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A directional compactification of the complex Fermi surface and isospectrality


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A DIRECTIONAL COMPACTIFICATION OF THE COMPLEX FERMI SURFACE AND ISOSPECTRALITY

D. BÄTTIG
1. Introduction and Theorems:

The content of this report is joint work with H. Knörrer and E. Trubowitz (ETH-Zürich, Switzerland), [BKT].

We consider a lattice \( \Gamma \subset \mathbb{R}^3 \) of maximal rank and \( L^2_{\mathbb{R}}(\mathbb{R}^3/\Gamma) \) the Hilbert-space of square-integrable real-valued functions on the torus \( \mathbb{R}^3/\Gamma \). Let \( q \) be in \( L^2_{\mathbb{R}}(\mathbb{R}^3/\Gamma) \).

For each \( k \in \mathbb{R}^3 \) the self-adjoint boundary value problem

\[
(-\Delta + q(x))\psi(x) = \lambda \psi(x)
\]

\[
\psi(x + \gamma) = e^{i(k,\gamma)} \psi(x) \quad \text{for all} \quad \gamma \in \Gamma
\]

has discrete spectrum, customarily denoted by

\[ E_1(k) \leq E_2(k) \leq E_3(k) \leq \cdots \]

The eigenvalue \( E_n(k), n \geq 1 \), defines a function of \( k \) called the \( n \)-th band function. It is continuous and periodic with respect to the lattice \( \Gamma^d := \{ b \in \mathbb{R}^3/(\gamma, b) \in 2\pi \mathbb{Z} \quad \text{for all} \quad \gamma \in \Gamma \} \).

dual to \( \Gamma \).

The physical Fermi surface for energy \( \lambda \) is the set

\[ F_{\text{phys},\lambda}(q) := \{ k \in \mathbb{R}^3 / E_n(k) = \lambda \quad \text{for some} \quad n \geq 1 \} \]

For example, if \( q(x) \) constant, then \( F_{\text{phys},\lambda}(q) \) is the union of the spheres

\[ \{ k \in \mathbb{R}^3 / (k_1 + b_1)^2 + (k_2 + b_2)^2 + (k_3 + b_3)^2 = \lambda - \text{constant} \} \]

with \( b = (b_1, b_2, b_3) \in \Gamma^d \).

**Theorem 1.** If \( q \) is in \( L^2_{\mathbb{R}}(\mathbb{R}^3/\Gamma) \) and if for a single \( \lambda \) in \( \mathbb{R} \) one of the components of \( F_{\text{phys},\lambda}(q) \) is a sphere (not necessarily centered at a point of the dual lattice), then \( q \) is constant.

Actually the same conclusion holds if \( F_{\text{phys},\lambda}(q) \) contains an algebraic component \( X \), which fulfills certain assumptions, (see section 3). These assumptions are fulfilled if \( X \) is an ellipsoid.

To prove Theorem 1 we complexify the Fermi surface. The (lifted) complex Fermi surface is defined by \( F_{\lambda}(q) := \{ k \in \mathbb{C}^3 / \text{there exists a non trivial solution} \psi \text{ in} H^2_{\text{loc}}(\mathbb{R}^3) \text{of} \ (-\Delta + q(x))\psi(x) = \lambda \psi(x) \text{ satisfying} \psi(x + \gamma) = e^{i(k,\gamma)} \psi(x) \text{ for all} \gamma \in \Gamma \} \).

Clearly, the dual lattice \( \Gamma^d \) acts on \( F_{\lambda}(q) \) by \( k \mapsto k + b \), be \( \Gamma^d \). Furthermore we have \( F_{\lambda}(q) \cap \mathbb{R}^3 = F_{\text{phys},\lambda}(q) \).

It is easy to show, using regularized determinants (see [KT]), that \( F_{\lambda}(q) \) is a complex analytic hypersurface in \( \mathbb{C}^3 \). The main purpose is to construct a directional compactification of \( F_{\lambda}(q) \) in the sense of [KT]. The above theorem follows from the analysis of the points added at "infinity".

To compactify \( F_{\lambda}(q) \) we first embed \( \mathbb{C}^3 \) in a quadric \( Q \) lying in \( \mathbb{P}^4 \). For each affine line \( g = \{ c + tb/t \in \mathbb{R} \} \) in \( \mathbb{R}^3 \), where \( b, c \in \Gamma^d \) and \( b \) is primitive, we blow-up two distinguished points of \( \mathbb{P}^4 \) that lie on the quadric \( Q \), to get, by using inverse limits, a space \( \mathcal{M} \). Denote by \( E_1(g) \) and \( E_2(g) \) the corresponding exceptional divisors.
Theorem 2.— The directional closure of $F_{\lambda}(q)$ in the space $M$ intersects $E_1(g)$ and $E_2(g)$ along curves both of which are isomorphic to the one-dimensional Bloch-variety

$$B(g) \quad \text{where} \quad q_g(x) = \sum_{n=-\infty}^{\infty} \hat{q}(n) e^{i(n, x)}, x \in g.$$

Here $\hat{q}(b)$ is the Fourier-coefficient $\int_{\mathbb{R}^3/\Gamma} q(x) e^{-i(b, x)} dx (b \in \Gamma^4$, without loss of generality we assume that $\mathbb{R}^3/\Gamma$ has volume one). Recall that in [KT] the complex one dimensional Bloch-variety for $p(x) \in L^2(\mathbb{R}/|b|\mathbb{Z})$ is

$$B(p) = \{(k, \lambda) \in \mathbb{C} \times \mathbb{C} / \text{there is a non-trivial function } \psi \text{ in } H^2_{\text{loc}}(\mathbb{R}) \text{ satisfying } -\psi''(x) + p(x) \psi(x) = \lambda \psi(x) \text{ and } \psi(x + |b|n) = e^{ik|b|n} \psi(x) \text{ for all } n \in \mathbb{Z}\}.$$

2. Sketch of the proof of Theorem 2

First we construct a compactification of $\mathbb{C}^3$, which serves as the ambient space for the directional compactification of $F_{\lambda}(q)$. This compactification will be independent of $q$. It’s construction is motivated by considering the free Fermi-surface $F_{\lambda}(0)$. $F_{\lambda}(0)$ is the union of the quadrics

$$\{k \in \mathbb{C}^3/(k_1 + b_1)^2 + (k_2 + b_2)^2 + (k_3 + b_3)^2 = \lambda \}, \quad b = (b_1, b_2, b_3) \in \Gamma^4.$$

If we compactify $\mathbb{C}^3$ in the naive way to $\mathbb{P}^3$ or $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ we would have to perform many blow-up’s before the components of $F_{\lambda}(0)$ are in general position at infinity. Instead we embed $\mathbb{C}^3$ in the complex projective 3-dimensional nonsingular quadric

$$Q := \{(k_1, k_2, k_3, y, z) \in \mathbb{P}^4/yz = k_1^2 + k_2^2 + k_3^2\}$$

by mapping $(k_1, k_2, k_3)$ to $(k_1, k_2, k_3, k_1^2 + k_2^2 + k_3^2, 1)$.

The image of the embedding is the complement of

$$Q_\infty := \{(k_1, k_2, k_3, y, z) \in Q/yz = 0\}.$$

The closures of the components of $F_{\lambda}(0)$ in $Q$ are the intersections of $Q$ with the hyperplanes $H_b$ in $\mathbb{P}^4$ given by

$$y + 2(k, b) + (b^2 - \lambda)z = 0, \quad b \in \Gamma^4.$$

If $b \neq b'$, then $H_b \cap H_{b'}$ is a plane in $\mathbb{P}^4$. It intersects $Q_\infty$ in the set $D_{b, b'}$, consisting of two points, given by the equations

$$z = 0, k_1^2 + k_2^2 + k_3^2 = 0, (k, b - b') = 0, \quad y + 2(k, b) = 0.$$

One checks that $D_{b, b'}$ and $D_{b', b''}$ are disjoint if $b, b', b'', b'''$ do not lie on a line and that $D_{b, b'} = D_{b', b''}$ if these four points of $\Gamma^4$ are on a line. Thus we can denote the points $D_{b, b'}$ by $D(g)$, where $g$ is the affine line through $b$ and $b'$. The group $\Gamma^4$ acts by translation on $\mathbb{C}^3$. This action extends to $Q$ and it maps $D(g)$ to $D(c + g)$ for $c \in \Gamma^4$.

IV-2
If $b$ and $b' \in \Gamma^4$ are different points on the line $g = c_1 + R c_2$ ($c_i \in \Gamma^4$) then $Q \cap H_b$ and $Q \cap H_{b'}$ have different tangent planes in the points of $D(g)$. Therefore we can separate $Q \cap H_b$ and $Q \cap H_{b'}$ by blowing-up the points of $D(g)$. Precisely, for each line $g = c_1 + R c_2$ ($c_i \in \Gamma^4$), let $\mathcal{M}(g)$ be the space obtained from $\mathbb{P}^4$ by blowing-up the points of $D(g)$, $Q(g)$ the strict transform of $Q$ in $\mathcal{M}(g)$ and $E_1(g)$, $E_2(g)$ the two exceptional divisors over the two points of $D(g)$. As compactification $\mathcal{M}$ of $C^4$ we take the inverse limit of all the spaces $\mathcal{M}(G)$, where $G$ is a finite set of affine lines and $\mathcal{M}(G)$ is obtained from $\mathbb{P}^4$ by blowing-up the points of $\cup_{g \in G} D(g)$, defined by the natural maps $\mathcal{M}(G_1) \rightarrow \mathcal{M}(G_2)$ for $G_2 \subset G_1$.

Using the action of $\Gamma^4$ we consider $\mathcal{M}(g)$ where $g$ passes through the origin, and after rotating and scaling we further assume that $g = t(1, 0, 0)$.

Then

$$D(g) = \{(0, \pm i, 1, 0, 0) \in \mathbb{P}^4\}$$

Consider now the exceptional divisor $E_1 := E_1(g)$ lying above the point $(0, i, 1, 0, 0)$, the other divisor is treated similarly. Near this point we take coordinates $(\frac{k_1}{k_3}, \frac{k_2}{k_3} - i, \frac{y}{k_3}, \frac{z}{k_3})$. In $\mathcal{M}(g)$ we have coordinates $(\ell_1, \ell_2, y', z)$ such that

$$\frac{k_1}{k_3} = z \ell_1, \frac{k_2}{k_3} - i = z \ell_2, \frac{y}{k_3} = z y', \quad k_3 = \frac{1}{z}$$

For convenience we perform the change of variables

$$y' = -\mu + \ell_1^2 + \lambda$$

In these coordinates the blow-up map $\pi : \mathcal{M}(g) \rightarrow \mathbb{P}^4$ is

$$k_1 = \ell_1, k_2 = \ell_2 + \frac{i}{z}, y = -\mu + \ell_1^2 + \lambda, \quad k_3 = \frac{1}{z}.$$

$Q(g)$ intersects $E_1$ in the plane $z = \ell_2 = 0$. The strict transform of the hyperplane $H_b, b \in \Gamma^4$, does not meet $E_1$ if $b_2 \neq 0$ or $b_3 \neq 0$. Further, the strict transform of $H_{(b_1, 0, 0)}$ intersects $E_1$ in

$$(\ell_1 + b_1)^2 - \mu = 0$$

Remember that the strict transform of $Q \cap H_b$ is the closure of a component of the free Fermi-surface $F_\lambda(0)$, and that the one-dimensional Bloch-variety for potential zero is

$$\cup_{n \in \mathbb{Z}} \{(\ell, \mu) \in \mathbb{C} \times \mathbb{C} / (\ell + n)^2 - \mu = 0\}.$$

This shows that for $q \equiv 0$ the union of the closures of the components of $F_\lambda(0)$ meets $E_1 \cap Q(g)$ along a curve isomorphic to the one-dimensional Bloch-variety for potential zero. Observe however that the closure of $F_\lambda(0)$ in $Q(g)$ is bigger than the union of the closures of its components. This indicates that it is necessary for the general case to restrict the way one takes limits to $E_1$, i.e. the directional closure in Theorem 2 is made precise by introducing a subset $\Sigma(g)$ of $\mathbb{C}^4$ such that the closure of $F_\lambda(q) \cap \Sigma(g)$ in $Q(g)$ intersects $E_1(g)$ and $E_2(g)$ along a curve each isomorphic to the Bloch-variety $B(q_g)$.
An equation for $F_\lambda(q)$ outside of the free Fermi-surface $F_\lambda(0)$ is given by (see [KT]), assuming without loss of generality $\hat{q}(0) = 0$,

$$\det_2(-\Delta_k + q - \lambda 1) \circ (-\Delta_k - \lambda 1)^{-1} = \det_2(\delta_{cb} + \frac{\hat{q}(c - b)}{(k + b)^2 - \lambda}) = 0.$$ 

This determinant can be computed by taking limits of finite principal minors. (It is not difficult to get an equation for $F_\lambda(q)$ on the whole $\mathbb{C}^3$, but to get the notations as small as possible we work with the above equation). In the coordinates $(\ell_1, \ell_2, \mu, z)$ of $\mathcal{M}(g)$ the entries of the matrix for $(-\Delta_k + q - \lambda) \circ (-\Delta_k - \lambda)^{-1}$ are

$$\delta_{cb} + \frac{\hat{q}(c - b)}{(k + b)^2 - \lambda}.$$

Block the matrix in the form

$$\begin{pmatrix}
A(\ell_1, \ell_2, \mu, z) & B(\ell_1, \ell_2, \mu, z) \\
C(\ell_1, \ell_2, \mu, z) & D(\ell_1, \ell_2, \mu, z)
\end{pmatrix} := \mathcal{F}(\ell_1, \ell_2, \mu, z)$$

With this notation $A(\ell_1, \ell_2, \mu, z) = (\delta_{c_1 b_1} + \frac{\hat{q}(c_1 - b_1, 0, 0)}{(\ell_1 + b_1)^2 - \mu}) c_1 b_1 \in \mathbb{Z}$. This is the matrix whose determinant describes the Bloch-variety of the averaged potential $q_g$ outside of $B(0)$. Furthermore on $Q(g) \cap E_1 = \{z = \ell_2 = 0\}$ the matrix $B = 0$ and $D = 1$.

The square of the Hilbert-Schmidt norm of

$$\mathcal{F}(\ell_1, \ell_2, \mu, z) - \mathcal{F}(\ell_1, 0, \mu, 0)$$

is bounded by

$$\|q\|_2^2 \sum_{b \in \mathbb{Z}(1,0,0)} \frac{1}{\ell_1^2 (ib_2 + b_3) + [(\ell_1 + b_1)^2 - \mu + 2\ell_2 b_2 + b_2^2 + b_3^2]^2}$$

Definition:

$$\Sigma(g) := \left\{(\ell_1, \ell_2, \mu, z) \in \mathbb{C}^4 / \sum_{b \in \mathbb{Z}(1,0,0)} \frac{1}{\ell_1^2 (ib_2 + b_3) + [(\ell_1 + b_1)^2 - \mu + 2\ell_2 b_2 + b_2^2 + b_3^2]^2} \right\}$$

$$+ \sum_{b \in \mathbb{Z}(1,0,0)} \frac{|\ell_1 + b_1|^2 + b_2^2}{\ell_1^2 (ib_2 + b_3) + [(\ell_1 + b_1)^2 - \mu + 2\ell_2 b_2 + b_2^2 + b_3^2]^4} < |z|^{1/5}$$

The restriction of $\det_2 \mathcal{F}$ to $\Sigma(g)$ is continous at $z = 0$:

$$\|\mathcal{F}(\ell_1, \ell_2, \mu, z) - \mathcal{F}(\ell_1, 0, \mu, 0)\|_{\text{Hilbert-Schmidt}} = O(\|q\|_2^2 |z|^{1/5})$$

Therefore we have:

$$F_\lambda(q) \cap \Sigma(g) \cap (Q(g) \cap E_1) \subset B(q_g). \quad (1)$$

To prove the converse we need information about the structure of $\Sigma(g)$ in the neighbourhood of any point of $Q(g) \cap E_1$:
Lemma 1.— For every point \( p = (\ell_1^*, \ell_2^*, \mu^*, 0) \) of \( E_1(g) \) and for all \( A > 0 \) there is a neighbourhood \( \mathcal{U} \) of \( p \) in \( \mathcal{M}(g) \) and an open set \( Z \subset \mathbb{C} \) having 0 as a cluster point such that

\[
T := \{ (\ell_1, \ell_2, \mu, z) \in \mathcal{U} / z \in Z, |\ell_2 - \ell_2^*| \leq A|z| \} \subset \Sigma(g)
\]

The proof of Lemma 1 is technical, very long and done by contradiction. One has to estimate the functions in the sums defining \( \Sigma(g) \) outside of little discs centered at

\[
z_1(\ell_1, \mu) := 2i(1 + \frac{(\ell_1 + b_1)^2 - \mu}{b_2^2 + b_3^2})^{-1}(b_2 + ib_3)^{-1}
\]

since

\[
\frac{2}{z}(ib_2 + b_3) + [(\ell_1 + b_1)^2 - \mu + 2\ell_2 b_2 + b_2^2 + b_3^2] =
\]

\[
= (b_2^2 + b_3^2)\frac{2i}{z} - (1 + \frac{(\ell_1 + b_1)^2 - \mu + 2\ell_2 b_2}{b_2^2 + b_3^2})(-b_2 + ib_3)^2.
\]

We do not know if \( \Sigma(g) \) is path-connected, i.e. if \( Z \) is.

Let us fix now a smooth point \( p = (\ell_1^*, 0, \mu^*, 0) \) of \( Q(g) \cap E_1 \cap \mathcal{B}(g_0) \). For simplicity we assume that \( p \) doesn’t lie on the free Bloch-variety \( \mathcal{B}(0) \) in \( Q(g) \cap E_1 \). By Lemma 1 there is a neighbourhood \( \mathcal{U} \) of \( p \) in \( \mathcal{M}(g) \) and an open subset \( Z \subset \mathbb{C} \) having 0 as a cluster point such that \( T \subset \Sigma(g) \).

It is easy to see (using the definition of \( \Sigma(g) \) and the fact that \( \det_2 \) is continuous in Hilbert-Schmidt norm) that we have

Lemma 2.— The restriction of the function

\[
f(\ell_1, \ell_2, \mu, z) := \det_2 F(\ell_1, \ell_2, \mu, z)
\]

to \( \overline{T} \) has the following properties:

i) \( f(p) = 0 \)

ii) There is a constant \( C \), such that

\[
|f(\ell_1, \ell_2, \mu, z) - f(\ell_1, \ell_2, \mu, 0)| \leq C|z|^{1/5}
\]

for all \( z \in Z, (\ell_1, \ell_2, \mu, z) \in \mathcal{U} \)

iii) For any \( z \in \overline{Z} \) the mapping \( f(., z) \) is differentiable and \( (\ell_1, \ell_2, \mu, z) \mapsto (\nabla(\ell_1, \ell_2, \mu) f)(\ell_1, \ell_2, \mu, z) \) is continuous on \( \overline{T} \).

iv) \( \frac{\partial f}{\partial \ell_1}(p) \) and \( \frac{\partial f}{\partial \mu}(p) \) are not both equal to zero.
We apply this lemma as follows:

Since \( Q(g) \) intersects \( E_1 \) transversally, we can choose \((\ell_1, \mu, z)\) as local coordinates on \( Q(g) \cap U =: V \) near \( p \) (observe that there exists an \( A > 0 \) such that \( |\ell_2| \leq A|z| \) for all points near \( p \) in \( Q(g) \)). Assume \( \frac{\partial F}{\partial \ell_1}(p) \neq 0 \) (the other case is treated similarly using \( \frac{\partial e_{1 \ell_1}}{\partial \ell_1}(p) = 0 \)) and consider the continuous mapping

\[
F : V \subset \mathbb{R}^4 \times \mathbb{Z} \to \mathbb{R}^4
\]

defined by

\[
F(\ell_1, \mu, z) := (f(\ell_1, \ell_2(\mu, z), \mu, z), \mu - \mu^*) .
\]

It is not difficult to apply the implicit function theorem to \( F \), by imitating its proof, to get a sequence \(((\ell_1, \mu)_k, z_k)_{k \in \mathbb{N}}\) in \( V \times Z \) with \( z_k \neq 0 \) converging to \(((0, 0), 0)\) such that \( F((\ell_1, \mu)_k, z_k) = 0 \). Therefore \( p \) lies in the closure of the zero-set of \( f \) in \((Q(g)-\text{strict transform of } Q_{\infty}) \cap T\), hence in the closure of \( F_\lambda(q) \cap \Sigma(g) \). From [Bo] one knows, that the equation defining the one-dimensional Bloch-variety \( B(q_g) \) is reduced. So the smooth points are dense in the zero-set of \( \ell_1, \mu, 0 \) and we get

\[
\overline{F_\lambda(q) \cap \Sigma(g) \cap (Q(g) \cap E_1)} \supset B(q_g)
\]

(1) and (2) imply the Theorem 2.

3. Sketch of the proof of Theorem 1

First we claim:

Assume that \( q \) is a real potential and that \( F_\lambda(q) \) contains an algebraic component \( X \). If the closure \( \overline{X} \) of \( X \) in \( Q \) contains of the curves \( \{(k, Y, 0) \in Q_\infty / (k, c) + y = 0\} \) with \( c \in \Gamma^4 \), then \( q \) is constant.

Proof:

For \( b \in \Gamma^4 - \{0\} \) let \( g_b \) be the line \( \{c + tb/t \in \mathbb{R}\} \). Then \( \overline{X} \) contains all the sets \( D(g_b), b \in \Gamma^4 \). By Lemma 1 the closure of \( X \cap \Sigma(g_b) \) in \( Q(g_b) \) meets \( E_1(g_b) \) and \( E_2(g_b) \) along a (non-empty) algebraic curve, namely the intersection of the strict transform of \( X \) with \( E_1(g_b) \) resp. \( E_2(g_b) \). Hence by Theorem 2 the Bloch-varieties of all the averaged potentials \( q_b, b \in \Gamma^4 \) each contain an algebraic component. As each \( q_b \) is real, Borg’s Theorem [Bo] implies that \( q_b \) is constant. Therefore \( q \) is constant. \( \square \)

The assumption of the claim is fulfilled if \( F_\lambda(q) \) contains a sphere around a point of \( \Gamma^4 \). Assume that \( F_\lambda(q)/\Gamma^4 \) is irreducible. Then, if \( X \) where any algebraic component of \( F_\lambda(q) \), by Theorem 2 there would be an affine line \( g \), such that \( X \cap \Sigma(g) \) intersects \( E_i(g)(i = 1, 2) \) along a curve, and one would deduce the fact that \( q \) is constant as above.

Theorem 1’ shows, under further assumptions on \( X \), one does not need the irreducibility of \( F_\lambda(q)/\Gamma^4 \) to conclude Theorem 1.

Theorem 1’:

Let \( q \in L_\mathbb{R}^2(\mathbb{R}^3/\Gamma) \). Assume that \( F_\lambda(q) \) contains an algebraic component \( X \) whose closure \( \overline{X} \subset Q \) is transversal to \( Q_\infty \) at almost every point of \( \overline{X} \cap Q_\infty \). Then \( q \) is constant.
This is the case if for example $X$ is a sphere or an ellipsoid.

For the proof of Theorem 1 it suffices to show that
\[ \overline{X} \cap Q_\infty \subseteq \cup_{b \in \Gamma^4} \{(k, y, 0) \in Q_\infty / \langle k, b \rangle + y = 0\}. \] (\ast)

Let $\mathcal{D} := \{(\kappa_1, \kappa_2, \kappa_3, 1, 0) \in Q_\infty / \text{there are } M, \tau \geq 0 \text{ such that for all } b \in \Gamma^4 - \{0\} \text{ one has} \langle \kappa, b \rangle \geq M|b|^{-\tau}, |\langle \kappa, b \rangle + 1| \geq M|b|^{-\tau}\}.

Then one shows (by blowing up the point $p \in \mathbb{P}^4$ and using the methods to prove the Theorem 2).

**Lemma 3.** Let $q \in L^2(\mathbb{R}^3 / \Gamma)$ and $p = (\kappa, 1, 0) \in \mathcal{D}$. Then there is no algebraic component of $F_\lambda(q)$, whose closure passes through $p$ and is transversal to $Q_\infty$ in this point.

If $C$ is a component of $\overline{X} \cap Q_\infty$ which is not contained in $\cup_{b \in \Gamma^4} \{(k, y, 0) \in Q_\infty / \langle k, b \rangle + y = 0\}$, then $C$ meets $\{(k, y, 0) \in Q_\infty / y = 0\}$ in only finitely many points, i.e.

\[ C' := \{(k, 1, 0) \in Q_\infty / \langle k, 1, 0 \rangle \in C\} \]

is an affine curve and by Lemma 3 $C' \cap \mathcal{D}$ consists of only finitely many points. One shows that this leads to a contradiction:

Let $\mathcal{D}_0$ be the set of points $(y_1, y_2, y_3) \in \mathbb{P}^2(\mathbb{R})$ which fulfil a diophantine estimate
\[ |\langle y, b \rangle| \geq \frac{K}{|b|^\tau} \text{ for all } b \in \Gamma^4 - \{0\} \]
with some $K, \tau \geq 0$. Clearly a point $(k, 1, 0) \in Q_\infty$ with $k \neq 0$ lies in $\mathcal{D}$ if its imaginary part $\text{Im}k$ represents a point of $\mathcal{D}_0$. Consider the map
\[ \pi_0 : C' - \{(0, 1, 0)\} \rightarrow \mathbb{P}^2(\mathbb{R}), (k, 1, 0) \mapsto \text{Im}k. \]

The image of $\pi_0$ intersects $\mathcal{D}_0$ in only finitely many points. On the other hand one easily verifies that $\mathbb{P}^2(\mathbb{R}) - \mathcal{D}_0$ has measure zero. Hence by Sard’s theorem $\pi_0$ does not have maximal rank anywhere. From this one can conclude that $C'$ is contained in a plane. Therefore it exists a $\gamma \in \mathbb{C}^3$ such that
\[ C \subseteq \{(k, y, 0) \in Q_\infty / \langle k, \gamma \rangle + y = 0\}. \]

Since $\pi_0$ has rank $\leq 1$, $\gamma$ is either purely real or purely imaginary. We discuss here the case $\gamma \in \mathbb{R}^3$. We may now assume that
\[ C' = \{(k, 1, 0) \in Q_\infty / \langle k, \gamma \rangle + 1 = 0\} = \{(k, 1, 0) \in \mathbb{P}^4 / k_1^2 + k_2^2 + k_3^2 = 0, \langle k, \gamma \rangle + 1 = 0\}. \]

We have to show : $\gamma \in \Gamma^4$, i.e. (\ast) is true.

So let $\gamma \notin \Gamma^4$. Consider for $k \in \mathbb{C}^3 - \{0\}$ with $k_1^2 + k_2^2 + k_3^2 = 0$ $v(k)$, the unit vector in $\mathbb{R}^3$ such that $Rek, Imk, v(k)$ form an oriented orthogonal basis.

Put $\mathcal{D}_1 := \{v \in \mathbb{R}^3 / |v| = 1, v \neq \frac{b}{|b|} \text{ for all } b \in \Gamma^4 - \{0\}\}$ and there are only finitely many be $\Gamma^4$ such that $|v - \frac{b}{|b|}| < \frac{1}{|b|^2} \}.

It is easy to see that the complement of $\mathcal{D}_1$ in the unit sphere $S^2$ has Lebesgue measure zero. Further one shows
Lemma 4.— For any $k \in \mathbb{C}^3 - \{0\}$ with $k_1^2 + k_2^2 + k_3^2 = 0$ and $v(k) \in \mathcal{D}_1$ there is a $K' > 0$ such that for all $b \in \Gamma^d - \{0\}$

$$|\langle k, b \rangle| \geq K'|b|^{-2}.$$ 

But the map $C' \to S^2, (k, 1, 0) \mapsto v(k)$ has maximal rank almost everywhere. Therefore for all points $(k, 1, 0)$ outside a set of Lebesgue measure zero in $C'$ there is a $K > 0$ such that $|\langle k, b \rangle| \geq K|b|^{-2}$ for all $b \in \Gamma^d - \{0\}$.

Now the map

$$\pi : C' \to P := \{x \in \mathbb{R}^3/(x, \gamma) + 1 = 0\}, (k, 1, 0) \mapsto \text{Re } k$$

is surjective and submersive. Thus Theorem 1' follows immediately (since then $C' \cap \mathcal{D}$ consists of infinitely many points) from

Lemma 5.— The set of points $x \in P$ for which there is $K, \tau > 0$ such that $|\langle x, b \rangle + 1| \geq K|b|^{-\tau}$ has positive Lebesgue measure.

4. Appendix

It is possible to show that $F_{\lambda}(q)/\Gamma^d$ for split potentials of the form $q(x) = p_1(x_1, x_2) + p_3(x_3)$ for a lattice $\Gamma = a_1 \mathbb{Z} + a_2 \mathbb{Z} + a_3 \mathbb{Z}$ with $\langle a_1, a_3 \rangle = \langle a_2, a_3 \rangle = 0$ is always irreducible. One uses three facts:

i) The Bloch-varieties $B(p_1)$ and $B(p_2)$ are irreducible (see [KT])

ii) The map $\Phi : B(p_1) \times B(p_2) \to B(p_1 + p_2)$ is surjective

iii) Introducing

$$\pi_1^{(\lambda)} : B(p_1) \to \mathbb{C}, (k_1, k_2, \lambda_1) \mapsto \lambda_1 - \frac{\lambda}{2}$$

$$\pi_2^{(\lambda)} : B(p_2) \to \mathbb{C}, (k_3, \lambda_2) \mapsto \frac{\lambda}{2} - \lambda_2$$

the Fermi-surface $F_{\lambda}(q)$ is the fibered product

$$B(p_1) \times_\lambda B(p_2) = \{((k_1, k_2, \lambda_1), (k_3, \lambda_2)) \in B(p_1) \times B(p_2)/\pi_1^{(\lambda)}(k_1, k_2, \lambda_1) = \pi_2^{(\lambda)}(k_3, \lambda_2)\}$$

Therefore we have

Theorem 3.— If $q \in L^2(\mathbb{R}^3/\Gamma)$ and the Fermi-surface $F_{\text{phys},\lambda}(q)$ is the same as $F_{\text{phys},\lambda}(q')$, where $q'$ is a split potential of the above form, then $q$ also splits.

Let us close this report by the remark that for the discrete periodic Schrödinger operator $F_{\lambda}(q)/\Gamma^d$ is always irreducible (see [B]).
Bibliography


