(|x'|^{-\mu-|\alpha|}) \quad \text{as } |x'| \to \infty, \quad \forall \alpha : |\alpha| \le K,
\end{aligned}
$$

$^{\dagger)}$These results are obtained in cooperation with my post-graduate E.Filippov.

$d = d' + d'', 0 < d' < d$. If $d' = 1$ then instead of $x' \in \mathbb{R}^{d'}$ one can consider $x' \in \mathbb{R}^+$.

Surely one can generalize our results to the case when there is more than one cusp and their exponents d''... and μ... are different.

For the sake of simplicity we consider second-order operator

$$A = \sum_{j,k} D_j g^{jk}(x) D_k + \sum_j (b_j(x) D_j + D_j b_j(x)) + c(x) \tag{3}$$

where $g^{jk} = g^{kj}, b_j, c \in C^K$ are real-valued functions and we assume that A is uniformly elliptic operator i.e.

$$\sum_{j,k} g^{jk} \xi_j \xi_k \geq |\xi|^2/c \quad \forall \xi \in \mathbb{R}^d, \tag{4}$$

$$D^\alpha g^{jk} = O(|x'|^{-|\alpha|}), \quad D^\alpha b_j = o(|x'|^{-|\alpha|+\mu}), \\ D^\alpha c = o(|x'|^{-|\alpha|+2\mu}) \quad \forall \alpha : |\alpha| \leq K \tag{5}$$

and

(6) At ∂X Dirichlet boundary condition is given[1)] and A with this condition is self-adjoint in $L^2(\mathbb{R}^d)$.

Let $N(\tau)$ be an eigenvalue counting function of A.

First af all there are results due to Ivrii-Fedorova [1]:

Theorem 1. *Let conditions* $(1) - (6)$ *be fulfilled. Then*

(i) If $\mu > \mu_1 = d'/(d'' - 1)$ *then*

$$N(\tau) = N^w(\tau) + O(\tau^{(d-1)/2}) \quad \text{as } \tau \to +\infty \tag{7}$$

where

$$N^w(\tau) = (2\pi)^{-d} \int_{A^w(x,\xi)<\tau} dx d\xi \asymp \tau^{d/2} \tag{8}$$

is a Weylian expression; moreover if

(H) *the set of all the points of* T^*X *periodic with respect to billiard flow has measure* 0

1)One can consider more general boundary conditions but it is necessary to be very careful with them.

then

$$(9) \qquad N(\tau) = N^w(\tau) + (\varkappa_1 + o(1))\tau^{(d-1)/2}$$

where $\varkappa_1 = -\frac{1}{4}(2\pi)^{1-d}\omega_{d-1}\mathrm{vol}_{d-1}\partial X$ *is the standard Weylian coefficient and volume is calculated in metrics induced at* ∂X *by a Riemann metrics* g^{jk}, ω_k *is a volume of the unit ball in* $\mathbb{R}^k$*;*

(ii) If $\mu = \mu_1$ *then*

$$N(\tau) = N^w(\tau) + O(\tau^{(d-1)/2}\log\tau) \quad \text{as } \tau \to +\infty$$

and if $\mu_0 < \mu < \mu_1$ *with* $\mu_0 = d'/d''$ *then*

$$N(\tau) = N^w(\tau) + O(\tau^{(d-q)/2}) \quad \text{as } \tau \to +\infty$$

with $q = d'/\mu + d'' \in (0,1)$ *and (7) remains true;*

(iii) If $\mu = \mu_0$ *then*

$$N(\tau) = N_1^w(\tau) + O(\tau^{d/2}) \asymp \tau^{d/2}\log\tau \quad \text{as } \tau \to +\infty$$

where $N_1^w(\tau)$ *is given by (7) with additional restriction* $\rho(x') > \tau^{-1/2}$ *in the definition of the domain of integration.*

Remark 2. Condition "$\mu > \mu_1$" is equivalent to "$\rho^{d''-1} \in L^2(\mathbb{R}^{d'})$" and is equivalent to "$\mathrm{vol}_{d-1}\partial X$ is finite". Condition "$\mu > \mu_0$" is equivalent to "$\rho^{d''} \in L^2(\mathbb{R}^{d'})$" and is equivalent to "$\mathrm{vol}_d X$ is finite".

So our goal is to treat the case $\mu \le \mu_1$. Let us notify that

$$N(\tau) = N_0(\tau) + N_1(\tau)$$

where

$$N_j(\tau) = \int \psi_j(x')e(x,x,\tau)dx,$$

$e(x,y,\tau)$ is the Schwartz kernel of the spectral projector of A, $\psi_0 \in C_0^K$ and $\psi_1 = 1 - \psi_0$ is supported in $|x'| > c$. Then there is the standard Weylian asymptotics for $N_0(\tau)$ with the remainder estimate $O(\tau^{(d-1)/2})$ and even $o(\tau^{(d-1)/2})$ provided (H). So we need to obtain asymptotics of $N_1(\tau)$.

Let us change variables in the cusp zone $\{|x'| \ge c\}$: $x'_{\text{new}} = x'$, $x''_{\text{new}} = x''/\rho(x')$; in the new variables X_1 intersected with $\{|x'| \ge c\}$ coincides with cylinder $\mathbb{R}^{d'} \times Y$ and one can treat A as a differential operator with respect to x' with operator-valued symbol acting in the auxiliary

Hilbert space $\mathbf{H} = L^2(Y)$. Moreover let us consider a ball $B(\bar{x}', \gamma(\bar{x}'))$ in $\mathbb{R}^{d'}$ with $\gamma(x') = |x'|/2$. Making dilatation $x'_{\text{new}} = (x' - \bar{x}')/\gamma(\bar{x}')$ and dividing A by τ for $\rho(x') \geq \tau^{-1/2}/C_0$ and by ρ^{-2} otherwise we come in frames of [3] with $h = 1/\gamma(\bar{x}')\tau^{1/2}$, $h = \rho(\bar{x}')/\gamma(\bar{x}')$ respectively and with $B = (h^2\Delta_{YD}+I)^{1/2}$ where δ_{YD} is a positive Dirichlet Laplacian in Y and with $\mathbf{H}_t = (h^{2m}(\Delta^m)_{YN} + 1)^{-t/2m}L^2(Y) \cap D(B^{t'})$ with $t' = \min(1, t_+)$ and with a large enough $m = m_l$ and arbitrary l. Moreover in the second case we are in frames of elliptical situation and in the first case we are in frames of microhyperbolic situation provided

$$D^{\alpha}_{x'}(g^{jk} - \bar{g}^{jk}) = o(|x'|^{-|\alpha'|}) \quad \forall \alpha : |\alpha| \leq K, \tag{10}$$

$$D^{\alpha}_{x'}(\rho(x')|x'|^{\mu}) = o(|x|^{-|\alpha|}) \quad \forall \alpha : |\alpha| \leq K \tag{11}$$

where $\bar{g}^{jk} = \text{const}$.

Applying results §4.3.3 of [4] to γ-admissible partition of unity we obtain

Theorem 3. *Let conditions* (1) – (6) *and* (10) – (11) *be fulfilled. Then*

$$N(\tau) = N^w_0(\tau) + N^{w\prime}_1(\tau) + O(R) \tag{12}$$

where

$$R = \begin{cases} \tau^{(d-1)/2} & \text{for } d'' > (d'-1)\mu \\ \tau^{(d-1)/2}\log\tau & \text{for } d'' = (d'-1)\mu \\ \tau^{(d'-1)(\mu+1)/2\mu)} & \text{for } d'' < (d'-1)\mu \end{cases} \tag{13}$$

where

$$N^{w\prime}_1 = (2\pi) - d' \int \psi_1(x')\mathbf{n}(x', \xi', \tau)dx'd\xi' \tag{14}$$

where $\mathbf{n}(x', \xi', \tau)$ *is an eigenvalue counting function for an operator* $\mathbf{a}(x', \xi')$ *in* $\mathbf{H}$ *with the Dirichlet boundary condition and* $\mathbf{a}(x', \xi')$ *we obtain from operator* A *(in the new co-ordinates) replacing* D' *by* ξ'.

Remark 4. Moreover in frames of theorem 3 with $d'' > (d'-1)\mu$ under condition (H) we can give an asymptotic formula for N with remainder estimate $o(\tau^{(d-1)/2})$. On the other hand in frames of theorem 3 with $d'' < (d'-1)\mu$ under some condition of the global nature we can give an asymptotic formula for N with remainder estimate $o(\tau^{(d'-1)(\mu+1)/2\mu})$. We don't discuss them here for a sake of simplicity.

We are not satisfied by asymptotic formula (12)–(14) because it gives a very complicated answer: we need to calculate an eigenvalue counting function for an auxiliary operator $\mathbf{a}$ depending on parameter (x',ξ'). Now our goal is to give an asymptotic formula in which only an eigenvalue for single auxiliary operator $\bar{\mathbf{a}}$ is included.

In order to do it we treat the different parts o cusp by different way. Let us consider first cusp $\{X, c\le |x'|\le \tau^{1/2\mu-\delta}\}$ with an arbitrary $\delta>0$. Let $\bar{x}'$ be in these frames and let us consider the ball $B(\bar{x}',\bar{\rho})$ with $\bar{\rho}=\rho(\bar{x}')$ and a function $\psi'(x')=\psi^0((x'-\bar{x}')/\rho(\bar{x}'))$ supported in this ball while ψ^0 is a smooth function supported in the ball $B(0,1)$. Then from §4.2 of [4] follows estimate

$$\text{(15)}\quad |\int (F_{t\to\tau}\chi_{T_1}(t)U(x,x,t)-\sum_{0\le n\le N-1}\kappa'_n\tau^{(d-n)/2-1})\psi'(x')dx-$$
$$\int_{\partial X}\sum_{1\le n\le N-1}\tilde{\kappa}'_n\tau^{(d-n)/2-1})\psi'(x')dS|\le C\tau^{(d-N)/2-1}\bar{\rho}^{d-N}$$

where $U(x,y,t)$ is a Schwartz kernel of operator $\exp it\tau^{-1}A$, χ is a smooth function supported in $[-1,1]$ and equal to 1 at $[-1/2,1/2]$, $\chi_T(t)=\chi(t/T)$, $T_1=\epsilon/h_1$, $h_1=\bar{\rho}^{-1}\tau^{-1/2}$, $\epsilon>0$ is a small enough constant and $\kappa'_n,\tilde{\kappa}'_n$ are Weylian coefficients such that

$$\text{(16)}\qquad |\kappa'_n|\le C\bar{\rho}^{-n},\quad |\tilde{\kappa}'_n|\le C\bar{\rho}^{1-n}$$

and N is arbitrary, dS is a Riemannian density at ∂X. Hence in the same frames for ψ' replaced by $\psi(x')=\psi^0((x'-\bar{x}')/\epsilon_1\bar{\gamma}))$ with a small constant $\epsilon_1>0$ and $\bar{\gamma}=\gamma(\bar{x}')$ left-hand expression in (15) doesn't exceed $C\bar{\gamma}^{d'}\bar{\rho}^{d''-N}\tau^{(d-N)/2-1}$.

On the other hand microhyperbolicity property for symbol $\mathbf{a}$ and theorem 2.1.14 [3] yields that if $\tilde{\chi}$ is supported in $[-1,-1/3]\cup[1/3,1]$ then

$$|\int (F_{t\to\tau}\tilde{\chi}_T(t)U(x,x,t)\psi(x')dx|\le C\tau^{-s}$$

with an arbitrary s and $T\in[T_1,T_2]$, $T_2=\epsilon/h_2$, $h_2=\bar{\gamma}^{-1}\tau^{-1/2}$; therefore the left-hand expression of (15) with T_1 and ψ' replaced by T_2 and ψ also doesn't exceed $C\bar{\gamma}^{d'}\bar{\rho}^{d''-N}\tau^{(d-N)/2-1}$. Applying Tauberian arguments we obtain that

$$|\int (e(x,x,\tau)-\sum_{0\le n\le N-1}\kappa_n\tau^{(d-n)/2-1})\psi(x')dx-$$
$$\int_{\partial X}\sum_{1\le n\le N-1}\tilde{\kappa}_n\tau^{(d-n)/2-1}\psi(x')dS|\le C\tau^{(d-N)/2}\bar{\gamma}^{d'-1}\bar{\rho}^{d''-N}$$

where $\kappa_n = \frac{2}{d-n}\kappa'_n$, for $n = d$ we should take logarithmic term etc.

We can sum the this estimate on partition of unity and then we obtain finally that

$$(17)\quad |\int (e(x,x,\tau) - \sum_{0\le n\le N-1} \kappa_n \tau^{(d-n)/2-1})\psi(x')dx -$$
$$\int_{\partial X} \sum_{1\le n\le N-1} \tilde{\kappa}_n \tau^{(d-n)/2-1}\psi(x')dS| \le C\tau^{(d-N)/2}\bar{\gamma}^{d'-1}\bar{\rho}^{d''-N} + R$$

where here $\psi(x') = \psi_1(x'/\zeta) - \psi_1(x')$ with $\zeta \in [c, \tau^{1/2\mu-\delta}]$, $\bar{\rho} = \zeta^{-\mu}$, $\bar{\gamma} = \zeta$, $N \ge 2$ and R is the remainder estimate in theorem 3. The problem is that we know the algorithm how to calculate κ'_n and $\tilde{\kappa}'_n$ with $n \ge 2$ but we don't know the simple formulae for them. There are two ways: we can fix δ such that contribution of all these terms was less than remainder estimate but it will give us extra restrictions when we consider the remaining part of cusp. The second way is to compare this formula with the formula for operator A^0 which in the "cylindrical" co-ordinate system equals

$$\sum_{1\le j,k\le d'} \bar{g}^{jk}D_jD_k + \sum_{d'+1\le j,k\le d} |x'|^{2\mu}\bar{g}^{jk}D_jD_k$$

in the domain $\{x', |x'| \ge c\} \times Y$ with the Dirichlet boundary conditions; the exact answer for this operator can be obtained by separation of variables. Using this exact answer and continuous dependence of the Weyl coefficients we obtain

Proposition 5. *Let conditions* (1) – (6) *and* (10) – (11) *be fulfilled. Moreover let us assume that*

$$(18)\quad D^\alpha(g^{jk} - \bar{g}^{jk}) = O(|x'|^{-|\alpha|-\nu}), \quad D^\alpha b_j = O(|x'|^{\mu-|\alpha|-\nu}),$$
$$D^\alpha c = O(|x'|^{2\mu-|\alpha|-\nu}), \quad \rho = |x'|^{-\mu} \quad \forall\alpha : |\alpha| \le K$$

with $\nu \ge \nu^* = \min(\mu+1, d'(\mu+1)-(d-1)\mu)$, $\bar{g}^{jk} = 0$ *for* $j \le d', k > d'$. *Then the absolute value of the difference between*

$$\int (e(x,x,\tau) - \kappa_0\tau^{d/2})\psi(x')dx - \int_{\partial X} \tilde{\kappa}_1\tau^{(d-1)/2}\psi(x)dS$$

and the same quantity calculated for $\bar{A}$ *doesn't exceed* R.

On the other hand in zone $\{|x'| \ge \zeta = \tau^{1/2\mu-\delta}\}$ the difference between operators A and $\bar{A}$ is small and using this fact and the microhyperbolicity property again one can prove

Proposition 6. *Let conditions* (1)–(6), (10), (11) *and* (17) *be fulfilled with* $\nu > \nu^*$. *Then*

$$(19)\qquad |\int (\mathbf{n}(x',\xi',\tau) - \bar{\mathbf{n}}(x',\xi',\tau))\psi_1(x'/\zeta)dx'd\xi'| \le CR.$$

These two propositions, properties of coefficients $\kappa\dots$ and $\tilde{\kappa}\dots$ and asymptotics of $N_0(\tau)$ yield

Theorem 7. *Let conditions* (1) – (6), (10), (11) *and* (17) *be fulfilled with* $\nu > \nu^*$. *Then*

$$\begin{aligned}(20)\quad N(\tau) = (2\pi)^{-d}\omega_d\tau^{d/2}\int \psi_0(x'/\zeta_1)\sqrt{g}dx- \\ \frac{1}{4}(2\pi)^{1-d}\omega_d\tau^{(d-1)/2}\int_{\partial X}\psi_0(x'/\zeta_1)dS+ \\ (2\pi)^{-d}\omega_d\tau^{d/2}\int(1-\psi_0(x'/\zeta_1))(\sqrt{g}-\sqrt{\bar{g}})dx- \\ \frac{1}{4}(2\pi)^{1-d}\omega_d\tau^{(d-1)/2}\int_{\partial X}(1-\psi_0(x'/\zeta_1))(dS-d\bar{S}+ \\ (2\pi)^{-d'}\iint(1-\psi_0(x'/\zeta_1))\bar{\mathbf{n}}(x',\xi',\tau)dx'd\xi' + O(R)\end{aligned}$$

where ψ_0 *is arbitrary compactly supported function equal* 1 *in the neighbourhood of* 0, $\zeta \in [C,\tau^\delta$, $\delta > 0$ *is a small enough exponent,* $g^{-1} = \det(g^{jk})$, *bar means that the corresponding object are calculated for operator* $\bar{A}$ *and* R *is given by* (13).

Remark 9. (i) It is easy to calculate the last term in the right-hand expression in (20) in terms of an eigenvalue counting function of operator

$$\Lambda = \sum_{d'<j,k\le d} \bar{g}^{jk}D_jD_k$$

in Y with the Dirichlet boundary condition.

(ii) In frames of theorem 7 with $d'' > (d'-1)\mu$ and condition (H) one can replace remainder estimate $O(\tau^{(d-1)/2})$ by $o(\tau^{(d-1)/2})$ provided $\zeta_1 \to \infty$ as $\tau \to \infty$. On the other hand in frames of theorem 7 with $d'' < (d'-1)\mu$ and certain condition of the global nature (which is equivalent to "$\mu \ne 1$ and all the eigenvalues of Λ are simple") one can replace remainder estimate $O(\tau^{(d'-1)(\mu+1)/2\mu})$ by $o(\tau^{(d'-1)(\mu+1)/2\mu})$.

(iii) If $\rho(x')$ coincides with $|x'|^{-\mu}$ only asymptotically (but quickly enough) one can obtain equality $\rho(x') = |x'|^{-\mu}$ by an appropriate change of variables.

2. Let us consider now Schrödiger operator

$$A = \Delta + V(x)$$

in $\mathbf{R}^d$ with a potential $V = V_0 + V_1$ where

$$(21)_0 \qquad |D^\alpha V_0| \le c\delta(x)^{(2n-|\alpha|)_+}(|x|+1)^{2m-2n-(2n-|\alpha|)_-},$$

$$(21)_1 \qquad |D^\alpha V_1| \le c(|x|+1)^{2q-|\alpha|} \quad \forall\alpha : |\alpha| \le K,$$

$$(22) \qquad V_0 \ge c^{-1}\delta(x)^{2n}(|x|+1)^{2m-2n},$$

(23) $\delta(x) = \mathrm{dist}(x, \Xi)$, $\quad \Xi \subset \mathbf{R}^d$ is a conical C^K-manifold,

$$(24) \qquad q \ge 0, \quad n > 0, m \ge n + q;$$

the last inequality means that the diameter of the canyon $\{V(x) \le \tau\}$ tends to 0 at infinity for every τ. Then the standard Weylian theory [5] yields

Theorem 10. *Let conditions* (21) – (24) *be fulfilled. Let us assume that* $m > q + n + qn$ *and* $qdn + pq + dn < pm$ *where* $p = \mathrm{codim}\,\Xi$. *Then*

$$(25) \qquad N(\tau) = N^w(\tau) + 0(R^w) \quad \text{as } \tau \to \infty$$

with

$$(26) \qquad R^w = \begin{cases} \tau^{(d-1)(m+1)/2m} & for \quad m(p-1) > n(d-1) \\ \tau^{(d-1)(m+1)/2m}\log\tau & for \quad m(p-1) = n(d-1) \\ \tau^{(d-p)(m+1)/2(m-n)} & for \quad m(p-1) < n(d-1), \end{cases}$$

$$(27) \qquad N^w(\tau) \asymp \begin{cases} \tau^{d(m+1)/2m} & for \quad mp > nd \\ \tau^{d(m+1)/2m}\log\tau & for \quad mp = nd \\ \tau^{(d-p)(m+1)/2(m-n)} & for \quad mp < nd \end{cases}$$

where in the case $mp \le nd$ *we restrict a domain integration in the standard definition of* $N^w(\tau)$ *by inequality* $|x| \le C_0\tau^{(n+1)/2(m-n)}$ *with a large enough constant* C_0.

Remark 11. In frames of this theorem with $m(p-1) > n(d-1)$ under stabilizing condition for V_0 and a standard condition to Hamiltonian flow remainder estimate $o(\tau^{d-1)(m+1)/2m})$ holds [5].

Let us assume now that

$$(28) \qquad m > q + n + q, \quad m(p-1) \le n(d-1).$$

Let us change variables x such that $\Xi = \{x'' = 0\}$; then in the new variables the coefficients of the senior part of A are not necessarily constant. Let us replace D' by ξ'in A; then we obtain operator-valued symbol $\mathbf{a}(x', \xi')$ in the auxiliary Hilbert space $\mathsf{H} = L^2(\mathbb{R}^p)$; let $\mathbf{n}(x', \xi')$ be its eigenvalue counting function.

Then the methods similar to used in the first part yield

Theorem 12. *Let conditions* (21) – (24) *and* (28) *be fulfilled and let us assume that*

$$(29) \quad < x', \partial_{x'} > V \ge c^{-1}\delta(x)^{2n}|x|^{2m-2n} - C|x|^{2q} \qquad \text{for} |x| \ge c, \delta(x) \le c^{-1}|x|.$$

Then

$$(30) \quad N(\tau) = (2\pi)^{-d}\omega_d \int \psi(x'/\zeta)(\tau - V)_+^{d/2}\,dx + (2\pi)^{p-d}\iint (1 - \psi(x'/\zeta))\mathbf{n}(x', \xi')\,dx'd\xi' + O(R)$$

where

$$(31) \qquad R = \begin{cases} \tau^{(d-1)(m+1)/2m} & for \quad mp > n(d-1) \\ \tau^{(d-1)(m+1)/2m}\log\tau & for \quad mp = n(d-1) \\ \tau^{(d-p)(m+1)/2(m-n)} & for \quad mp < n(d-1), \end{cases}$$

$\zeta \in [C\tau^{1/2m}, C\tau^{1/2m+\delta}]$ *is arbitrary,* $\delta > 0$ *is small enough,* $\psi \in C_0^K$ *equals to* 1 *in the neighbourhood of* 0 *and asymptotic formula* (27) *remains true for* $N(\tau)$.

All the comments to theorem 3 remain true and under certain additional conditions we can obtain the similar asymptotical formulae in which an auxiliary operator $\mathbf{a}$ depends on the parameter of the smaller dimension (normally we obtain $\dim\Xi - 1$-dimensional parameter instead of $2\dim\Xi$-dimensional).

3. Eigenvalue asymptotics with accurate remainder estimates for maximally hypoelliptic operators with symplectic characteristics are presented in [6].

References

[1]V.Ivrii, S.Fedorova. Dilatations and the asymptotics of the eigenvalues of spectral problems with singularities. Funct. Anal. and Appl., 20, no 4, 277-281 (1986).

[2] V.Ivrii, E.Filippov, A.Kachalkina. Spectral asymptotics with accurate remainder estimates. Intern. Conf. Integral Equat. and Inverse Probl., Varna, 1989, Pitman Res. Notes in Math. Sci., Longman Sci.& Tech. (to appear).

[3] V.Ivrii, Semiclassical microlocal analysis and precise spectral asymptotics. Preprint 1. Ecole Polytechnique, Preprint M964.1190, November 1990.

[4] V.Ivrii, Semiclassical microlocal analysis and precise spectral asymptotics. Preprint 2. Ecole Polytechnique, Preprint M969.0191, January 1991.

[5]V.Ivrii, Estimations pour le nombre de valeurs propres negatives de l'opera- teurs de Schrödinger avec potentiels singuliesrs. C.R.A.S. Paris , sér 1, 302, no 13, p.467–470, no 14, p.491–494, no 15, p.535–538 (1986)

[6] V.Ivrii, Precise eigenvalue asymptotics for transversally elliptic operators. Curent Topics in Partial Differential Equations, Kunokuniya Co Ltd., Tokyo, 1985.

CENTRE MATHEMATIQUE
ECOLE POLYTECHNIQUE
91128 Palaiseau Cedex