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HYPOELLIPTIC OPERATORS WITH DOUBLE CHARACTERISTICS

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I want to discuss pseudo-differential operators of the form $P = P_m(x,D) + P_{m-1}(x,D) + \dots$ such that in a conic neighborhood Γ of $(x_o,\xi_o) \in \mathbb{R}^{2n}$, $P_m(x,\xi) \geq 0$ and vanishes to exactly second order on a submanifold Σ of codimension 1 transverse to the fiber axis. P is a classical pseudo-differential operator of order m. These assumptions imply that $P_m = QU^2$ where $Q \neq 0$ and $d_\xi U \neq 0$ near (x_o,ξ_o), Q and U are real and homogeneous of order m-2 and 1 respectively. Recall that the definition of subprincipal symbol of P is

$$P^s_{m-1}(x,\xi) = P_{m-1}(x,\xi) - \frac{1}{2i}\sum_{j=1}^{n}\frac{\partial^2 P_m}{\partial x_j \partial \xi_j} \quad .$$

The first two results I want to talk about concern necessary and sufficient conditions for P to be locally solvable and hypoelliptic.

Theorem 1 : Suppose that $\mathrm{Re}\, P^s_{m-1} \neq 0$ in Γ and that whenever $\mathrm{Re}\, P^s_{m-1} < 0$ at a point of Σ then along the null bicharacteristic of U through that point $\mathrm{Im}\, P^s_{m-1}$ has only zeros of even order $\leq k$.

(A) Then given any $\varphi \in S^o(\Gamma)$, $\varphi(x_o,\xi_o) \neq 0$ there exist operators E_i, and R_i, $i = 1,2$ such that

(1) $E_i : H_s(\Gamma) \to H_{s+m-2+(k+2)/2(k+1)}(\Gamma)$ and

(2) $R_i : H_s(\Gamma) \to H_{s+m-2+1/k+1}(\Gamma)$ are bounded and

(3) $PE_1 = \varphi(x_1D)I + R_1$

(4) $E_2P = \varphi(x_1D)I + R_2$

(B) P is hypoelliptic and locally solvable in Γ

In the converse direction there is

Theorem 2 : If at $(x_o,\xi) \in \Sigma$, $\mathrm{Re}\, P^s_{m-1}(x_o,\xi_o) < 0$ and $\mathrm{Im}\, P^s_{m-1}$ changes sign and has a zero of finite order on the null bicharacteristic of U through (X_o,ξ_o) then $P(x,D)$ is not locally solvable at x_o.

Note that since the hypotheses of theorem 2 also hold for P^*, P is also not hypoelliptic.

Theorem 2 is due to P. Wenston [5] for partial differential operators. Theorems 1 and 2 are due to P. Popivanov [2] and myself [1].

By means of Fourier integral operators the problem of studying P may be reduced to considering

$$Q = D_t^2 + a(t,x,D_x) + b(t,x,D_x,D_t)D_t \tag{5}$$

where $a \in S^1$, $b \in S^o$. This is a consequence of the invariance of the assumptions and conclusions of theorem 2 under multiplication by real elliptic factors or conjugation by Fourier integral operators. Noting that $a(t,x,\xi)$ is the sub-principal symbol of Q the assumptions of theorem 1 become that if $\operatorname{Re} a < 0$ then $\operatorname{Im} a(t,x,\xi)$ has only zeros of even order $\leq k$ as a function of t. This, of course, implies that a has constant sign.

To understand the hypotheses of theorems 1 and 2 consider the ordinary differential operator

$$L = D_t^2 + a(t,x,\xi) \tag{6}$$

depending on $(x,\xi) \in \mathbb{R}^{2n}$ as parameters. Considering functions which are oscillatory in x, the local solvability of Q^* is related to whether or not an estimate of the form

$$|||u||| \leq C\,||Lu|| \tag{7}$$

holds for some pair of norms on $C_o^\infty(\mathbb{R})$. Using the Green-Liouville approximation, $Lu \sim 0$ has solutions

$$u_{\pm}(t,x,\xi) = \frac{1}{a^{1/4}(t,x,\xi)}\, e^{\pm\int_{T(x,\xi)}^{t}\sqrt{a(t',x,\xi)}\,dt'} \tag{8}$$

If $\operatorname{Re} a < 0$ and $\operatorname{Im} a(t,x,\xi)$ changes signs at $T(x,\xi)$ then $\operatorname{Re}\sqrt{a}$ will also have a change of sign. This means that $Lu \sim 0$ has a solution in $\mathcal{S}$ and consequently an estimate of form (7) cannot hold.

To construct parametrices under the assumptions of theorem 1, I will use the notion of vector-valued pseudo-differential operators. The idea of using pseudo-differential operators whose symbols are operators between Hilbert space originated in Trèves [4]. I shall use an extension of this idea due to Sjöstrand [3] which allows the norms on the Hilbert spaces to vary.

Let H_1 and H_2 be a pair of Hilbert spaces whose norms depend on $\xi \in \mathbb{R}^n$ such that

$$(9) \qquad c\|u\|_{H_i} \leq \|u\|_{H_i(\xi)} \leq C(1+|\xi|)^{m_i}\|u\|_{H_i}$$

for $i = 1,2$. Let $\mathcal{L}(H_1(\xi),H_2(\xi))$ be the space of bounded operators from $H_1 \to H_2$ with the uniform operator norm, also varying with ξ. I will define the symbol in class $S^m_{\rho,\delta}(\Omega \times \mathbb{R}^n; H_1(\xi),H_2(\xi))$ to be the class of functions $A(x,\xi) : \Omega \times \mathbb{R}^n \to \mathcal{L}(H_1(\xi),H_2(\xi))$ such that

$$(10) \qquad \|\partial_x^\alpha \partial_\xi^B A(x,\xi)\|_{\mathcal{L}(H_1(\xi),H_2(\xi))} \leq C\,(1+|\xi|)^{m+\delta|\alpha|-\rho|\beta|}$$

The corresponding class of operators given by

$$A(x,D)u(x) = (2\pi)^{-n}\int e^{ix\xi}A(x,\xi)\hat{u}(\xi)d\xi$$

will be denoted by $L^m_{\rho,\delta}(\Omega,H_1(D),H_2(D))$, where $\Omega \subset \mathbb{R}^n$. These operators are at least maps

$$A(x,D) : C_o^\infty(\Omega,H_1) \to C^\infty(\Omega,H_2)$$

Furthermore, the standard calculus of pseudo-differential operators still holds for these operators.

The special case of such operators I will use is obtained by taking $H = L^2(\mathbb{R},dt)$ and $B(\xi)$ to be the completion of $C_o^\infty(\mathbb{R})$ in the norm

$$\|u\|_{B(\xi)} = \|(1+|D_t|^2+|D_x|)^{(k+2)/2(k+1)}u\|_{L^2}$$

An example of the boundlessness of results for this class of operators is that if $A \in L^o_{\rho,\delta}(\mathbb{R}^n;B(D),H)$ and $0 \leq \delta < 1/2 < \rho \leq 1$ then for any compact subset K of Ω

$$\|Au\|_{L^2(\mathbb{R}^{n+1})} \leq C\|(1+|D_t|^2+|D_x|)^{(k+2)/2(k+1)}u\|$$

for all $u \in C_o^\infty(K \times \mathbb{R})$.

To construct a parametrix of $Q = D_t^2 + a(t,x,D_x)$ define

$$(11)\quad e(x,\xi,t,s) = \begin{cases} \dfrac{1}{2(a(t,x,\xi)a(s,x,\xi))^{1/4}} \, e^{-\int_t^s \sqrt{a(t',x,\xi)}dt'} & \text{if } t \le S \\ e(x,\xi,s,t) & \text{if } s \le t \end{cases}$$

Suppose $\operatorname{Re} a < 0$, and $\operatorname{Im} a \ge 0$, then $a^{1/2}$ is a smooth function of all its variables. Choose the square root in (11) so that $\operatorname{Re} a^{1/2} \ge 0$. Define the corresponding integral operator with kernel $g\,e$ as

$$(12)\qquad E_g(x,\xi)f(t) = \int_{-\infty}^{\infty} g\, e(x,\xi,t,s)f(s)ds$$

where $g \in S^o(\mathbb{R}^{2n+2})$. My candidate for a parametrix of Q is $E_1(x,D)$.

A calculation will show that

$$(13)\qquad L(x,t,D_t)\ \ E(x,\xi) = I + E_g$$

where

$$(14)\qquad g = \left(\frac{a''}{4a} - \frac{5}{16}\ \frac{a'^2}{a^2}\right) \in S^o$$

Equation (13) may be thought of as a statement about the multiplication of the symbols of Q and $E(x,D)$. The crucial step in proving theorem 1 is

<u>Lemma 3</u> : With the above assumptions if $g \in S^o(\mathbb{R}^{2n+2})$ then $E_g(x,\xi) \in S^o_{\rho,\delta}(\Omega;H,B(\xi))_s$ where

$$\frac{1}{2} - \frac{1}{2(k+1)} = \delta < 1/2 < \rho = \frac{1}{2} + \frac{1}{2(k+1)}$$

To prove lemma 3, I must show that

$$(15)\qquad \left\| (1 + (D_t)^2 + |\xi|)^{(k+2)/2(k+1)} Eu \right\|_{L^2} \le C \|u\|_{L^2}$$

and similar estimates for $\partial_x^\alpha\, \partial_\xi^{\beta} E(x,\xi)$. I will need the following lemmas.

<u>Lemma 4</u> : Suppose $k(t,s)$ is a measurable function on $\mathbb{R}^2$ such that

$$\int |k(t,s)|\,ds \le B \qquad \text{and} \qquad \int |k(t,s)|\,dt \le B .$$

Then $\mathcal{K} f(t) = \int k(t,s)f(s)ds$ is a bounded operator on $L^2(\mathbb{R})$ and $\|\mathcal{K}\| \le B$.

Lemma 5 : If $0 \le a(t,X,\xi) \in S^0(\mathbb{R}^{n+1} \times \mathbb{R}^n)$ and has zeros in t of order $\le k$, then there is a constant $c > 0$ such that

$$c|t-s|^{k+1}|\xi| \le \int_t^s a(t',x,\xi)dt' \tag{16}$$

For a proof see the appendix of [4].

Lemma 6 : Given $C > 0$ there is a constant $C' > 0$ such that for any complex number z $|\operatorname{Im} z| \le C|\operatorname{Re} z|$ implies $|\operatorname{Re} z^{1/2}| \ge C'|\operatorname{Im} z|/|\operatorname{Re} z|^{1/2}$.

To prove (15), I will estimate

$$I = \int_t^\infty |e(x,\xi,t,s)|\,ds \quad \text{and} \quad \int_{-\infty}^t |e(x,\xi,t,s)|\,ds .$$

Using the definition of E,

$$I \le \int_t^\infty \frac{1}{|a(t)a(s)|^{1/4}} e^{-\int_s^t \sqrt{a}\,dt} ds . \tag{17}$$

Since $\operatorname{Re} a \ne 0$ $|a(s)\,a(t)|^{1/4} \ge C|\xi|^{1/2}$. Combining, lemmas 5 and 6 give

$$\operatorname{Re} \int_t^s \sqrt{a(t')}dt' \ge C|\xi|^{1/2}|t-s|^{k+1}$$

Using the last two inequalities in (17), I get

$$I \le C|\xi|^{-1/2}\int_t^\infty e^{-c|\xi|^{1/2}|t-s|^{k+1}} ds = C|\xi|^{-\frac{1}{2}-\frac{1}{2(k+1)}} .$$

Lemma 4, then yields the bound

$$\| \, |\xi|^{(k+1)/2(k+2)} E(x,\xi)u\| \le C\|u\| .$$

Since $D_t^2 E = -a(t,x,\xi)E +$ bounded operators and $a \in S^1$ we also may estimate twice as many t-derivatives as x-derivatives. This gives (15).

I am now in a position to complete the proof of theorem 1 by applying the calculus of pseudo-differential operators. Since D_t^2 is independent of t the symbol of $D_t^2 \circ E(x,D_x)$ is D_t^2 applied to the symbol of E. To compute the composition of a(t,x,D) and E, consider a as being in the class $S^1(\Omega \times \mathbb{R}^n ; B(\xi)\ B(\xi))$. I then have

$$a(t,x,D_x) \circ E(x,D_x) = (aE)(x,D) \bmod L'^{-\min(\rho,1-\delta)}(H,B(\xi)).$$

Combining the above observations with (13) gives

$$Q(t,x,D_t,D_k) \circ E(x,D_x) = I + R \quad ,$$

where $R \in L^{k/2(k+1)}(\Omega;H,B(D))$.

For R,there will be the estimate

$$\|(1+|D_t|^2+|D_x|)^{1/k+1}Ru\| \leq C\|u\|, \quad u \in C_o^\infty(\mathbb{R}^{n+1}) .$$

This completes the proof of theorem 1.

An extension of the argument used for theorem 1 will give

<u>Theorem 7</u> : Let $L = D_t^2 + a(t,x,\xi) + b(t,x,D_t,D_x)D_t + \ldots$ where $a \in S^1$, $b \in S^o$ and suppose that $\mathrm{Re}\, a \neq 0$ and if $\mathrm{Re}\, a < 0$ then Im a has a constant sign as a function of t and Im a never vanishes on an open t-interval for fixed x and ξ.

Then, L is locally solvable and for any $\varepsilon > 0$ there is a neighborhood ω of sufficiently small diameter such that

$$\|u\|_{s+m-3/2} \leq \varepsilon \|PL\, u\|_s + C\|u\|_{s+m-2} \quad , \quad u \in C_o^\infty(\omega) .$$

Operators whose sub-principal symbol vanishes on the characteristic variety may also be treated. Now let

$$Q = D_t^2 + b(t,x,D_x,D_t)$$

where $b(0,x,\xi) = t^k a(t,x,\xi)$, $a \neq 0$.

A parametrix for L may be attempted to be constructed in the same way as the one in theorem 1 was . Consider the ordinary differential operator $L = D_t^2 + t^k a(t,x,\xi)$. A solution of $Lu \sim 0$ may be sought of the form

$$u = g(t,x,\xi)\, V(\phi(t,x,\xi))$$

where V is a solution of $-V'' + s^k V(s) = 0$. This will force that

$$\phi = \left((k+2/2 \int_0^t \sqrt{t^k a}\, dt\right)^{2/k+2} ,$$

and $g = \phi_t^{-1/2}$. Then $Lu = \left(-\frac{d^2 g}{dt^2}\right) V(\phi)$.

If $Lu \sim 0$ has two independent solutions which increase and decrease exponentially on opposites sides of the t-axis, a parametrix may again be constructed. For instance, I have shown

<u>Theorem 8</u> : If either $a(0,x_0,\xi_0)$ is not real, ϕ or k is even and $a(0,x_0,\xi_0) > 0$ then L is hypoelliptic and locally solvable and has left and right parametrices.

<u>Theorem 9</u> : If $a(0,x_0,\xi_0)$ is real and Im a has a zero of first order then Q is hypoelliptic.

<u>Theorem 10</u> : If $a(0,x,\xi)$ is real and $\operatorname{Im} b(0,x,\xi) \neq 0$, then

$$Q = D_t^2 + t^k(a(t,x,D_x) + t^\ell b(t,x,D_x))$$

is locally solvable when $\ell < k+2$.

I don't know what the situation is for the operators of theorem 10 if $\ell \geqslant k+2$, nor do I have non-solvability results if $k > 1$ in theorem 8 etc...

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