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ÉQUATIONS AUX DÉRIVÉES PARTIELLES

SPECTRAL ANALYSIS OF NON-COMPACT MANIFOLDS USING COMMUTATOR METHODS

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I INTRODUCTION : THE PROBLEM AND THE RESULTS

The work described here is joint work with R.G. Froese and the details appear in [1]. The spectrum of the Laplace-Beltrami operator on compact manifolds has been extensively studied. Since the spectrum is discrete, much work, for example, has been directed towards describing the asymptotic distribution of eigenvalues [2] and estimation of the lowest-lying eigenvalues [3]. When the manifold $\mathcal{M}$ is non-compact, the spectrum of a second-order elliptic operator L becomes much richer in the sense that the pure point spectrum of L, $\sigma_{pp}(L)$, and the continuous spectrum of L, $\sigma_c(L)$, are, in general, non-empty. Here we are interested in the questions :

(1) What is the nature of the essential spectrum of L, $\sigma_{ess}(L)$? , i.e. find $\inf \sigma_{ess}(L)$ and describe $\sigma_{ac}(L)$ and $\sigma_{sc}(L)$, the absolutely continuous and singular continuous spectra of L ;

(2) How can we characterize $\sigma_{pp}(L)$? ; for example, for which manifolds $\mathcal{M}$ do we have $\sigma_{pp}(L) \cap [\inf \sigma_{ess}(L), \infty) = \phi$? ; and, if L has eigenvalues, what can be said about the behavior of the eigenfunctions? are the eigenvalues stable under perturbation?

We describe here results on question (1) for a large class of manifolds $\mathcal{M}$ and second-order elliptic operators L. This family includes, for example, the Laplace-Beltrami operator on finite and on infinite volume hyperbolic manifolds (see Section 2). The results, described in Section 4, include the Mourre estimate and related bounds which imply a limiting absorption principle and the absence of singular continuous spectrum. In the last section, work-in-progress on the second question will be briefly discussed. Our main tool is the method of local positive commutators, the so-called Mourre theory [4], which has proved to be very powerful for the spectral analysis of Schrödinger operators on $\mathbf{R}^n$. This is described in Section 3.

Questions (1) and (2) for the Laplace-Beltrami operator have been addressed for various families of manifolds $\mathcal{M}$. Part of this work has been motivated by the study of the Eisenstein series associated with various hyperbolic manifolds (see Section 5). When $\mathcal{M}$ is the quotient of hyperbolic space, these problems were studied by Selberg [5], Lax and Phillips [6] and Patterson [7], and others. More recently, Perry [8] and Agmon [9] have applied the method of stationary scattering theory to these problem and Mazzeo and Melrose [10] have studied them using microlocal analysis.

The manifolds $\mathcal{M}$ and the operators L to which our theory applies are described as follows. $\mathcal{M}$ is a non-compact manifold having the form

$$\mathcal{M} = \mathcal{K} \cup \mathcal{U}_1 \cup \ldots \cup \mathcal{U}_s$$

where $\mathcal{K}$ is compact and $\mathcal{U}_a, a = 1, \ldots, s$, has the form of a generalized cylinder :

$$\mathcal{U}_a \cong \mathbf{R}^+ \times \mathcal{M}_a$$

where $\mathcal{M}_a$ is compact. As it makes no difference for our proofs, we assume $s = 1$, i.e. :

$$\mathcal{M} = \mathcal{K} \cup \mathcal{U} \quad \text{and} \quad \mathcal{U} \cong \mathbf{R}^+ \times \mathcal{M}_1$$

We call $\mathcal{U}$ an "end". There is a smooth density ω on $\mathcal{M}$ such that $\omega\lceil\mathcal{U} = dr{\cdot}\mu$ where μ is a smooth density on $\mathcal{M}_1$ and r is the distinguished coordinate on $\mathcal{U}$.

We let $\mathcal{H} \equiv L^2(\mathcal{M},\omega)$. In a formal sense, the subspace $L^2(\mathcal{U},\omega)$ of $\mathcal{H}$ has a constant fibre direct integral decomposition :

$$L^2(\mathcal{U},\omega) \cong \int^{\oplus} L^2(\mathcal{M}_1,\mu)dr.$$

The operators L which we consider are perturbations of operators L_0 which respect this decomposition. We call a second-order elliptic operator L_0 on $\mathcal{M}$ **separable** if there exists some $R > 1$ such that :

$$L_0 \lceil C_0^\infty(\{(r,\theta) \in \mathcal{U} | r > R\}) = -D_r^2 + h(r)P + q(r) \tag{1.1}$$

where h and q are smooth functions on $\mathbf{R}^+, h \geq 0$, which satisfy conditions described below, and P is a second-order elliptic operator on the compact manifold $\mathcal{M}_1$ (we also assume that $C_0^\infty(\mathcal{M})$ is a core for L_0). The operators L which we consider can be written as

$$L = L_0 + E$$

where L_0 is separable and E is a second-order symmetric operator whose coefficients are smooth and relatively L_0-small. In a local coordinate chart on $\mathcal{M}_1$, E has the general form :

$$E = \tilde{D}_i^* e^{ij}(r,\theta)\tilde{D}_j + e^i(r,\theta)\tilde{D}_i + D_r f^i(r,\theta)\tilde{D}_i$$

$$+D_r f(r,\theta) + e(r,\theta) + \quad \text{(adjoint)} \tag{1.2}$$

where $\tilde{D}_i = \mu(\theta)^{-1/2}(\partial/\partial\theta_i)\mu(\theta)^{1/2}$ and the functions e^{ij}, e^i, f^i, f and e satisfy certain growth conditions relative to h ; for example, $\|(P+1)^{1/2}e^{ij}(r,\theta)\|_{L^\infty(\mathcal{M}_1)} = 0(< r >^{-2} h)$, $\|(P+1)^{1/2}f^i(r,\theta)\|_{L^\infty(\mathcal{M}_1)} = 0(< r >^{-2} h^{1/2})$, $\|f(r,\theta)\|_{L^\infty(\mathcal{M}_1)} = 0(< r >^{-2})$, etc.

II SOME EXAMPLES

We give 3 examples of manifolds $\mathcal{M}$ and operators L which are included in the framework described in Section 1.

Example 1 Finite Volume Hyperbolic Manifold

Let $\mathbf{H}^2$ denote the upper half plane with the Poincaré metric $ds^2 = y^{-2}(dx^2 + dy^2)$. $SL(2,\mathbf{R})$ acts as a group of isometries on $\mathbf{H}^2$. The discrete subgroup $SL(2,\mathbf{Z})$ has a fundamental domain F shown in figure 1. By identifying points of F equivalent under the action of $SL(2,\mathbf{Z})$, F has the structure of a complete Riemannian manifold $\mathcal{M}^2$ with the metric induced from the metric on $\mathbf{H}^2$. Note that the hyperbolic volume of F, $Vol(F) \alpha \int_{1/2}^{\infty} y^{-2}dy$ is finite and that $\mathcal{M}^2 = \mathcal{K} \cup \mathcal{U}$ where $\mathcal{K} = \{(x,y) \in F | y < a \ , \quad \text{any} \quad a > 1\}$ and $\mathcal{U} \cong S^1 \times [a,\infty)$. Let L be the Laplacian on $\mathcal{M}^2$ with Hilbert space $L^2(\mathcal{M}^2, y^{-2}dx\ dy)$. Then, we have

$$L = -y^2(\partial_x^2 + \partial_y^2)$$

and introducing $r \equiv \log y$, $r \in [\log(\sqrt{3}/2), \infty)$, this is equivalent to

$$\tilde{L} = -D_r^2 + D_r - e^{2r}\partial_x^2$$

acting on $L^2(\mathcal{M}^2, e^{-r} dr\ dx)$. Finally, by the unitary transformation :

$$Ug = e^{-r/2}g$$

we arrive at

$$L\lceil C_0^\infty(\mathcal{U}) = -D_r^2 + e^{2r}P + \frac{1}{4}$$

acting on $L^2(\mathcal{M}^2, dr\ dx)$, with $h(r) = e^{2r}$, $q(r) = 1/4$, and $P = -\partial_x^2$, an elliptic operator on S^1. Hence L is a separable operator.

Example 2 Infinite Volume Hyperbolic Manifold

We now consider $\mathbf{H}^n$, n-dimensional hyperbolic space, with the Poincaré metric. Let Γ be a discrete subgroup of hyperbolic isometries on $\mathbf{H}^n$ which is geometrically finite and such that the fundamental domain $F = \mathbf{H}^n/\Gamma$ has infinite hyperbolic volume. For simplicity we assume that Γ has no parabolic elements (this amounts to assuming that F has no ends or "cusps" equivalent to the type appearing in Example 1). F may appear as in figure 2 where $\mathcal{K}$ and $\mathcal{U}_i$ are identified. By identifying points on the boundary of F according to the action of Γ, we obtain a complete Riemannian manifold $\mathcal{M}^n$. Let L be the Laplacian on $\mathcal{M}^n$. To describe L on an end $\mathcal{U}_i$, we follow Perry [8] and introduce local coordinates $(\tau, \theta), \tau \in \mathbf{R}^+, \theta \in \mathcal{M}_i$. Then

$$L\lceil C_0^\infty(\mathcal{U}_i) = -D_\tau^2 + e^{-2\tau}P_\theta + (\frac{n-1}{2})^2 + E_i \tag{2.1}$$

where E_i is a second-order differential operator on $\mathcal{U}_i$ having the form

$$E = e^{-\tau}p(D_\tau, e^{-\tau}\partial_\theta)$$

with p a second order differential operator (with coefficients satisfying certain uniformity estimates). We define $L_0 \equiv L - \sum_{i=1}^{s} E_i$ so by (2.1), L_0 is separable with $h(\tau) = e^{-2\tau}$ and $\widehat{q(\tau)} = (\frac{n-1}{2})^2$.

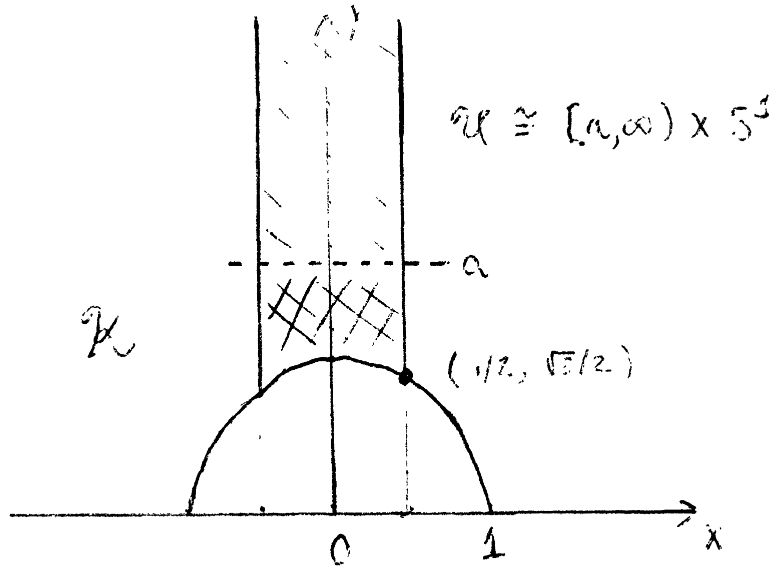

Figure 1

Example 3 Complete Riemannian Manifolds whose metrics are almost warped products on ends

Let $(\mathcal{M}, g)$ be a complete Riemannian manifold, $\mathcal{M} = \mathcal{K} \cup \mathcal{U}$, and L the Laplacian on $L^2(\mathcal{M}, g^2 dx_1 \ldots dx_n)$ with $g^2 = (det[g_{ij}])^{1/2}$. We assume that on $\mathcal{U} \cong \mathbf{R}^+ \times \mathcal{M}_1$, g has the form (in local coordinates (r, θ)) :

$$[g_{ij}] = h(r)^{-1}[k_{ij}(\theta)] + [e_{ij}]$$

where $[k_{ij}]$ is a metric on $\mathcal{M}_1$ and $[e_{ij}]$ a "small" perturbation. When $[e_{ij}] = 0$ for all $r > R$, the metric is a warped product on $\mathcal{U}$ and $g^2 = h^{-n/2}(det[k_{ij}])^{1/2}$. We define L_0 to be the Laplacian with this metric. By the unitary transformation

$$Uf = wf$$

where w is a smooth function equal to 1 on $\mathcal{K}$ and $h^{-n/4}$ on $\mathcal{U}$ for $r > R$, L_0 is equivalent to

$$L_0 \lceil C_0^\infty(\mathcal{U}, r > R) = -D_r^2 + h(r)P + q(r)$$

where

$$q(r) = \frac{g''}{g}(r) = \frac{n}{4}(\frac{n}{4} + 1)(\frac{h'}{h})^2 - \frac{n}{4}(\frac{h''}{h}).$$

Now consider a perturbation $[e_{ij}]$ which does not necessarily vanish outside of a compact set. It can easily be seen that after the unitary transformation

$$\tilde{U} f = \tilde{w} f$$

with $\tilde{w}$ equal to 1 on $\mathcal{K}$ and $g(0,\theta)/g(r,\theta)$ on $\mathcal{U}$, the Laplacian has the form $L_0 + E$, with L_0 as above, acting on $L^2(\mathcal{M}, \omega)$ where, in local coordinates on $\mathcal{M}_1$,

$$\omega = dr(det[g_{ij}(0,\theta)])^{1/2} d\theta_1 \ldots d\theta_{n-1}.$$

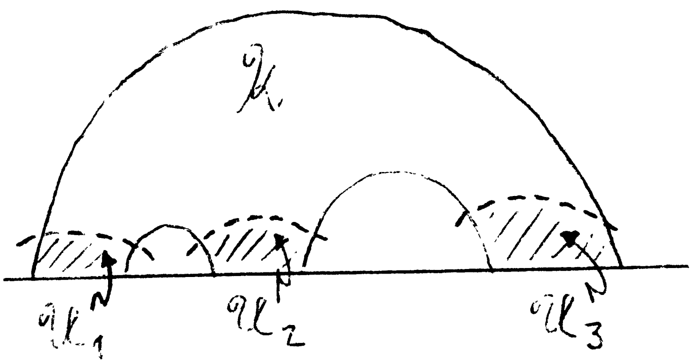

Figure 2

III THE MOURRE THEORY AND ITS CONSEQUENCES

Our analysis of L described above is based upon a method of E. Mourre [4], [11]. The method depends upon the existence of a (skew-adjoint) operator A, called a **conjugate operator** for L, such that L and A satisfy the following properties (these are stated imprecisely, see [11] for the exact statement). Let $\mathcal{H}_5(L)$, $s > 0$, be the s^{th} Sobolev space associated with L, i.e. the closure of $D((1+|L|)^{s/2})$ with the norm $\|\psi\|_s \equiv \|(|L|+1)^{s/2}\psi\|$; $\mathcal{H}_{-s}(L) \equiv \mathcal{H}_s(L)^*$, $s \geq 0$.

Boundedness

(1) The form $[L, A]$ extends to a bounded operator from $\mathcal{H}_{+2}(L) \to \mathcal{H}_{-1}(L)$.

(2) The form $[[L, A], A]$ extends to a bounded operator from $\mathcal{H}_{+2}(L) \to \mathcal{H}_{-2}(L)$.

Positivity

(3) For each $\lambda \in \mathbf{R}$, except possibly in a discrete set $\mathcal{I}(L)$, there exists an interval $\Delta \ni \lambda$, a constant $\alpha > 0$, and a compact operator K such that

$$E_\Delta(L)[L, A]E_\Delta(L) \geq \alpha E_\Delta(L) + K \quad (*)$$

where $E_\Delta(L)$ is the spectral projector for L and the interval Δ.

The condition (*) is called the Mourre Estimate.

In our case, as $L = L_0 + E$, we first construct a conjugate operator for L_0 satisfying (2) and (3) and

(1') the form $[L_0, A]$ extends to a bounded operator from $\mathcal{H}_{+2}(L_0) \to \mathcal{H}$.

Having found an A satisfying (1'), (2) and (3), sufficient conditions on the coefficients of E (see the end of Section 1) will insure that

- E is relatively L_0 bounded with bound < 1
- $(L - z)^{-1} - (L_0 - z)^{-1}, Im\ Z \neq 0$, is compact
- $[E, A] : \mathcal{H}_{+2}(L_0) \to \mathcal{H}_{-1}(L_0)$ is bounded
- $[[E, A], A] : \mathcal{H}_{+2}(L_0) \to \mathcal{H}_{-2}(L_0)$ is bounded

- $f(L)[E,A]f(L)$, $f \in C_0^\infty(\mathbf{R})$, is compact.

Hence one can pass from the Mourre estimate for L_0 and A to the Mourre estimate for L with the same A.

The problem, therefore, is to find a conjugate operator A for L_0. Unlike the Schrödinger operator case, the main difficulties already appear for the unperturbed operator L_0. As the examples of Section 2 show, there is a wide variation in the behavior of h as $r \to \infty$. We divide the operators L_0 into 3 classes according to this behavior : the choice of A depends crucially upon this. Note that in Example 3, h^{-1} measures the "size" of the manifold $\mathcal{M}$ at infinity, hence the terminology below.

(A) $h(r) \to 0$ as $r \to \infty$; $\mathcal{M}$ is large at infinity. We distinguish 2 cases :

$$(Ai) \qquad h(r) = 0(e^{-r^\beta}), 0 < \beta \leq 1$$

$$(Aii) \qquad h(r) = 0(p(r)^{-1}),\ p \quad \text{a polynomial}$$

(B) $h(r) \to \infty$ as $r \to \infty$; $\mathcal{M}$ is small at infinity.

(C) $h(r) \to h(\infty) < \infty$; $\mathcal{M}$ has a "constant volume" at infinity.

The conjugate operators to be constructed for each class will be supported in the end $\mathcal{U}$: the local compactness property of L_0 guarantees that no singular spectrum is contributed from $\mathcal{H}$. Case (Aii) is the easiest in the sense that it is the closest to the one-body Schrödinger operator case : A is effectively the generator of the dilation group in r. Technically, case (Ai) is the hardest.

Results For each of the cases (A) - (C) we prove that $\inf \sigma_{ess}(L) = q(\infty)$ and we construct a conjugate operator A for L_0, and hence for L, such that (1) - (3) hold. It is then a consequence of Mourre's theory that a limiting absorption principle holds for L. For each $\lambda \notin \sigma_{pp}(L)$ and for which (*) holds for some open interval $\Delta \ni \lambda$, there exists a constant $C > 0$ such that

$$(3.1) \qquad \limsup_{\delta \downarrow 0} \sup_{\mu \in \Delta} \|(1+|A|)^{-\alpha}(L-\mu-i\delta)^{-1}(1+|A|)^{-\alpha}\| \leq C$$

for $\alpha > \frac{1}{2}$. As a consequence, $\sigma_{sc}(L) \cap \Delta = \phi$. For cases (A) - (B), the Mourre estimate holds for all points above $q(\infty) = \inf \sigma_{ess}(L)$ and hence $\sigma_{sc}(L) = \phi$. In case (C), the set of exceptional points $\mathcal{I}(L) = \{h(\infty)\lambda_n + q(\infty) | \lambda_n \in \sigma(P)\}$ at which the Mourre estimate fails is non-empty and discrete so $\sigma_{sc}(L) = \phi$. It also follows from Mourre's theory that L has finitely many eigenvalues of finite multiplicity in any interval on which (*) holds. Hence the eigenvalues can accumulate only at $\inf \sigma_{ess}(L)$ in cases (A) and (B) and also at $\mathcal{I}(L)$ in case (C).

IV SKETCH OF THE CONSTRUCTION OF A AND THE PROOFS

We sketch the ideas behind the construction of A in the various cases described in Section 3. As L_0 has the separable form (1.1) only on the end $\mathcal{U}$, A will be supported in $\mathcal{U}$ for $r > R$, R sufficiently large. The compact piece of $\mathcal{M}$ can be neglected because L_0 is locally compact (as follows from ellipticity and the smoothness of the coefficients). Let $\chi \in C_0^\infty(\mathbf{R}^+)$ such that $\chi \geq 0$, monotone, and $\chi = 0$ for $r < 1$ and $\chi = 1$ for $r > 2$; set $\chi_R = \chi(r/R)$.

Case (Ai)

. Consider $h(r) = e^{-r}$ and $q(r) = 0$ so $L_0 \lceil C_0^\infty(\mathcal{U}) = -D_r^2 + e^{-r}P$. Suppose we try $A_0 = \chi_R^2 r\ D_r + D_r\ r\chi_R^2$, then

$$(4.1) \qquad [L_0, A_0] = 2\chi_R(-D_r^2 + re^{-r}P)\chi_R + \quad \text{(remainder)}$$

As $re^{-r} >> e^{-r}$ on $supp\chi_R$ this is positive in the sense of Mourre but it is not relatively L_0-bounded for the same reason. To reduce the size of the commutator, we try $\tilde{A}_0 = \chi_R^2 D_r + D_r\chi_R^2$ and obtain :

$$[L_0, \tilde{A}_0] = 2\chi_R e^{-r} P\chi_R + \quad \text{(remainder)}$$

This is L_0-bounded but not positive in the sense of Mourre. A basic problem above is that P is unbounded. Let us write λ for P as r and P commute. Returning to (4.1), we see that if we restrict ourselves to a region of (r, λ)-space where $\lambda r\ e^{-r} \leq C_1(1 + \lambda e^{-r})$ or, equivalently, where $\lambda\ e^{-r} < C_2$, for some $C_1, C_2 > 0$, it then follows that the first term on the right in (4.1) is L_0-bounded provided we add a cut-off function ξ to A_0 supported in the region $hP < C$. This almost works except that there are remainder terms like $r\chi_R\xi' D_r^2$ which are not L_0-bounded since $C_1 < \lambda e^{-r} < C_2$ does not imply that r is bounded. Both of these problems are solved if we modify r in a P-dependent manner and add a cut-off function ξ to A_0 which is supported in the region $hP < C$. We now take

$$(4.2) \qquad A = [r - \log(P+1)]\chi_R^2\xi^2 D_r + D_r\chi_R^2\xi^2[r - \log(P+1)]$$

and find

$$(4.3) \qquad [L_0, A] = 2\chi_R\xi(-D_r^2 + [r - \log(P+1)]e^{-r}P)\chi_R\xi + \quad \text{(remainder)} \quad .$$

Now $[r - \log(P+1)]\lceil supp(\xi\chi_R) > \delta > 0$ so we have positivity, $[r - \log(P+1)]\lceil supp(\xi'\chi_R)$ is bounded so the remainder is negligible and

$$0 \leq [r - \log(P+1)]e^{-r}\xi \leq [r - \log(P+1)]e^{\ -[r - \log(P+1)]}\ \xi$$

so the commutator is L_0-bounded. A in (4.2) is basically the conjugate operator we construct. Now, to finish a proof of the Mourre estimate (*), we take $f \in C_0^\infty(\mathbf{R})$ and multiply (4.3) on both sides by $f(L_0)$. After some manipulations, we obtain :

$$f(L_0)[L_0, A]f(L_0) \geq \alpha\ f(L_0)^2 + K - c\|\chi_R(\xi - 1)f(L_0)\|$$

where K is compact. We prove that as $supp(f)$ shrinks around a point λ_0 and the constants R and C such that $(1-\xi)$ projects onto $h(P+1) > C$, are taken sufficiently large, $\|\chi_R(\xi-1)f(L_0)\| \to 0$. This can be understood classically as f restricts the energy $\xi_r^2 + h\lambda + q \approx \lambda_0$ where as $(1-\xi)$ restricts $h\lambda > C$. Hence, the supports of f and $(1-\xi)$ become disjoint.

Case (Aii)

It is easily seen that in this case a conjugate operator for L_0 is $A = \chi_R^2 r D_r + D_r r \chi_R^2$ since $|rp(r)'p(r)^{-2}| \leq Cp(r)^{-1}$ for $r > R$, R sufficiently large. In fact, this situation is simply part of a more general case for which L_0 is not necessarily separable on $\mathcal{U}$. For example, suppose $L_0 = -D_r^2 + E$ on $\mathcal{U}$ where E is as in (1.2). If the coefficients of E satisfy conditions like

$$c_1[e^{ij}] \leq -r[e^{ij}] \leq c_2[e^{ij}]$$

and

$$-b_1[e^{ij}] \leq r^2[e^{ij''}] \leq b_2[e^{ij}]$$

etc., for constants $c_i, b_i > 0$, then the simple form of A given above works.

Case B

To see the idea, consider $L_0 \lceil C_0^\infty(\mathcal{U}) = -D_r^2 + e^r P$. Let $\{\lambda_n\} = \sigma(P)$. Then L_0 on the end $\mathcal{U}$ is a direct sum $\bigoplus_{n=0}^\infty(-D_r^2 + e^r\lambda_n)$. For $n > 0$, $-D_r^2 + e^r\lambda_n$ has compact resolvent and hence discrete spectrum. For $n = 0$, however, if $\lambda_0 = 0$, the operator $-D_r^2$ has continuous spectrum $[0,\infty)$. From this we see that (1) we need only control the commutator on the $n=0$ subspace, and (2) L may have eigenvalues embedded in $\sigma_c(L)$. Concerning point (1), let P_0 be the projection for P onto the $n=0$ eigenspace. Then with $A = P_0\chi_R^2 r D_r + D_r \chi_R^2 r P_0$,

$$\begin{aligned}[L_0, A] &= 2P_0\chi_R(-D_r^2)\chi_R P_0 + \qquad \text{(remainder)}\\ &= 2P_0\chi_R L_0 \chi_R P_0 + \qquad \text{(remainder)}\end{aligned}$$

An argument similar to the one given in Case (Ai) shows that $\|(1-P_0)\chi_R f(L_0)\|$ can be made small by taking R large and shrinking $suppf$. Concerning point (2), it is known that for certain case (for instance, Example 2) there are embedded eigenvalues ; we comment on this in Section 5.

Case C

When $h(\infty) < \infty$, the eigenvalues of $h(\infty)P + q(\infty)$ on $L^2(\mathcal{M}_1, \mu)$ form an exception set $\mathcal{I}(L_0)$ at which the Mourre estimate fails. To construct A, consider $\lambda > q(\infty) = \inf \sigma_{ess}(L_0), \lambda \notin \mathcal{I}(L_0)$ and take N such that $\lambda_N h(\infty) + q(\infty) < \lambda < \lambda_{N+1}h(\infty) + q(\infty)$. Set $P_N = \sum_{i=0}^N P_i$ and define $A = P_N\chi_R^2 r D_r + D_r r \chi_R^2 P_N$ so

$$[L_0, A] \geq P_N\chi_R(L_0 - \lambda_N h(\infty) - q(\infty))\chi_R P_N + \qquad \text{(remainder)}$$

where we assumed that $|rh'| \to 0$, $|rq'| \to 0$. The remainder term $\|\chi_R(1-P_N)f(L_0)\|$ is controlled in the same way as above : $suppf \subset (\lambda_N h(\infty) + q(\infty), \lambda_{N+1}h(\infty) + q(\infty))$ and is made small whereas $(1-P_N)\chi_R hP > \lambda_{N+1}h\chi_R \to \lambda_{N+1}h(\infty)$ by taking R large. Note that in this case A depends upon the point where the Mourre estimate is to be computed.

V WORK-IN-PROGRESS AND APPLICATIONS

We conclude with some remarks about work-in-progress [12] and applications [13].

A. Eigenvalues

The existence of eigenvalues for L above $\inf \sigma_{ess}(L)$ can be studied using the results presented here and the method of Froese and Herbst [14]. This method combines the virial theorem and the Mourre estimate to establish isotropic L^2-exponential bounds on eigenfunctions. When the eigenfunction decays faster than any exponential, one can many times use a unique continuation theorem or a positivity estimate to conclude that it is identically zero. This is the case for manifolds which are large at infinity, i.e. in Case A we prove that L has no embedded eigenvalues.

When $h(r) \to \infty$ as $r \to \infty$, Case B, it is known that embedded eigenfunctions exist in certain cases (Example 2 is discussed in [15]). However, we can show that if ψ is an eigenfunction corresponding to an embedded eigenvalue, then $P_i\psi$ decays faster than any exponential (where P_i projects onto the λ_i-eigenspace of P). Moreover, we have $\lim_{R\to\infty} \|\chi_R P_0 \psi\| / \|\chi_R(1 - P_0)\psi\| = 0$ which can be interpreted as a generalization of the cusp form condition known to hold for eigenfunctions of the Laplacian on finite volume hyperbolic space, i.e. that $P_0\psi = 0$.

B. Stability of Eigenvalues and Resonances

The stability of embedded eigenvalues for the Laplacian on finite volume hyperbolic manifolds in 2-dimensions was extensively studied by Colin de Verdière [16]. He showed that they are unstable under generic C_0^∞-perturbations of the metric. In such a situation, one expects that the eigenvalues dissolve into spectral resonances of the operator. We study this situation using the analytic family of operators $L(\theta)$, $|Im\theta| < \pi/2$, constructed from L using the unitary group $U(\theta) \equiv exp(i\theta A)$, $\theta \in \mathbf{R}$, where A is a conjugate operator for L, by continuing $U(\theta)LU(\theta)^{-1}$ from $\theta \in \mathbf{R}$. The resonances of L are complex eigenvalues of $L(\theta)$ lying in $\mathbf{C}^-$. It is an easy application of perturbation theory and the theory of resonances to prove that if L has embedded eigenvalues then a C_0^∞-perturbation of the metric generically causes these eigenvalues to dissolve into resonance of the perturbed operator. We also believe that in Case B the Laplacian generically has spectral resonances, but this seems much harder to prove.

C. Meromorphic Continuation of the Eisenstein Series

A first step fowards the meromorphic continuation of the Eisenstein series associated with a hyperbolic manifold $\mathbf{H}^n/\Gamma$ is the continuation of the resolvent kernel for the Laplacian L. Using the spectral deformation group $U(\theta)$ introduced in B above, it follows from the analyticity of $L(\theta)$ and a calculation of $\sigma_{ess}(L(\theta))$, that matrix elements of the resolvent of L between vectors from a dense set of analytic vectors for $U(\theta)$ have meromorphic continuations across $\sigma_c(L)$. Since it can be shown that $L(\theta)$ is analytic on the strip $|Im\theta| < \pi/2$ and that $\sigma_{ess}(L(\theta)) = e^{-2\theta}[q(\infty), \infty)$ these continuations extend to $\mathbf{C}^-$. From this information we hope to derive the corresponding results for the resolvent kernel.

D. Scattering Theory for the Wave and Schrödinger Equations

With S. DeBièvre, we are studying scattering theory on manifolds $\mathcal{M}$ of the type described here : $\mathcal{M} = \mathcal{K} \cup \mathcal{U}_1 \cup \ldots \cup \mathcal{U}_s$. Physically, each end $\mathcal{U}_i$ appears as a geometric channel into which a particle may scatter. Of particular interest is the wave equation on $\mathcal{M}$:

$$\partial_t^2 u = -Lu + Vu \tag{5.1}$$

where $L \geq 0$ is an operator of the type considered here and V is a short-range potential on $\mathcal{M}$. To study questions like asymptotic completeness for this equation, we extend the Mourre theory for L to a form applicable to (5.1). This amounts to finding a conjugate operator for $L^{1/2}$, which is formally $L^{1/2}A + AL^{1/2}$, where A is a conjugate operator for L. It follows from the limiting absorption principle (3.1) and the theory of smooth operators, that the wave operators, which compare the dynamics given in (5.1) to that given by a separable operator $L_{0,i}$ on the end $\mathcal{U}_i$:

$$\partial_t^2 w_i = -L_{0,i} w_i$$

exist and are complete. Other situations of physical interest which fit into the framework given here include obstacle scattering on unbounded domains in $\mathbf{R}^n$ and scattering on static space-times, like the Schwarzschild metric on $\mathbf{R} \times \mathbf{R} \times S^2$.

BIBLIOGRAPHIE

[1] R.G. FROESE, P.D. HISLOP, Spectral Analysis of Second-Order Elliptic Operators on Non-compact Manifolds, to appear in Duke Mathematical Journal (1988).

[2] I. CHAVEL, **Eigenvalues in Riemannian Geometry**. New York : Academic Press 1984.

[3] J. CHEEGER, D. EBIN, **Comparison Theorems in Riemannian Geometry**. Amsterdam : North Holland Publishing Co. 1975.

[4] E. MOURRE, Absence of Singular Continuous Spectrum for certain selfadjoint Operators. Commun. Math. Phys. **78**, 391-408 (1981).

[5] A. SELBERG, Harmonic Analysis and Discontinuous Groups in Weakly Symmetric Riemmannian Spaces with Applications to Dirichlet Series. J. Ind. Math. Soc. **20**, 47-87 (1956).

[6] P. LAX, R.S. PHILLIPS, Translation Representation for Automorphic Solutions of the non-Euclidean Wave Equation I, II, III. C.P.A.M. **37**, 303-328 (1984); **37**, 779-813 (1984); **38**, 179-208 (1985).

[7] S.J. PATTERSON, The Laplacian Operator on a Riemann Surface I, II, III. Compositio Math. **31**, 83-107 (1975); **32**, 71-112 (1976); **33**, 227-259 (1976).

[8] P.A. PERRY, The Laplace Operator on a Hyperbolic Manifold I. Spectral and Scattering Theory. J. Funct. Anal. **75** 161-182 (1987) (1988).

[9] S. AGMON, Spectral Theory of Schrödinger Operators on Euclidean and non-Euclidean Spaces. C.P.A.M. **39**,par.3 par.16, Number S, Supplement (1986).

[10] R. MAZZEO, R. MELROSE, Meromorphic Extension of the Resolvent on Complete spaces with Asymptotically Constant Negative Curvature. J. Funct. Anal. **75**, 260-310 (1987).
[11] H.L. CYON, R.G. FROESE, W. KIRSCH, B. SIMON, **Schrödinger Operators, with Application to Quantum Mechanics and Global Geometry**. Berlin : Springer-Verlag 1987.
[12] R.G. FROESE, P.D. HISLOP, Analysis of the Point Spectrum of Second Order Elliptic Operators on Non-compact Manifolds, preprint (1988).
[13] S. DEBIEVRE, P.D. HISLOP, Scattering Theory for the Wave and Schrödinger Equations on Non-compact Manifolds, preprint (1988).
[14] R. FROESE, I. HERBST, Exponential Bounds and Absence of positive Eigenvalues for N-Body Schrödinger Operators. Commun. Math. Phys. **87**, 429-447 (1982).
[15] P. LAX, R.S. PHILLIPS, **Scattering Theory for Automorphic Functions**. Annals of Mathematics Studies 87. Princeton : Princeton University Press 1976.
[16] Y. COLIN DE VERDIERE, Pseudo-Laplacians II. Ann. Inst. Fourier **33**, 87-113 (1983).

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