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ÉQUATIONS AUX DÉRIVÉES PARTIELLES

INVERSE SPECTRAL PROBLEM FOR THE
SCHRÖDINGER OPERATOR WITH PERIODIC
MAGNETIC AND ELECTRIC POTENTIALS.

G. ESKIN

Consider the Schrödinger operator

$$(1) \quad H = \left(i \frac{\partial}{\partial x_1} + A_1(x) \right)^2 + \left(i \frac{\partial}{\partial x_2} + A_2(x) \right)^2 + V(x),$$

where $x = (x_1, x_2) \in \mathbf{R}^2$, A_1, A_2 and V are periodic in x with respect to some lattice L . $\vec{A}(x) = (A_1(x), A_2(x))$ is called the magnetic (vector) potential, $V(x)$ is the electric (scalar) potential and

$$(2) \quad B(x) = \text{curl} \vec{A} = \frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1}$$

is called the magnetic field.

The Schrödinger equation

$$(3) \quad H\psi = \lambda\psi$$

describes the spectrum of the electron in the periodic electromagnetic field (see [1]). Denote by $\text{Spec}_0 H$ the periodic spectrum of H i.e. when the eigenfunctions $\psi(x)$ are periodic :

$$\psi(x + d) = \psi(x), \forall d \in L .$$

We shall study the following problem : **Recover $B(x)$ and $V(x)$ from $\text{Spec}_0 H$.**

We shall assume that

$$(4) \quad \text{div} \vec{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} = 0$$

It follows from (2) that

$$(5) \quad \int \int_{T^2} B(x) dx_1 dx_2 = 0 ,$$

where $T^2 = \mathbf{R}^2/L$. Using (5) and the Fourier series expansions one can easily show that there are unique A_1, A_2 satisfying (2) and (4) and such that

$$(6) \quad \int \int_{T^2} A_k(x) dx_1 dx_2 = 0, k = 1, 2 .$$

Therefore the problem of recovering $B(x)$ and $V(x)$ from $\text{Spec}_0 H$ is equivalent to the recovering of A_1, A_2 and $V(x)$ where A_1, A_2 satisfy (4) and (6). This work is a continuation of Eskin-Ralston-Trubowitz (see [2] and [3]) where the case $\vec{A} = 0$ was studied. We shall use some results and constructions from [2] and [3]. However the case when $\vec{A} \neq 0$ requires new methods.

Denote by $\text{Spec}_k H$ the Floquet spectrum of H , i.e.

$$(7) \quad H\varphi_n(x) = \lambda_n(k)\varphi_n(x), k \in \mathbf{R}^2/L' ,$$

where $\varphi_n(x + d) = e^{2\pi i k \cdot d} \varphi_n(x), \forall d \in L, L'$ is the dual lattice.

Repeating the proof of Theorem 6.2 in [2] we obtain :

Theorem 1.— Assume that $\vec{A}(x)$ and $V(x)$ are real analytic and the lattice L has the following property :

$$(8) \quad |d| = |d'| \quad \text{implies} \quad d = \pm d' \quad \text{for any} \quad d, d' \in L .$$

Then $\text{Spec}_0 H$ determines $\text{Spec}_k H$ for any $k \in \mathbf{R}^2/L'$.

As in [2] denote by S the set of all “directions” in L' i.e. for any $\delta \in L'$ there is $\delta_0 \in S$ such that $\delta = m\delta_0$, $m \in \mathbf{Z}$ and $k\delta_0 \notin S$ for any $k \neq 1$.

Any periodic function $\vec{A}(x)$ has the following decomposition

$$(9) \quad \vec{A}(x) = \sum_{\delta \in S} \vec{A}_\delta \left(\frac{x \cdot \delta}{|\delta|} \right),$$

where

$$\vec{A}_\delta(s) = \sum_{n=-\infty}^{\infty} \vec{a}_{\delta n} e^{2\pi i n |\delta| s},$$

$$\vec{a}_{\delta n} = \frac{1}{|T^2|} \int \int_{T^2} \vec{A}(x) e^{-2\pi i n (\delta \cdot x)} dx,$$

$|T^2|$ is the area of $T^2 = \mathbf{R}^2/L$.

Talk arbitrary $\delta_0 \in S$. There is a basis $(d_0, d^{(0)})$ in L such that $d_0 \cdot \delta_0 = 0$, $d^{(0)} \cdot \delta_0 = 1$. Denote

$$(10) \quad A_{\delta_0}(s) = \vec{A}_{\delta_0}(s) \cdot \frac{d_0}{|d_0|}.$$

The following theorem holds :

Theorem 2.— Knowing the Floquet spectrum $\text{Spec}_k H$ for all $k \in \mathbf{R}^2/L$ one can recover the following integrals

$$(11) \quad H_{\delta_0}(\mu) = \frac{1}{\sqrt{2}} \int_0^{|\delta_0|^{-1}} \frac{ds}{\sqrt{\mu + 4A_{\delta_0}(s)}}$$

for all $\delta_0 \in S$ and $\mu > -4 \min_s A_{\delta_0}(s)$.

The proof of Theorem 2 is based on the study of the asymptotics of the Green function for the nonstationary Schrödinger equation

$$(12) \quad i \frac{\partial G}{\partial x_0} = H G(x, y, x_0), x_0 > 0 ,$$

$$(12') \quad G(x, y, 0) = \delta(x - y), x \in \mathbf{R}^2, y \in \mathbf{R}^2 .$$

As in [2] it is easy to find out that the Floquet spectrum of H determines the integrals $\int \int_{T^2} G(x+d, x, x_0) dx, dx_2$ for any $d \in L$. Indeed the trace formula gives

$$(13) \quad \sum_{n=1}^{\infty} e^{-i\lambda_n(k)x_0} = \int \int_{T^2} G_k(x, x, x_0) dx,$$

where $G_k(x, y, x_0)$ satisfies (12), (12') for $x \in T^2, y \in T^2$ and the Floquet boundary conditions

$$(14) \quad G_k(x+d, y, x_0) = e^{2\pi i d \cdot k} G_k(x, y, x_0), \forall d \in L.$$

The Green function $G_k(x, y, x_0)$ can be represented in the form

$$(15) \quad G_k(x, y, x_0) = \sum_{d \in L} e^{-2\pi i d \cdot k} G(x+d, y, x_0).$$

Substituting (15) into (13) we get that $\int_{T^2} \int G(x+d, x, x_0) dx$ are the Fourier coefficients of $\sum_{n=1}^{\infty} \exp(-i\lambda_n(k)x_0)$. The main result of the work is the following theorem :

Theorem 3.— *The following asymptotics holds as $N \rightarrow \infty$:*

$$(16) \quad \int \int_{T^2} G(x + Nd_0 + md^{(0)}, x, \frac{\tau_0}{N_1}) dx = \\ = -\frac{iN_1|d_0|}{\pi(2\tau_0^3)^{1/4}} \exp\left(-\frac{iN_1^3}{4\tau_0} - i\frac{N_1}{\sqrt{2\tau_0}} S_0\left(\sqrt{\frac{\tau_0}{2}}\right)\right) (a_0(\tau_0) + o\left(\frac{1}{N_1}\right)),$$

where $N_1 = N|d_0| + m\left(\frac{d_0}{|d_0|} \cdot d^{(0)}\right)$, $m > 0$ is fixed, $N \rightarrow \infty$,

$$(17) \quad \frac{d}{d\tau} S_0(\tau_m) = E(\tau_m),$$

$$(18) \quad \int_0^{|\delta_0|^{-1}} \frac{ds}{\sqrt{E(\tau_m) + 4A_{\delta_0}(s)}} = \frac{\sqrt{2\tau_m}}{m}, \tau_m = \sqrt{\frac{\tau_0}{2}},$$

τ_0 is arbitrary and sufficiently small. There is an explicit expression for $a_0(\tau_0)$ and the further terms in the asymptotic expansion (16) can be found.

The proof of the Theorem 3 consists of the following three steps :

1) Make change of variables

$$(19) \quad s = x \cdot \frac{\delta_0}{|\delta_0|}, t = \frac{d_0}{|d_0|} \cdot x$$

and substitute in (12)

$$(20) \quad G = e^{i\gamma(s,t)} g$$

with appropriate choice of $\gamma(s, t)$ such that $g(s, t, s', t', x_0)$ will satisfy

$$(21) \quad i \frac{\partial}{\partial x_0} g = -\frac{\partial^2 g}{\partial s^2} - \frac{\partial^2 g}{\partial t^2} + 2iA_{\delta_0}(s) \frac{\partial g}{\partial t} + 2iA_3(s, t) \frac{\partial g}{\partial s} + C(s, t)g,$$

where $A_{\delta_0}(s)$ is the same as in (10).

2) Construct $g(s, t, s', t', x_0)$ as a kernel of a Fourier integral operator with nonhomogeneous phase function :

$$(22) \quad g = \frac{1}{(2\pi)^2} \int \int a(x_0 \wedge, s, t, \xi, \eta) e^{-i\wedge L(x_0 \wedge, s, t, s', t', \xi, \eta)} d\xi d\eta,$$

where

$$(23) \quad \wedge = (\xi^2 + \varepsilon_0^4 \eta^4 + \varepsilon_0^{-8})^{\frac{1}{4}},$$

ε_0 is small and fixed, L satisfies the eiconal equation

$$(24) \quad L_\tau - L_s^2 - L_t^2 - 2\wedge^{-1} A_{\delta_0}(s)L_t - 2\wedge^{-1} A_3(s, t)L_s = 0,$$

$$(24') \quad L(0, s, t, s', t', \xi, \eta) = (s - s')\eta \wedge^{-1} + (t - t')\xi \wedge^{-1},$$

$\tau = x_0 \wedge$, $a = a_0 + a_1 + \dots + a_N$ where a_k satisfy corresponding transport equations.

3) Compute the trace of the Fourier integral operator by the stationary phase method. The stationary points form a curve corresponding to the whirling motion of the pendulum

$$(25) \quad \frac{d^2 s}{d\tau^2} - 4 \frac{dA_{\delta_0}(s)}{ds} = 0.$$

Such curves are defined by their period τ_m and they satisfy the following conditions

$$(26) \quad s(\tau_m) = y + m|\delta_0|^{-1}, p(\tau_m) = \eta,$$

where $s(0) = y$, $p(0) = \eta$ are the initial conditions and $p(\tau) = \frac{1}{2} \frac{ds(\tau)}{d\tau}$. For each sufficiently small τ_m there is such a curve and $\tau_m \rightarrow 0$ as $E(\tau_m) \rightarrow \infty$ and vice versa. Here

$$(27) \quad E(\tau_m) = \frac{1}{2} \left(\frac{ds}{d\tau} \right)^2 - 4A_{\delta_0}(s) = 2\eta^2 - 4A_{\delta_0}(y)$$

is the energy. It is enough to consider the case $m = 1$ since $\tau_m = m\tau_1$ and $E(\tau_m) = E(\tau_1)$. The Theorem 2 follows from (18) if we take $\mu = E(\tau_1)$ and denote by $H_{\delta_0}(\mu)$ the function inverse to $E(\tau_1)$.

Remark 1 The asymptotics of the integral $\int \int_{T^2} G(x + Nd_0 + md^{(0)}, \frac{\tau_0}{N_1}) dx$ is more difficult in the case $m = 0$. It requires a Maslov's type global construction of the Fourier integral operators. The stationary points in this case form curves corresponding to the periodic trajectories of the pendulum (25). We didn't consider the case $m = 0$ since the spectral invariants obtained by the asymptotics for $m = 0$ can be easily obtained from (11) by the analytic continuation in μ .

Now we shall apply Theorem 2 and 3 to the inverse spectral problem.

Note that $H_{\delta_0}(\mu)$ can be extended analytically to the whole complex plane μ with the cut along the real axis from $-\infty$ to $-4\min_s A_{\delta_0}(s)$. Using this analytic continuation we can find the following functions

$$(28) \quad H_{\delta_0}^{(1)}(\mu) = \int_{\mu + YA_{\delta_0}(s) < 0} (-\mu - 4A_{\delta_0}(s))^{-\frac{1}{2}} ds ,$$

$$(29) \quad H_{\delta_0}^{(2)}(\mu) = \int_{\mu + 4A_{\delta_0}(s) > 0} (\mu + 4A_{\delta_0}(s))^{\frac{1}{2}} ds .$$

The spectral invariant $H_{\delta_0}^{(2)}(\mu)$ can be used to prove the following theorem :

Theorem 4.— Assume that $A_{\delta_0}(s)$ is even and real analytic. Assume that $A_{\delta_0}(s)$ has $2m$ local maxima and minima : $0, s_0^\pm, \dots, s_{m-1}^\pm$ where $s_{m-1}^+ = \frac{1}{2}|\delta_0|^{-1}$ and $s_k^- = -s_k^+$. Assume that $A_{\delta_0}''(0) \neq 0$ and $A_{\delta_0}(0) \neq A_{\delta_0}(s_k^\pm)$ for $0 \leq k \leq m-1$. Then there is at most $2m$ even real analytic functions having the same spectral invariant $H_{\delta_0}^{(2)}(\mu)$ as $A_{\delta_0}(s)$.

Computing the first term in the asymptotic expansion (16) that depends on $V(x)$ and using Theorem 4 one can prove the following theorem on the rigidity of isospectral deformations :

Theorem 5.— Let $\vec{A}^{(t)}(x) = (A_1^{(t)}(x), A_2^{(t)}(x))$ and $V^{(t)}(x)$ be continuous family of even real analytic magnetic and electric potentials, $0 \leq t \leq 1$. Assume that the lattice L satisfies the condition (8) and $A_{\delta_0}^{(0)}(s)$ for all $\delta_0 \in S$ satisfies the same conditions as in Theorem 4. Assume that the periodic spectrum of $H^{(t)}$ is independent of t , $0 \leq t \leq 1$ where $H^{(t)}$ is the Schrödinger operator corresponding to $\vec{A}^{(t)}(x), V^{(t)}(x)$. Then $\vec{A}^{(t)}(x) = \vec{A}^{(0)}(x), V^{(t)}(x) = V^{(0)}(x)$ for all $t \in (0, 1]$.

The asymptotic formula (16) shows that some kind of quantum mechanical semiclassical asymptotics appears in the direction δ_0 . The same semiclassical nature of the problem appears when one considers the asymptotics of eigenvalues for the operator H with periodic boundary conditions.

Let n be an integer, $n \rightarrow \infty$. Denote

$$(30) \quad \xi_n = \frac{2\pi}{|d_0|}n, h_n = \frac{1}{\sqrt{|\xi_n|}} = \sqrt{\frac{|d_0|}{2\pi n}}, h_n \rightarrow 0.$$

Let $\mu_{m,n}$ be the approximative eigenvalues for the semiclassical eigenvalue problem

$$(31) \quad -h_n^2 \frac{d^2 \varphi(s)}{ds^2} + (\mu_{m,n} - 2A_{\delta_0}(s) + h_n^2 C_{\delta_0}(s))\varphi(s) = 0(h_n^N),$$

where $h_n \rightarrow 0$ and $C_{\delta_0}(s)$ has the same relation to $C(s, t)$ as $\vec{A}_{\delta}(s)$ to $\vec{A}(x)$ (see (9) and (21)).

It is known (see [4]) that

$$(32) \quad \mu_{m,n} = \mu_{m,n,0} + h_n^2 \mu_{m,n,1} + \dots,$$

where $\mu_{m,n,0}$ satisfies the Bohr-Sommerfeld quantization condition

$$(33) \quad \int_{2A_{\delta_0}(s) > \mu_{m,n,0}} (2A_{\delta_0}(s) - \mu_{m,n,0})^{\frac{1}{2}} ds = \pi h_n (m + \frac{1}{2}), m \in \mathbf{Z}.$$

Theorem 6.— Let $\lambda_{m,n} = \xi_n^2 - \xi_n \mu_{m,n}$. There exists a subsequence $\lambda_{m,n}^*$ in $\text{Spec}_0 H$ such that

$$(34) \quad |\lambda_m^* - \lambda_m| \leq C h_n$$

for all n sufficiently large and m such that

$$(35) \quad 0 < C_1 \leq (m + \frac{1}{2})h_n \leq C_2.$$

The approximative eigenfunctions (quasimodes) have the following form

$$(36) \quad (H - \lambda_{m,n})[e^{i\gamma(s,t) + i\xi_n t} (w_{m,n}(s) + h_n w_{m,n,1}(s, t) + h_n^2 w_{m,n,2}(s, t))] = 0(h_n),$$

where $w_{m,n}(0) = 1$, $w_{m,n,k}(s, t) = 0(1)$, $k = 1, 2$, $\gamma(s, t)$ is the same as in (20).

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