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M. S. BAOUENDI

L. P. ROTHCHILD

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ECOLE POLYTECHNIQUE

F-91128 PALAISEAU Cedex (France)

Tél. (1) 69.41.82.00

Télex ECOLEX 691.596 F

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ÉQUATIONS AUX DÉRIVÉES PARTIELLES

CR MAPPINGS BETWEEN REAL
HYPERSURFACES IN COMPLEX SPACE.

M.S. BAOUENDI L.P. ROTHCHILD

Let M be (a germ) of a smooth real hypersurface in $\mathbf{C}^{n+1}$ containing the origin defined by $\rho(Z, \bar{Z}) = 0$, where ρ is a smooth real valued function satisfying $\rho(0, 0) = 0$, $d\rho(0) \neq 0$. We may assume $\frac{\partial \rho}{\partial \bar{Z}_{n+1}}(0) \neq 0$. By a CR function h defined on M we mean a germ of a smooth function h at 0 satisfying $L_j h = 0$, $j = 1, \dots, n$, with

$$L_j = \frac{\partial}{\partial \bar{Z}_j} - \frac{\rho_{\bar{Z}_j}(Z, \bar{Z})}{\rho_{\bar{Z}_{n+1}}(Z, \bar{Z})} \frac{\partial}{\partial \bar{Z}_{n+1}}.$$

If M' is another hypersurface of $\mathbf{C}^{n+1}$, a smooth mapping $H : M \rightarrow M'$, $H(0) = 0$, is called CR if $H = (h_1, \dots, h_{n+1})$, where the h_j 's are CR functions defined on M . We shall give some local geometric and analytic properties of such mappings. We refer to [4] and [5] for complete details.

If M is real analytic, after a holomorphic change of coordinates we can assume that

$$(1) \quad \rho(Z, 0) = \alpha(Z) Z_{n+1}, \quad \alpha(0) \neq 0.$$

If M is only smooth such a change of variables can be done formally (i.e. in formal power series of Z). If (1) is satisfied we say that Z_{n+1} is a **transversal (holomorphic or formal) coordinate for M** . If $Z' = (Z'_1, \dots, Z'_{n+1})$ are coordinates in $\mathbf{C}^{n+1}$ such that Z'_{n+1} is transversal to M' , and if $H = (h_1, \dots, h_{n+1})$ is a CR map from M to M' given by $Z'_j = h_j(Z)$, for $Z \in M$, we say that h_{n+1} is a **transversal component of H** .

If j is a CR function defined on M , we associate to j a formal power series $J(Z)$, $Z = (Z_1, \dots, Z_{n+1})$, such that the Taylor series of j at 0 coincides with $J(Z)|_M$. If Z_{n+1} is transversal to M , we write (z, w) instead of Z (i.e. $w = Z_{n+1}$), $z \in \mathbf{C}^n$, $w \in \mathbf{C}$. Similarly if h_{n+1} is a transversal component of H , we write $H = (f, g)$, $f = (f_1, \dots, f_n)$ (i.e. $h_j = f_j$, $1 \leq j \leq n$, $h_{n+1} = g$), and $F_j(z, w)$, $G(z, w)$ the associated formal power series. It follows from (1) that

$$(2) \quad G(z, w) = w G_1(z, w),$$

where $G_1(z, w)$ is another power series.

If H is a CR mapping as above then it is said to be of finite multiplicity if

$$(3) \quad \dim_{\mathbf{C}} \mathcal{O}[[z]] / (F(z, 0)) < \infty,$$

where $\mathcal{O}[[z]]$ is the ring of formal power series in n indeterminates $z_1, \dots, z_n$ and $(F(z, 0))$ is the ideal generated by $(F_1(z, 0), \dots, F_n(z, 0))$. The number given by the left hand side of (3) is called the **multiplicity** of H at 0.

As in Baouendi-Jacobowitz-Treves [3] in the real analytic case, and D'Angelo [1] in the smooth case, we say that M is **essentially finite** at 0 if

$$(4) \quad \dim_{\mathbf{C}} \mathcal{O}[[z]] / (a_\alpha(z)) < \infty,$$

with $\rho(z, 0, \zeta, 0) = \sum_{\alpha} a_\alpha(z) \cdot \zeta^\alpha$, and $(a_\alpha(z))$ the ideal generated by the power series $a_\alpha(z)$ for all $\alpha \in \mathbf{Z}_+^n$. Note that it follows from (1) that $a_\alpha(0) = 0$. The number given by the left hand side of (4) is called the **essential type** of M at 0 and is denoted by $\text{ess. type}_0 M$.

We are now ready to state our main results.

Theorem 1.— Let $H : M \rightarrow M'$ be a smooth CR mapping defined near 0, with M and M' C^∞ hypersurfaces in $\mathbb{C}^{n+1}$. Let w be any (formal) transversal coordinate for M and G any (formal) transversal coordinate of H . Assume that M is essentially finite at 0.

(i) If $G \equiv 0$ then either H is not of finite multiplicity at 0, or M' is not essentially finite at 0.

(ii) If $G \not\equiv 0$ then $\frac{\partial G}{\partial w}(0) \neq 0$, H is of finite multiplicity and M' is essentially finite.

In addition, if M and M' are real analytic and H is holomorphic, then $G \not\equiv 0$ if and only if H maps any neighborhood of 0 in M onto a neighborhood of 0 in M' .

Theorem 2.— Let $H : M \rightarrow M'$ be a smooth CR map if, either M is essentially finite and $G \not\equiv 0$, or M' is essentially finite and H of finite multiplicity then

$$(5) \quad \text{ess. type}_0 M = (\text{mult}_0 H) \times (\text{ess. type}_0 M'),$$

with all three integers in (5) being finite.

The proofs of Theorems 1 and 2 could be found in [4] and [5]. Several tools of commutative algebra such as the Nullstellensatz and Nakayama's lemma are used in these proofs.

If $M, M' \subset \mathbb{C}^{n+1}$ are real analytic and $H : M \rightarrow M'$ is a smooth CR map, we are interested in the following question : when is H the restriction of a (local) holomorphic mapping in $\mathbb{C}^{n+1}$? Several results could be found in the literature starting with Lewy [9] and Pincuk [10] when M and M' are strictly pseudoconvex and H is a diffeomorphism. Recent results closely related to ours (Theorem 3) have been independently proved by Diederich and Fornaess [8].

Before stating our extension results we need to introduce another definition. If $H = (f_1, \dots, f_n, g)$ is a CR map as above, with g a transversal imponent, and if (z, w) are coordinates for M such that w is transversal to M , we say that H is **totally degenerate** if

$$(6) \quad \det \left(\frac{\partial F_j}{\partial z_k}(z, 0) \right) = 0 ,$$

i.e. the formal power series defined by the left hand side of (6) is 0. We have the following result.

Theorem 3.— Let $H : M \rightarrow M'$ be a smooth CR map, $H(0) = 0$, where M and M' are real analytic hypersurfaces in $\mathbb{C}^{n+1}$, and g a transversal CR component. Then H extends holomorphically to a neighborhood of 0 in $\mathbb{C}^{n+1}$ if any of the following conditions holds :

(i) M is essentially finite and g is not flat at 0.

(ii) M' is essentially finite and H is of finite multiplicity at 0.

(iii) M' is essentially finite and H is not totally degenerate at 0.

Note that it follows from Theorems 1 and 2 that (i) $\Leftrightarrow$ (ii). We can also show that (i) and (ii) imply (iii). However condition (iii) is weaker than (i) and (ii) as is shown by the following example. Let M and M' be embedded in $\mathbf{C}^3$ given by $M = \{(z, w) : \operatorname{Im} w = |z_1|^2 + |z_1 z_2|^2\}$, and $M' = \{(z', w') : \operatorname{Im} w' = |z'_1|^2 + |z'_2|^2\}$, and $H = (f_1, f_2, g)$ with $f_1(z, w) = z_1$, $f_2(z, w) = z_1 z_2$ and $g = w$. Here M' is essentially finite, M is of finite type (but not essentially finite), H is not totally degenerate but not of finite multiplicity at 0.

When $n = 1$, (i.e. $M, M' \subset \mathbf{C}^2$), then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) ; in this case Theorem 3 was proved by the authors jointly with S. Bell [2]. Theorem 3 generalizes the result of [3] which deals with the diffeomorphic case. A complete proof could be found in [4] and [5].

We give some corollaries of Theorem 3.

Corollary 1.— *Let $\mathcal{H} : D \rightarrow D'$ be a proper holomorphic mapping between two bounded domains in $\mathbf{C}^{n+1}$ with real analytic boundaries. If $\mathcal{H} \in C^\infty(\bar{D})$, and if at every $p \in \partial D$ a transversal component of $\mathcal{H}$ at p is not flat at p , then $\mathcal{H}$ extends as a proper holomorphic mapping from a neighborhood of $\bar{D}$ into a neighborhood of $\bar{D}'$.*

Using the result of Bell-Catlin [6] and Diederich-Fornaess [7] Corollary 1 yields :

Corollary 2.— *If $\mathcal{H} : D \rightarrow D'$ is a proper holomorphic mapping between two bounded pseudoconvex domains in $\mathbf{C}^{n+1}$ with real analytic boundaries, then the conclusion of Corollary 1 holds.*

Several other corollaries of Theorems 1, 2 and 3 could be found in [4] and [5].

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M.S. Baouendi
Purdue University
W. Lafayette, IN 47907
U.S.A.

L.P. Rothschild
University of California,
San Diego La Jolla, CA 92093
U.S.A.