Séminaire Équations aux dérivées partielles - École Polytechnique

A. Melin

The Lippman-Schwinger equation treated as a characteristic Cauchy problem

Séminaire Équations aux dérivées partielles (Polytechnique) (1988-1989), exp. nº 4, p. 1-7
<http://www.numdam.org/item?id=SEDP_1988-1989____A4_0>

CENTRE
DE
MATHEMATIQUES

Unité associée au C.N.R.S. n° 169

ECOLE POLYTECHNIQUE

F-91128 PALAISEAU Cedex (France)
Tél. (1) 69.41.82.00
Télex ECOLEX 691.596 F

Séminaire 1988-1989

ÉQUATIONS AUX DÉRIVÉES PARTIELLES

THE LIPPMAN-SCHWINGER EQUATION TREATED AS A CHARACTERISTIC CAUCHY PROBLEM.

A. MELIN

Exposé n°IV

22 Novembre 1988

THE LIPPMANN-SCHWINGER EQUATION TREATED AS A

CHARACTERISTIC CAUCHY PROBLEM.

by Anders Melin

Introduction.

We shall consider a real-valued function $v \in C^\infty(\mathbf{R}^n)$ when $n > 1$ is odd. In order to have sufficiently regular scattering data associated to the Schrödinger operator $H_v = -\Delta_x + v(x)$ we shall assume that v satisfies the following short-range condition:

$$\int_{\mathbf{R}^n} (1+|x|)^{2-n+|\alpha|}|v^{(\alpha)}(x)|\,dx < \infty \quad \text{for any} \quad \alpha. \tag{1}$$

The class of such potentials will be denoted $\mathcal{V}$. By using polar coordinates in the frequency variables one may write the the Lippmann-Schwinger equation on the form

$$(-\Delta_x + v(x))\phi(x,\theta,k) = k^2\phi(x,\theta,k), \quad x \in \mathbf{R}^n,\ \theta \in S^{n-1},\ k \in \mathbf{R}. \tag{2}$$

One has also to impose some condition on $\phi(x,\theta,k)$ as $|x| \to \infty$ in order to obtain a unique solution of (2). We shall always consider ϕ as a perturbation of the function $\phi_0(x,\theta,k) = e^{ik\langle x,\theta\rangle}$ which solves (2) when $v = 0$. Moreover, ϕ will be a continuous function of $k \in \mathbf{R} \setminus 0$ with a meromorhic extension to the upper half-plane. If $0 < \Im k$ is small then

$$\phi = \phi_0 - (H_0 - k^2)^{-1}(v\phi),$$

where $(H_0 - k^2)^{-1}$ is the L^2- bounded inverse of $H_0 - k^2$. In the case of a compactly supported potential v this leads to the formula

$$\phi(x,\theta,k) - \phi_0(x,\theta,k) = 2^{-1}\left(\frac{4\pi}{ik|x|}\right)^{(n-1)/2} e^{ik|x|}T(k, x/|x|, \theta) + O(|x|^{-(n+1)/2}), \tag{3}$$

where T is the scattering amplitude. We also remark that ϕ can be defined in terms of the distribution kernels of the wave operators $W_\pm = \lim_{t\to\pm\infty} e^{itH_v}e^{-itH_0}$, and one often calls the solutions of (2) generalized eigenfunctions.

In this note we show how ϕ, or rather its Fourier transform w.r.t. the variable k, can be obtained as the solution of a characteristic Cauchy problem for the differential operator $\Delta_x - \partial_t^2 - v(x)$. This viewpoint will give us extra information about ϕ and enables us to prove that

$$e^{-ik\langle x,\theta\rangle}\phi(x,\theta,k) = 1 + \int_0^\infty w_\theta(x,t)e^{ikt}\,dt, \tag{4}$$

where $w_\theta(x,t)$ is a smooth function. In particular we shall recover an identity which is usually referred to as the miracle (cf [N1, N2, C]). We also remark that part of the discussions here can be carried over to the case of more general short range potentials.

Construction of ϕ by means of intertwining operators.
We shall first consider the equation

$$(\Delta_x - \Delta_y - v(x))A_\theta(x,y) = 0. \tag{5}$$

In [M5] it was proved that this equation has a solution which is supported in the set $\langle y-x,\theta\rangle \geq 0$ and given by a series

$$\sum_0^\infty U_{N,\theta}(x,y),$$

where

$$(\Delta_x - \Delta_y)U_{N+1,\theta}(x,y) = v(x)U_N(x,y), \quad U_0(x,y) = \delta(x-y).$$

In order to describe the regularity of the solution one introduces the set $\mathcal{P}_\lambda$ of all semi-norms

$$p(U) = \sup_x \int_{\mathbf{R}^n} e^{-\lambda\langle y-x,\theta\rangle}|(\partial_x+\partial_y)^\alpha(\langle x,\partial_x\rangle + \langle y,\partial_y\rangle)^\beta U(x,y)|\,dy.$$

Then for each v which satisfies (1) there is a $\lambda = \lambda_v \geq 0$ so that

$$\sum_1^\infty p(U_{N,\theta}) < \infty, \quad p \in \mathcal{P}_\lambda. \tag{6}$$

Moreover, for each $m \geq 0$ there is a positive integer $N(m)$ so that

$$\sum_{N(m)}^\infty p(\partial_x^\alpha \partial_y^\beta U_{N,\theta}) < \infty, \quad |\alpha+\beta| \leq m, \ p \in \mathcal{P}_\lambda. \tag{7}$$

In order to make the $U_{N,\theta}$ unique one also has to introduce some conditions at infinity which will exclude from the considerations functions which are constant in the direction of (θ,θ). We shall not discuss these details here.

Next we introduce

$$V_{N,\theta}(x,t) = \int_{\langle y-x,\theta\rangle = t} U_{N,\theta}(x,y)\,dy,$$

and we let $\mathcal{Q}_\lambda$ be the family of semi-norms

$$q(V) = \sup_x \int_0^\infty e^{-\lambda t}|\partial_x^\alpha(\langle x,\partial_x\rangle + t\partial_t)^\beta V(x,t)|\,dt.$$

It follows from (6) and (7) then that

$$\sum_1^\infty q(V_{N,\theta}) < \infty, \quad q \in \mathcal{Q}_\lambda, \tag{6'}$$

and

$$\sum_{N(m)}^{\infty} q(\partial_x^\alpha \partial_t^\beta V_{N,\theta}) < \infty, \quad |\alpha| + \beta \le m, \ q \in \mathcal{Q}_\lambda, \tag{7$'$}$$

It was proved in [M5] that

$$\phi(x,\theta,k) = \int A_\theta(x,y) e^{ik\langle y,\theta\rangle}\, dy.$$

Hence if $V_\theta(x,t) = \sum_0^\infty V_{N,\theta}(x,t)$ we must (in view of the definition of $V_{N,\theta}$) have

$$e^{-ik\langle x,\theta\rangle}\phi(x,\theta,k) = \int e^{itk} V_\theta(x,t)\, dt. \tag{8}$$

It follows from (6)$'$ that the integrand is continuous w.r.t. x and integrable w.r.t. t when $\Im k$ is large enough. Moreover, $t \ge 0$ in the support of V_θ, and $V_\theta(x,t) = \delta(t)$ if $v = 0$.

The main result.

It follows immediately from (6)$'$ that $V_\theta(x,t)$ is a smooth function of x and t when $t > 0$, and the next result implies that one may write $V_\theta(x,t) = \delta(t) + Y_+(t) w_\theta(x,t)$, where Y_+ is the Heaviside function and $w_\theta(x,t)$ is smooth when $x \in \mathbf{R}^n$ and $t \ge 0$.

Theorem 1. *There is a positive number λ such that*

$$\sup_{x,\theta} \int_{+0}^{\infty} e^{-\lambda t} |\partial_x^\alpha \partial_t^\beta V_\theta(x,t)| \langle x\rangle^{-\beta}\, dt < \infty \tag{9}$$

for any α and β. (Here $\langle x\rangle = (1+|x|^2)^{1/2}$.)

We have already seen that $V_\theta(x,t)$ is smooth when $t > 0$, and it follows from (6)$'$ also that we need only consider the integral over the interval $(0,1)$ in (9). Moreover, the estimates (7)$'$ imply that it it suffices to prove a similar result for each of the $V_{N,\theta}$. Hence Theorem 1 results from the following

Theorem 1$'$. *If $N \ge 1$, then*

$$\sup_{x,\theta} \int_{+0}^{1} \langle x\rangle^{-\beta} |\partial_x^\alpha \partial_t^\beta V_{N,\theta}(x,t)|\, dt < \infty. \tag{10}$$

We have now come to the point where one has to study the wave equation. In fact, the equation $(\Delta_x - \Delta_y) U_{N+1,\theta}(x,y) = v(x) U_{N,\theta}(x,y)$ implies that

$$\mathcal{L}_\theta V_{N+1,\theta}(x,t) = v(x) V_{N,\theta}(x,t), \tag{11}$$

if $\mathcal{L}_\theta = \Delta_x - 2\langle\theta,\partial_x\rangle\partial_t$. We observe that $\mathcal{L}_\theta$ is obtained from the wave operator $\Delta_x - (\partial_{t_0})^2$, after the substitution $t_0 = t + \langle x,\theta\rangle$. Hence $t \ge 0$ is a characteristic half-space for $\mathcal{L}_\theta$. We let G_θ be the image of the fundamental solution of $\Delta_x - 2\langle\theta,\partial_x\rangle\partial_t$ with support in the set $t_0 \ge 0$ under the substitution above. Then $t \ge 0$ in the support of G_θ.

We shall consider approximate solutions of the equation $\mathcal{L}_\theta V(x,t) = v(x) V_{N,\theta}(x,t)$, which is solved by $V_{N+1,\theta}$. The construction of such a solution will be similar to the methods of geometrical optics used in microlocal analysis, and an exact solution will then be obtained after convolving the error term $\mathcal{L}_\theta V(x,t) - v(x) V_{N,\theta}(x,t)$ with some fundamental solution Q_θ of $\mathcal{L}_\theta$ with support in the set $t \ge 0$. The following result shows that one has to take $Q_\theta = G_\theta$ if one hopes to obtain good bounds for the solutions.

Proposition 2. *Assume that $\mathcal{L}_\theta u(x,t) = 0$ and that $t \geq 0$ in the support of u. If $u(x,t)$ is temperate w.r.t. x then*

$$u(x,t) = \sum_{|\alpha| \leq \mu(t)} f_\alpha(t) x^\alpha,$$

where $f_\alpha \in \mathcal{D}'(\mathbf{R})$ and the integer valued function $\mu(t)$ is locally bounded.

PROOF: We may assume that $\theta = e_n$. The function $G(x,y) = u(x, y_n - x_n)$ then solves the equation $(\Delta_x - \Delta_y)G(x,y) = 0$, and $y_n \geq x_n$ in its support. The proof of Theorem 3.5 of [M5] shows then that

$$G(x,y) = G(x,y_n) = \sum_0^\infty g_j(x', y_n - x_n) x_n^j,$$

where $x' = (x_1, \ldots, x_{n-1})$ and $g_j \in \mathcal{D}'(\mathbf{R}^n)$. Then

$$u(x,t) = \sum_0^\infty g_j(x',t) x_n^j,$$

where only finitely many of the g_j are $\neq 0$ when t stays in any bounded open set ω. The equation $(\Delta_x - 2\partial_{x_n}\partial_t)u = 0$ implies that

$$\Delta_{x'} g_j(x',t) - 2(j+1)\partial g_{j+1}(x',t)/\partial t + (j+2)(j+1) g_{j+2}(x',t) = 0, \quad j = 0, 1, \ldots.$$

Hence, when t is in any ω as above, then $\Delta_{x'}^N g_j(x',t) = 0$ for any j if N is large enough. Since the g_j are temperate in x', this implies that they are polynomials in this variable and the proposition follows.

Let Γ_0 be the cone $|x| = t_0$ and Γ be its image under the substitution $t_0 = t + \langle x, \theta \rangle$, i.e. Γ is defined by $t = |x| - \langle x, \theta \rangle$. The half-plane $B : t \geq 0$ corresponds to $B_0 : t_0 \geq \langle x, \theta \rangle$, which intersects Γ_0 only along the ray $\{(t_0\theta, t_0); t_0 \geq 0\}$. Hence $-\Gamma_0$ intersects B_0 only along the opposite ray $\{(t_0\theta, t_0); t_0 \leq 0\}$, and any distribution u_0 supported in B_0 and vanishing over a conic neighbourhood of $\gamma = \mathbf{R}_-\theta$ has to vanish identically if it satisfies the wave equation $(\Delta_x - (\partial_{t_0})^2)u_0(x,t_0) = 0$. In fact, if (x,s) is in the wave cone Γ_0, (y,t) belongs to the support of u_0 and $x+y$, $s+t$ belong to bounded sets, then $|y|$ can not tend to infinity unless $y/|y|$ tends to $-\theta$. In (x,t) space this implies that $w = G_\theta * u$ is defined if $t \geq 0$ in the support of u and u vanishes over a conic neighbourhood of the ray $\gamma = \mathbf{R}_-\theta$, and w is the unique solution of the equation $\mathcal{L}_\theta w = u$ in the space of such distributions.

In order to have $G_\theta * u$ defined on a larger space we introduce the following definition:

Definition 3. *$\mathcal{D}'_{G_\theta}$ is the space of all u in $\mathcal{D}'(\mathbf{R}^n \times \mathbf{R})$ such that $\lim_{j\to\infty} G_\theta * (\chi_j u)$ exists for any sequence $\chi_j \in C_0^\infty(\mathbf{R}^n \times \mathbf{R})$ such that $\|\chi_j\|_{L^\infty}$ is bounded, χ_j converges pointwise and $\|\chi_j^{(\alpha)}\|_{L^\infty} \to 0$ as $j \to \infty$ if $\alpha \neq 0$.*

If $u \in \mathcal{D}'_{G_\theta}$, and the sequence χ_j above tends to 1 then we define $G_\theta * u$ as the limit of $G_\theta * (\chi_j u)$. This limit is independent of the choices made. Moreover, $\mathcal{D}'_{G_\theta}$ is invariant under differentiation, and $u \to G_\theta * u$ is a left-inverse for $\mathcal{L}_\theta$ on this space in the sense that $u = G_\theta * \mathcal{L}_\theta u$ when $u \in \mathcal{D}'_{G_\theta}$.

One can show that $u \in \mathcal{D}'_{G_\theta}$ if $t \geq 0$ in its support and if it decays as $|x|^{-1-\varepsilon}$ over some conic neighbourhood of γ for some positive ε. One can even allow less restrictive conditions on the decay of u, however, since we are dealing with potentials in the class $\mathcal{V}$, we shall only consider the following conditions:

Definition 4. *Let $f \in C^\infty(\mathbf{R}^n)$. Then we say that $f \in \mathcal{V}_\theta$ if $af \in \mathcal{V}$ for any $a \in S^0(\mathbf{R}^n)$ such that the support of a is contained in some cone $\varepsilon|x| \leq -\langle x, \theta\rangle$, where $\varepsilon > 0$.*

Here the condition that $a \in S^0(\mathbf{R}^n)$ means that $\langle x\rangle^{|\alpha|} a^{(\alpha)}(x)$ is bounded for any α. It is easy to see that $\mathcal{V}$ and $\mathcal{V}_\theta$ are Fréchet spaces and they are also S^0– modules.

We let $C^\infty(\overline{\mathbf{R}_+}) \otimes \mathcal{V}_\theta$ be the space of smooth maps from $\overline{\mathbf{R}_+}$ to $\mathcal{V}_\theta$. Set $Y_{+,j}(t) = t^j Y_+(j)/j!$. If j is a non-negativ integer, $p = 0$ or 1, then $\mathcal{W}_{\theta,j,p}$ is the space of all functions on the form $Y_{+,j}(t)\langle x\rangle^p U(x,t)$, where $U \in C^\infty(\overline{\mathbf{R}_+}) \otimes \mathcal{V}_\theta$.

Theorem 5. *$\mathcal{L}_\theta$ is bijective from $\mathcal{W}_{\theta,j+1,1}$ to $\mathcal{W}_{\theta,j,0}$ if $j \geq 0$.*

By combining this result with some uniqueness statements obtained from Proposition 2 one can easily prove Theorem 1′ now by induction over N. We leave out these details and discuss instead the proof of the theorem above.

It is clear that $\mathcal{L}_\theta$ maps $\mathcal{W}_{\theta,j+1,1}$ into $\mathcal{W}_{\theta,j,0}$. The injectivity of the map follows since one can show that $\mathcal{W}_{\theta,0,1}$ is contained in $\mathcal{D}'_{G_\theta}$. Hence convolution with G_θ gives us a left inverse.

In order to give the main ideas of the proof of the surjectivity of the map $\mathcal{L}_\theta$ in the theorem we consider the corresponding situation when $j = -1$ so that $Y_{+,j}(t) = \delta(t)$. This leads us to discuss the equation

$$\mathcal{L}_\theta V(x,t) = v(x)\delta(t) \tag{12}$$

when $v \in \mathcal{V}$. We first construct an approximate solution. We set

$$v_j(x) = (2^{j+1} j!)^{-1} \int_0^\infty \Delta_x^j v(x - s\theta)\, ds.$$

Then

$$\begin{aligned} &2\langle \theta, \partial_x\rangle v_j(x) = \Delta_x v_{j-1}(x), \quad j > 0, \\ &2\langle \theta, \partial_x\rangle v_0(x) = v(x), \end{aligned} \tag{13}$$

and $\langle x\rangle^{-1} v_j(x) \in \mathcal{V}_\theta$.

Choose $\zeta(t) \in C_0^\infty(\mathbf{R}^n)$ such that $\zeta(t) = 1$ in a neighbourhood of the origin. If the sequence $1 \leq L_j$ grows sufficiently fast, then it is true that the series

$$w(x,t) = \sum_0^\infty Y_{+,j}(t)\zeta(L_j t) v_j(x)$$

converges in C^∞ and defines an element in $\mathcal{W}_{\theta,0,1}$. Moreover, it follows from (13) that

$$r(x,t) = \mathcal{L}_\theta w(x,t) - v(x)\delta(t)$$

is a smooth function of t with values in $\mathcal{V}_\theta$. Moreover, it vanishes when $t \leq 0$. Since the dimension is odd, one has a simple explicit formula for G_θ which allows one to conclude that $w_r(x,t) = (G_\theta * r)(x,t)$ is a smooth function of t with values in $\mathcal{V}_\theta$ after multiplication by $\langle x\rangle^{-1}$. Hence by subtraction w_r from w we have obtained a solution $V(x,t)$ of (12) such that $V \in \mathcal{W}_{\theta,0,1}$.
Remark. The proof shows that $V_\theta(x,\varepsilon) \to v_0(x)$ as $\varepsilon \to 0$. Hence it follows from (13) that

$$2\langle\theta,\partial_x\rangle V_\theta(x,\varepsilon) \to v(x) \quad \text{as} \quad \varepsilon \to 0.$$

This phenomenon was discovered by R.G. Newton and called the miracle by him ([N1, N2]).

Remarks about the case of exponentially decaying potentials.

We shall finally discuss a situation when the potential is exponentially decreasing. Let a be a positive number and assume that one has the estimates

$$|v^{(\alpha)}(x)| \leq C_\alpha e^{-(2a+\varepsilon)|x|} \tag{14}$$

for every α and some positive ε. In this case it turns out that $V_\theta(x,t)$ will be exponentially decaying in the variable t except for some contributions to V_θ that are due to bound states and resonances:

Theorem 6. *Assume that v satisfies (14). Then there is a finite set $Z \subset \{k \in \mathbf{C}; \Im k \geq -a\}$ so that*

$$V_\theta(x,t) = \delta(t) + Y_+(t)a(x,\theta,t) + \sum_{z\in Z}\sum_{\mu\leq\mu(z)} t^\mu e^{-itz} a_{z,\mu}(x,\theta), \tag{15}$$

where for some constants $C_{\alpha,\beta}$ and C_α

$$\int_0^\infty |\partial_x^\alpha \partial_t^\beta a(x,\theta,t)| e^{at}\, dt \leq C_{\alpha,\beta}\langle x\rangle^\beta e^{a(|x|-\langle x,\theta\rangle)},$$

and

$$|a_{z,\mu}^{(\alpha)}(x,\theta)| \leq C_\alpha e^{a(|x|-\langle x,\theta\rangle)}.$$

All estimates are uniform in θ.

Remark. It is also possible to prove smoothness w.r.t. θ.

REFERENCES

[C] *Cheney, M.*, Inverse scattering in dimension 2. J. Math. Phys. 25(1984), 94-107.

[F] *Faddeev, L.D.*, Inverse problems in quantum scattering theory, II. J. Sov. Math., 5(1976), 334-396.

[M1] —"—,Méthodes d'entrelacements dans le problème de scattering inverse. Journées équations aus dérivées partielles, Saint -Jean -de Monts, 1986.

[M2] —"—, Some problems in inverse scattering theory. Journées équations aus dérivées partielles, Saint -Jean -de Monts, 1987.

[M3] —"—, Some mathematical problems in inverse potential scattering. Séminaire sur les équations aus dérivées partielles 1986-1987, Ecole polytechnique, no XX, 1987.

[M4] —"—, Intertwining methods in the theory of inverse scattering. Int. J. of Quantum Chemistry 31 (1987), 739-746.

[M5] —"—,Intertwining methods in multi-dimensional scattering theory I. Univ. of Lund and Lund Inst. of Tech, Dept of Math. preprint series, 1987:13.

[N1] *Newton, R.G.*, Inverse scattering II. Three dimensions. J. Math. Phys. 21(1980), 1698-1715.

[N2] —"—,An inverse spectral theorem in three dimensions. SIAM–AMS Proceedings 14, 81-90.

Anders Melin
Department of Mathematics
University of Lund
Box 118,S–221 00 Lund, Sweden.