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NONLINEAR EFFECTIVELY HYPERBOLIC EQUATIONS

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§ 0. INTRODUCTION :

We shall give some notes about the effective hyperbolicity. We think there are two ways to explain a notion. The first one is to find out an equivalent notion from a different point view. And the other one is to give some interesting examples of applications of the notion. For the first one, we have a proposition for single partial differential operators that to be effectively hyperbolic is equivalent to be strongly hyperbolic. The strong hyperbolicity means the stability of solvability (well posedness) of the Cauchy problem under changing lower terms. The effective hyperbolicity defined later is a geometrical notion on the principal symbols of partial differential operators. The sufficient part of the above proposition, that is, the Cauchy problem is well posed if it is effectively hyperbolic, is very important for applications and does not always require that partial differential operators are single and even linear. So, we may replace the effective hyperbolicity for the strict hyperbolicity used usually and frequently because the previous notion is wider than the later.

Many partial differential operators in applications are non linear. So, we have to extend the results in the linear cases to ones in the non linear cases in order to find out interesting examples. In the present stage, this extension is very easy because we know already a very famous abstract theorem, so called the Nash-Moser implicit function theorem.
Here, we explain that this theorem is also applicable to our cases. We shall remark we needed some minor change of expression of the Nash-Moser implicit function theorem in order to obtain a sharper result for the non linear Cauchy problem making it possible to apply directly to the Monge-Ampère equation.

§ I. RESULTS OF LINEAR CASES :

Let P_m be a polynomial of homogeneous order m in $\xi \in \mathbb{R}^{n+1}$ with coefficients C^∞-functions in $x \in \mathbb{R}^{n+1}$. We assume it is normal in ξ_o,

$$P_m = \xi_o^m + \dots .$$

Définition 1. We call P_m effectively hyperbolic (on an open set in x) if P_m is hyperbolic in ξ_o and if at the critical points of the characteristics

$\{P_m = \nabla P_m = 0\}$, the fundamental matrices F of P_m have non zero real eigenvalues, where the fundamental matrix F is defined by

$$F = \begin{pmatrix} P_{m_{\xi x}} , & P_{m_{\xi\xi}} \\ -P_{m_{xx}} , & -P_{m_{x\xi}} \end{pmatrix} , \quad P_{m_{\xi x}} = \partial_\xi \partial_x P_m \quad , \text{etc...}$$

Let us consider a system of partial differential operator P of order m with a diagonal principal part, namely,

$$P = P_m I + Q , \tag{1.1}$$

where the lower term $Q = (q_{ij})$ is a system such that order $q_{ij} \leqslant \alpha_i - \alpha_j + m-1$ with respect to a multi-index α of integers.

Theorem 1 . Let P_m in (1.1) be effectively hyperbolic on a neighborhood of the origin. Then, there exist a conic domain $\Omega = \{x; x_o + \lambda|x'| < 0 , \lambda > 0\}$ and a constant $\varepsilon_o > 0$ such that for all ε $(0 \leqslant \varepsilon < \varepsilon_o)$, the Cauchy problem

$$\begin{cases} Pu = f & \text{on } \{x > 0\} \cap \Omega_\varepsilon \\ u = 0 & \text{on } \{x_o \leqslant 0\} \cap \Omega_\varepsilon \end{cases} \tag{1.2}$$

has a unique C^∞-solution u on Ω_ε for any C^∞- datum f on Ω_ε supported on $\{x_o \geqslant 0\}$, where $\Omega_\varepsilon = \Omega + (\varepsilon, 0, \ldots, 0)$. Moreover, for some suitably fixed ℓ , they satisfy the estimate for all $s \geqslant 0$

$$\|u\|_s \leqq C_s(\|f\|_{s+\ell} + \|a\|_{s+\ell}\|f\|_\ell) , \tag{1.3}$$

where $\|\cdot\|_s$ is the Sobolev norms on Ω_ε and a stands for coefficients of the partial differential operator P . We should remark that the constants C_s are uniform on P belonging to a suitable neighborhood of a fixed effectively hyperbolic operator in the space of hyperbolic operators with the type (1.1) on a neighborhood of the origin.

When we usually call the Cauchy problem (1.2) well posed, we do not require the estimate (1.3), especially, the existence of the constant ℓ independent of s . However, almost all cases which we know wellposed, have the type (1.3) of estimates, and this is important to use the Nash-Moser implicit function theorem. So, we introduce a notation for convenience.

Let us consider the problem (1.2) for a set $\mathcal{P}$ of general system P of partial differential operators.

Definition 2. We call a set $\mathcal{P}$ strongly wellposed if the conclusion of Theorem 1 holds for any element P of $\mathcal{P}$, especially, if ε_o, Ω and constants C_s, ℓ of the estimate (1.3) are common on $\mathcal{P}$.

Remark. Let P be a r $\times$ r system of partial differential operators. Define the principal symbol P_{pr} as usual. Assume that det P_{pr} is effectively hyperbolic. This is reduced to Theorem 1 so that it is strongly wellposed.

§ II. NONLINEAR CASES :

We consider a nonlinear system ;

$$(2.1)\qquad \begin{aligned} Pu &= (P_i)_{i=1,\dots,r} \\ P_i &= P_i(x,\eta_{j\alpha};|\alpha|\leqslant\bar\alpha_i-\bar\beta_j\ ,\ j=1,\dots,r)\Big|_{\eta_{j\alpha}=\partial^\alpha u_j} \\ &= P_i(x,\partial^\alpha u)\ , \\ u &= (u_1,\dots,u_r)\ , \end{aligned}$$

where $\bar\alpha$ and $\bar\beta$ are multi-indices of integers.
The linearization DP of this system is given by

$$DP\ \varphi = \Big(\sum_{j,\beta}\frac{\partial}{\partial\eta_{j\beta}}P_i(x,\partial^\alpha u)\partial^\beta\varphi_j\Big)_{i=1,\dots,r}$$

and the principal symbol $P_{pr} = (P_{ij})$

$$P_{ij} = \sum_{\beta;\,|\beta|=\bar\alpha_i-\bar\beta_j}\frac{\partial}{\partial\eta_{j\beta}}P_i(x,\partial^\alpha u)\xi^\beta\ .$$

Now we extend the strong well posedness to this type of system.

Definition 3. We call a nonlinear system P (2.1) strongly wellposed if there exist two linear system Q and R satisfying the following conditions ;

1) Coefficients of Q and R are also functions of (x,η_β) and $(x,\eta_\beta,\zeta_\beta)$, respectively, where we put the unknown function $\partial^\beta u$ in η_β and the parameter function $\partial^\beta h$ in ζ_β.

2) The linearization DP are decomposed as

$$DP = Q + R \quad\text{with } h = Pu\ .$$

3) There exists a neighborhood U of u = 0 in $C^\infty \cap \{u;\ u=0 \text{ at } x_o < 0\}$

such that $\{Q;\ u \in U\}$ is strongly wellposed in the sense of Definition 2.

4) $R \equiv 0$ at $h \equiv 0$.

We shall later explain by an example the reason why we do not define simply as $\{DP\}$ are strongly well posed. Roughly speaking, we assume the existence of a parametrix of DP toward $Pu = 0$. Under this assumption we can conclude the following unique extension theorem of a solution.

Theorem 2. Let a nonlinear operator P(2.1) be strongly wellposed. If there exists a neighborhood Ω_0 of the origin such that $u = 0$ is a solution of $Pu = 0$ at $\{x_0 < 0\} \cap \Omega_0$, then there exists uniquely a C^∞ solution u of $Pu = 0$ on a neighborhood Ω of the origin such that $u = 0$ on $\{x_0 < 0\} \cap \Omega$.

Corollary 3 . We assume that the principal symbol P_{pr} of P (2.1) are decomposable as well as Definition 3 such that

$$P_{pr} = Q_0 + R_0 \quad \text{with} \quad h = Pu \quad ,$$

where Q_0 and R_0 satisfy the conditions for Q and R in 1) and 4) of Definition 3 , respectively, and where Q_0 is one of the type of effectively hyperbolic operators treated in Theorem 1 or uniformly reducible to one of this type for example, det Q_0 is effectively hyperbolic, for all u of U in 3).
Then, the conclusion of Theorem 2 holds.
This Theorem 2 is no more than a translation from the abstract Nash-Moser's theorem into the category of the Cauchy problem. However, it requires the improvement of expression of the Nash-Moser's theorem, because we assume a weaker condition than as usual, namely, we assume the existence of parametrices of the linearization DP instead of the existence of exact inverses.

We follow the expression of the Nash-Moser's theorem by L. Hörmander [1] . $\{\mathbb{H}^s\}_{s \geq 0}$ is a Banach scale, in other words, interpolation spaces by means of a smoothing operator.

Let $\Phi(u)$ be a nonlinear operator on $\mathbb{H}^\infty = \bigcap_s \mathbb{H}^s$. Assume the existence of the first and second derivatives in u and their estimates as similar as (2.2). The different point is to assume that the right (left, resp.) parametrix ψ of the first derivative $D\Phi$ exists on a neighborhood of the origin of (u,Φ) such that

$$D\Phi.\psi(\psi D\Phi, \text{resp.}) = I + \Theta \quad ,$$

where Θ depends on u and Φ , and satisfies

(2.2) $$\|\Theta\varphi\|_s \leqslant C_s[(1+\|u\|_s)\|\Phi\|_m\|\varphi\|_m + (1+\|u\|_m)\ (\|\Phi\|_{s+m}\|\varphi\|_m + \|\Phi\|_m\|\varphi\|_{s+m})]$$

for a fixed m and for all s .

Then, there exist neighborhood V , W of $u = 0$ and $\Phi = 0$ such that

$$\Phi(V) \cap W = \emptyset \quad \text{or} \quad \Phi(V) \ni 0$$
$$(\Phi(0) = 0 \Rightarrow \Phi^{-1}(0) \cap V = \{0\}, \text{ resp.}) .$$

Therefore we can conclude the existence and the uniqueness only for $\Phi(u) = 0$. In the proof, we need only an addition of the error terms in the argument of the existence of Θ , which is estimated as well as other error terms are. If considering the scale basing on the Sobolev spaces or the Hölder spaces of functions on Ω_ε supported on $\{x_0 \geqslant 0\}$, then the strong wellposedness assures the existence of the right and left parametrix ψ . The non essential cases will be excluded since the norm $\|\Phi(0)\|_s$ on Ω_ε tends to 0 as ε tends to 0 .

§ III. AN EXAMPLE :

The Monge-Ampère equation

$$u_{xx}u_{yy} - (u_{xy})^2 = f(x,y)$$

is elliptic if $f > 0$, strictly hyperbolic if $f < 0$ and of Tricomi type if $\nabla f \neq 0$ at the points $f = 0$ and if it is kowalevskian. They are very classical and well known. So we treat more singular cases. Since we are treating the Cauchy problems, we consider the case where $f \leqslant 0$. In this case, the equation is hyperbolic if it is kowalevskian. More generally, we consider

$$\Phi = \det A - f = 0 ,$$

where $A = \partial^2 u + C(x,u,\partial u)$, C is a symmetric matrix and f is also a function in $(x,u,\partial u)$. A typical example is the Gauss curvature $K(x)$ of a hypersurface $\{y = u(x)\}$, $x \in \mathbb{R}^n$

$$\det(\partial^2 u) = K(x)\ (1+|\partial u|^2)^{(n+2)/2} .$$

Now, we prepare some notation to state the result.

w.r.t. $A_{ij} = (a_{k\ell})_{k,\ell \neq i,j}$ = minor matrix w.r.t. a_{ij} .

$A^{co} = (a^{co}_{ij})$ $a^{co}_{ij} = (-1)^{i+j} \det A_{ji}$: cofactor matrix .

The equation is

$$(3.1)\qquad \begin{cases} \Phi = \det A - f = 0 \\ \text{the initial condition at } x_o = 0 . \end{cases}$$

If we take the Fréchet derivative in u , then

$$D\Phi\varphi = \mathrm{Tr}(A^{co}\partial^2\varphi) + \text{lower term}$$
$$= A^{co}(\partial)\,\varphi + \dots \quad .$$

We assume the following (3.2-4).

(3.2) $\qquad f \leqslant 0 \quad$ always.

(3.3) $\qquad A_{oo}(u)\big|_{x_o = 0} > 0$.

(3.4) $\quad L^2 f(x,u,\partial u) < 0$ at $\{f = 0\} \cap \{x_o = 0\}$, where L is a vector field defined by

$$L = \sum_{j=o}^{n} a_{o_j}^{c_o}\left(\frac{\partial}{\partial x_j}\right) .$$

Theorem 4. Let $\tilde{u}$ be a formal solution of (3.1) at $x_o = 0$ and satisfy (3.2-4) on a neighborhood of x = 0 . Then there exists a unique C^∞ solution u of $\Phi(u) = 0$ on a neighborhood of x = 0 such that $u - \tilde{u}$ is flat at $x_o = 0$ $(\partial^\alpha(u-\tilde{u})\big|_{x_o=o} = 0, \forall\alpha)$.

Example. $u = \frac{1}{2}x^2 - \frac{1}{12}y^4$.

$$\Phi(u) = u_{xx}u_{yy} - (u_{xy})^2 + y^2 = 0 ,$$
$$D\Phi(u)\varphi = \partial_y^2\varphi - y^2\partial_x^2\varphi ,$$
$$f = -y^2 \text{ and } L = \frac{\partial}{\partial y} .$$

In fact, let us put eigenvalues and eigenprojections of A^{co} by θ_j and p_j $(i = 0,\dots,n)$, and λ_j are eigenvalues of A. Then $\theta_j = \pi_{k\neq j}\lambda_k$ and eigen spaces of θ_j and λ_j are the same, so their projection is p_j . The assumption (3.3) gives us the existence of n positive eigenvalues, so we denote them by $\lambda_j > 0$ $(j = 1,\dots,n)$.
Since $\det A = \lambda_o \dots \lambda_n = f \leqslant 0$ at $u = \tilde{u}$ and $x_o = 0$, so $\lambda_o \leqslant 0$, there. Hence λ_o his near non positive axis if u is near $\tilde{u}$, that is, remainding eigenvalue λ_o is separated from the others λ_j $(j \geqslant 1)$.

Therefore $\theta_o = \lambda_1 \ldots \lambda_n$ and P_o are defined smoothly. So A^{co} is written as

$$A^{co} = \Sigma_{j=o}^{n} \theta_j P_j = \lambda_1 \ldots \lambda_n P_o + \ldots + \lambda_o \ldots \lambda_{n-1} P_n$$

$$= \theta_o P_o + \lambda_o \ldots \lambda_n (\Sigma_{j=1}^{n} \lambda_j^{-1} P_j)$$

$$= \theta_o P_o + (\det A)\, E$$

$$= \theta_o P_o + f\, E + \Phi\, E\ ,\quad E = \Sigma_{j=1}^{n} \lambda_j^{-1} P_j\ .$$

Then we can decompose

$$A^{co}(\partial) = Q_o(\partial) + R_o(\partial)$$

as

$$Q_o(\partial) = \theta_o P_o(\partial) + f\, E(\partial)$$

and

$$R_o(\partial) = \Phi . E(\partial).$$

Here $Q_o(\partial) \sim L^2 + f$. (an elliptic operator with respect to the transversal directions to L).

Since $f \leqslant 0$, and $L^2 f \neq 0$ at $f = 0$, so Q_o is effectively hyperbolic. And also $R_o(\partial) = 0$ at $\Phi = 0$. Therefore $A^{co}(\partial)$ satisfies the conditions of Corollary 3. Using some speciality of the Monge-Ampère equation, we conclude the existence theorem from the unique extension theorem.

Other examples of effectively hyperbolic operators will be found in N. Iwasaki [2] , where you can see the further informations about this note.

BIBLIOGRAPHY :

[1] L. Hörmander, The boundary problems of physical geodesy, Arch. Rat. Mech. Anal., 62 (1976), 1-52.

[2] N. Iwasaki, The strongly hyperbolic equation and its applications, to appear in Pattern and Wave (Studies math. appl.) North-Holland Publishing Company, 1986.

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