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ON THE EULER EQUATIONS OF INCOMPRESSIBLE PERFECT FLUIDS
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R. TEMAM

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ON THE EULER EQUATIONS OF INCOMPRESSIBLE PERFECT FLUIDS

Roger TEMAM (*)

INTRODUCTION.

Let Ω be a bounded domain of $\mathbb{R}^3$ with smooth boundary Γ . The motion of an incompressible perfect fluid filling Ω is governed by the Euler equations

$$(0.1) \quad \frac{\partial u}{\partial t} + \sum_{j=1}^3 u_j \frac{\partial u}{\partial x_j} + \text{grad } \pi = f \quad \text{in } \Omega \times (0, T),$$

$$(0.2) \quad \text{div } u = 0 \quad \text{in } \Omega \times (0, T)$$

$$(0.3) \quad u \cdot n = 0 \quad \text{on } \Gamma \times (0, T)$$

$$(0.4) \quad u(x, 0) = u_0(x) \quad \text{in } \Omega,$$

where $f = f(x, t)$, $u_0 = u_0(x)$ are given, $u(x, t) = u = (u_1, u_2, u_3)$ and $\pi = \pi(x, t)$ are the unknowns, the velocity vector and the pressure; n is the unit outward normal on Γ .

The problem of existence and uniqueness of solutions of the Euler equations has been considered by several authors and most recently by T. Kato [4], [5], D. Ebin and J. Marsden [3], J.P. Bourguignon and H. Brezis [2]. In [4] T. Kato proves the existence of a global solution in the two dimensional case and in [5] the existence of a local solution in the three dimensional case, for $\Omega = \mathbb{R}^3$. The existence of a local solution in the general case, i.e. Ω a domain of $\mathbb{R}^3$ with a boundary, was then proved by D. Ebin and J. Marsden [3] using techniques of Riemannian Geometry on infinite dimensional manifolds, and by J.P. Bourguignon and H. Brezis [2] who give an alternate proof of the local existence, more analytical but relying still on geometrical techniques.

Our purpose here is to give a new short proof of this result, using a new local a priori estimate and standard techniques in partial differential equations. Our proof is essentially an extension of that of T. Kato [5] to bounded domain, with a suitable treatment of the boundary terms which do not appear in [5].

The author thanks J. Marsden for interesting discussions on this problem.

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PLAN.

1. A priori estimates of the solutions of the Euler Equations.
2. The existence and uniqueness result.

1. A PRIORI ESTIMATE OF THE SOLUTIONS OF THE EULER EQUATIONS.1.1. Notations.

We will use classical notations and results concerning the Sobolev spaces:

$W^{s,p}(\Omega)$, s integer, $1 \leq p < \infty$, is the Sobolev spaces of real valued L^p functions on Ω , such that all their derivatives up to order s belong to $L^p(\Omega)$. If $p = 2$, we write $H^s(\Omega) = W^{s,2}(\Omega)$.

We write (f,g) , $|f|$, the scalar product and the norm in $L^2(\Omega)$, $((f,g))_m$ and $\|f\|_m$, the scalar product and the norm in $H^m(\Omega)$,

$$((f,g))_m = \sum_{|\alpha| \leq m} (D^\alpha f, D^\alpha g),$$

where D^α is a multi-index derivation, $\alpha = \{\alpha_1, \alpha_2, \alpha_3\}$. The norm in $L^p(\Omega)$ is denoted $|f|_p$ and $\|f\|_{m,p}$ denotes that of $W^{m,p}(\Omega)$. The same notations will be used also for the norms and scalar products in $L^2(\Omega)^3$, $H^m(\Omega)^3$,

We assume that the boundary of Ω is a two dimensional manifold of class $\mathcal{C}^r$ with r sufficiently large so that the usual embedding theorems hold. In particular : $W^{m,p}(\Omega) \subset L^r(\Omega)$ where $\frac{1}{r} = \frac{1}{p} - \frac{m}{3}$ if $m < \frac{3}{p}$, $1 \leq r < \infty$ is arbitrary if $m = \frac{3}{p}$, $r = \infty$ if $m > \frac{3}{p}$ (in this case $W^{m,p}(\Omega)$ is even a space of Hölderian functions).

We recall also that if $m > \frac{3}{p}$, (and Ω is smooth), $W^{m,p}(\Omega)$ is an algebra for the pointwise multiplication of functions (see [2], [3]).

Let

$$X_m = \{v \in H^m(\Omega)^3, \operatorname{div} v = 0, v \cdot n = 0 \text{ on } \Gamma\}$$

$$X_{m,p} = \{v \in W^{m,p}(\Omega)^3, \operatorname{div} v = 0, v \cdot n = 0 \text{ on } \Gamma\}.$$

For $m = 0$, X_0 is a closed subspace of $L^2(\Omega)^3$ and we denote P the orthogonal projection in $L^2(\Omega)^3$ on X_0 . We recall that P is also a linear continuous operator

from $W^{m,p}(\Omega)^3$ into itself ($m \geq 1$). Indeed if $v \in W^{m,p}(\Omega)^3$, then $(I-P)v = \text{grad } \pi$, where π is solution of the Neuman problem

$$(1.1) \quad \begin{cases} \Delta \pi = \text{div } v & (\in W^{m-1,p}(\Omega)) \\ \frac{\partial \pi}{\partial n} = v \cdot n & (\in W^{m-\frac{1}{p},p}(\Gamma)) \end{cases}$$

and $\pi \in W^{m,p}(\Omega)$ by the classical results of regularity for the Neuman problem (Agmon Douglis Nirenberg [1]).

1.2. Representation of π as a functional of u .

We will now assume that u and π are solutions of (0.1)-(0.4) and we will establish an energy inequality satisfied by u . We assume at present that u and π are classical solutions of (0.1), (0.4) as smooth as necessary for the subsequent calculations to make sense.

The following result will be useful

Lemma 1.1. If u and π satisfy (0.1)-(0.3), then

$$(1.2) \quad \Delta \pi = \text{div } f - \sum_{i,j} D_j u_i \cdot D_i u_j, \text{ in } \Omega$$

$$(1.3) \quad \frac{\partial \pi}{\partial n} = f \cdot n + \sum_{i,j} u_i u_j \phi_{ij} \text{ on } \Gamma,$$

the functions ϕ_{ij} depending only on Γ , $D_i = \frac{\partial}{\partial x_i}$, $n = \{n_1, n_2, n_3\}$.

Proof. We get (1.2) by applying the divergence operator on both sides of (0.1).

Taking then the scalar product of each side of (0.1) with n , we get on Γ :

$$(1.4) \quad \frac{\partial \pi}{\partial n} = f \cdot n - \sum_{i,j} u_i (D_i u_j) n_j.$$

Since Γ is a smooth manifold, we can locally represent it by an equation

$$\phi(x) = 0,$$

and on the corresponding part of Γ (say Γ_0),

$$n(x) = \frac{\text{grad } \phi(x)}{|\text{grad } \phi(x)|}$$

(ϕ is a smooth function in some neighborhood Ω_0 of Γ_0).

Then

$$|\text{grad } \phi(x)| u_i(x) (D_i u_j(x)) \cdot n_j(x) = u_i(x) (D_i u_j(x)) D_j \phi(x) .$$

Since

$$u(x) \cdot n(x) = 0 \quad \text{on } \Gamma ,$$

we have

$$u(x) \cdot \text{grad } \phi(x) = 0$$

when $\phi(x) = 0$ and the gradients of these two functions are therefore parallel on Γ_0 :

$$D_i(u \cdot \text{grad } \phi) = k D_i \phi \quad \text{on } \Gamma_0 .$$

$$\sum_{j=1}^3 D_i u_j \cdot D_j \phi = - \sum_{j=1}^3 u_j \cdot D_{ij} \phi + k D_i \phi .$$

Whence with (0.3)

$$\sum_{i,j=1}^3 u_i \cdot D_i u_j \cdot D_j \phi = - \sum_{i,j=1}^3 u_i \cdot u_j \cdot D_{ij} \phi$$

and (1.3) follows with

$$(1.5) \quad \phi_{ij}(x) = \frac{D_{ij} \phi(x)}{|\text{grad } \phi(x)|}$$

1.3. Quadratic estimation of π in term of u .

Lemma 1.2. If u and π satisfies (0.1)-(0.3) then for each $t > 0$, for $m > \frac{5}{2}$,

$$(1.6) \quad \|\text{grad } \pi(t)\|_m \leq c_1 \{ \|f(t)\|_m + \|u(t)\|_m^2 \}$$

and for $m > 1 + \frac{3}{p}$,

$$(1.7) \quad \|\text{grad } \pi(t)\|_{m,p} \leq c_2 \{ \|f(t)\|_{m,p} + \|u(t)\|_{m,p}^2 \} ,$$

the constant c_1 depending only m and Ω , c_2 depending on p, m , and Ω .

Proof. We infer from (1.2), (1.3) and [1] that

$$\|\text{grad } \pi\|_{m,p} \leq c_0 \left\{ \|\text{div } f - \sum_{i,j} D_j u_i \cdot D_i u_j\|_{m-1,p} + \|f \cdot n + \sum_{i,j} u_i u_j \phi_{ij}\|_{W^{m-\frac{1}{p},p}(\Gamma)} \right\}.$$

By the triangle inequality and obvious majorations for f , it remains to estimate

$$\left\| \sum_{i,j} D_j u_i \cdot D_i u_j \right\|_{m-1,p} \quad \text{and} \quad \left\| \sum_{i,j} u_i u_j \cdot \phi_{ij} \right\|_{W^{m-\frac{1}{p},p}(\Gamma)}.$$

Since $m > 1 + \frac{3}{p}$, $W^{m-1,p}(\Omega)$ is an algebra and

$$\|D_j u_i \cdot D_i u_j\|_{m-1,p} \leq c_3 \|D_j u_i\|_{m-1,p} \|D_i u_j\|_{m-1,p}$$

(c_0, c_3 depend on m, p and Ω).

For the boundary term we write

$$\left\| \sum_{i,j} u_i u_j \phi_{ij} \right\|_{W^{m-\frac{1}{p},p}(\Gamma)} \leq c_4 \left\| \sum_{i,j} u_i u_j \right\|_{W^{m-\frac{1}{p},p}(\Gamma)}$$

where c_4 depends only on m, p , and the ϕ_{ij} i.e. Γ . Observing that $m - \frac{1}{p} > \frac{2}{p}$, we see that $W^{m-\frac{1}{p},p}(\Gamma)$ is an algebra and hence

$$\left\| \sum_{i,j} u_i u_j \right\|_{W^{m-\frac{1}{p},p}(\Gamma)} \leq c_5 \|u|_{\Gamma}\|_{W^{m-\frac{1}{p},p}(\Gamma)}^2$$

$\ll$ (by the trace theorems)

$$\ll c_6 \|u\|_{W^{m,p}(\Omega)}^2.$$

1.4 A priori estimate for $p = 2$.

Let α be a multi index, $|\alpha| \leq m$. We apply the operator D^α on each side of (0.1). We then multiply by $D^\alpha u$, integrate over Ω and add these equalities for $|\alpha| \leq m$. We obtain

$$\frac{1}{2} \left(\frac{d}{dt} \right) \|u\|_m^2 = - \sum_{j=1}^3 \left((u_j \frac{\partial u}{\partial x_j}, u) \right)_m - ((\text{grad } \pi, u))_m + ((f, u))_m.$$

The first term on the right can be majorized using T. Kato [4] {(2.2) p.298⁽¹⁾} and we find

$$\left| \sum_{j=1}^3 \left(u_j \frac{\partial u}{\partial x_j}, u \right)_m \right| \leq c' \|u\|_m^3,$$

where c' depends only on m .

For the other terms, we clearly have

$$\begin{aligned} ((f, u))_m &\leq \|f\|_m \|u\|_m, \\ -((\text{grad } \pi, u))_m &\leq \|\text{grad } \pi\|_m \|u\|_m \\ &\leq (\text{by Lemma 1.2}) \\ &\leq c_1 \{ \|f\|_m + \|u\|_m^2 \} \|u\|_m. \end{aligned}$$

Whence

$$\frac{1}{2} \left(\frac{d}{dt} \right) \|u\|_m^2 \leq c'_1 \|u\|_m^3 + c'_2 \|f\|_m \|u\|_m$$

$$c'_1 = c' + c_1, \quad c'_2 = 1 + c_1,$$

$$(1.8) \quad \left(\frac{d}{dt} \right) \|u\|_m \leq c'_1 \|u\|_m^2 + c'_2 \|f\|_m.$$

so that

$$(1.9) \quad \|u(t)\|_m \leq y(t), \quad 0 < t < T_0,$$

where y is the solution of the differential equation

$$(1.10) \quad \begin{cases} \frac{dy(t)}{dt} = c'_1 y(t)^2 + c'_2 \|f(t)\|_m, \\ y(0) = \|u_0\|_m, \end{cases}$$

and $(0, T_0)$, $0 < T_0 \leq +\infty$, is the interval of existence of y ; T_0 depends only on c'_1 , c'_2 , and the H^m -norms of the data f , u_0 .

⁽¹⁾ See (1.12) below giving a more general result using L^p norms, $p \neq 2$.

In conclusion if Ω is a smooth bounded domain, if u and π are smooth solutions of (0.1)-(0.4) and $m > \frac{5}{2}$, then the estimate (1.9) holds.

1.5. A priori estimate for $p \neq 2$.

We rapidly establish an estimate similar to (1.9), involving the norms in $W^{m,p}(\Omega)$, $m > 1 + \frac{3}{p}$.

We apply the operator D^α on each side of (0.1), we multiply by $|D^\alpha u|^{p-2} D^\alpha u$, integrate over Ω and add these equalities for $|\alpha| \leq m$. This leads to

$$(1.11) \quad \frac{1}{p} \left(\frac{d}{dt} \right) \|u\|_{m,p}^p = - \sum_{|\alpha| \leq m} (D^\alpha(\psi + \text{grad } \pi - f), |D^\alpha u|^{p-2} D^\alpha u),$$

where $\psi = \sum_j u_j \frac{\partial u}{\partial x_j}$. We prove hereafter that

$$(1.12) \quad \left| \sum_{|\alpha| \leq m} (D^\alpha \psi, |D^\alpha u|^{p-2} D^\alpha u) \right| \leq c_7 \|u\|_{m,p}^3.$$

From (1.7) and Holder inequality we see then that the right hand side of (1.11) is less than

$$c_7 \|u\|_{m,p}^3 + c_2 \{ \|f\|_{m,p} + \|u\|_{m,p}^2 \} \|u\|_{m,p} + \|f\|_{m,p} \|u\|_{m,p}.$$

Whence

$$(1.13) \quad \frac{d}{dt} \|u\|_{m,p} \leq c'_3 \|u\|_{m,p}^2 + c'_4 \|f\|_{m,p}$$

with

$$c'_3 = c_2 + c_7, \quad c'_4 = 1 + c_2.$$

We conclude from (1.11) that

$$(1.14) \quad \|u(t)\|_{m,p} \leq z(t), \quad 0 < t < T_1,$$

where z is the solution of

$$(1.15) \quad \begin{cases} \frac{dz}{dt}(t) \leq c'_3 z(t)^2 + c'_4 \|f(t)\|_{m,p} \\ z(0) = \|u_0\|_{m,p}, \end{cases}$$

and $(0, T_1)$ is the interval of existence of z .

There remains to establish (1.12).

Proof of (1.12). Application of the Leibnitz rule gives

$$(1.16) \quad D^\alpha \psi = (u \cdot \text{grad}) D^\alpha u + \sum_{\alpha < \beta \leq \alpha} c_{\alpha, \beta} (D^\beta u \cdot \text{grad}) D^{\alpha - \beta} u$$

Because of (0.2), (0.3), the contribution of the first term of (1.16) is zero, for each α . The contribution of the subsequent terms is less than

$$\sum_{\alpha < \beta \leq \alpha} |c_{\alpha, \beta}| | (D^\beta u \cdot \text{grad}) D^{\alpha - \beta} u |_p |D^\alpha u|_p.$$

In order to prove (1.12) it is then sufficient to show that

$$(1.17) \quad | (D^\beta u_i) (D_i D^{\alpha - \beta} u_j) |_p \leq c \|u\|_{m,p}^2,$$

for each i, j, α, β , $1 \leq i, j \leq 3$, $1 \leq |\alpha| \leq m$, $0 < \beta \leq \alpha$.

Let us show (1.17). We set $g = D^\beta u_i$, $h = D_i D^{\alpha - \beta} u_j$ and we observe that

$$g \in W^{m-|\beta|, p}(\Omega) \subset L^\rho(\Omega),$$

$$h \in W^{m-|\alpha|+|\beta|-1}(\Omega) \subset L^\sigma(\Omega),$$

$$|g|_p \leq c |g|_{m-|\beta|, p} \leq c \|u\|_{m,p},$$

$$|h|_\sigma \leq c |h|_{m-|\alpha|+|\beta|-1} \leq c \|u\|_{m,p},$$

for the values of ρ and σ given by Sobolev inclusion theorems ($m-|\beta| \geq 0$, $m-|\alpha|+|\beta|-1 \geq 0$ as $|\alpha| \geq |\beta| \geq 1$). If ρ or σ is infinite then we just write

$$|gh|_p \leq |g|_\infty |h|_p \leq c |g|_{m-|\beta|, p} |h|_p \leq c \|u\|_{m,p}^2$$

or

$$|gh|_p \leq |g|_p |h|_\infty \leq c |g|_p |h|_{m-|\alpha|+|\beta|-1} \leq c \|u\|_{m,p}^2.$$

If $|\beta| = m - \frac{3}{p}$, $\rho \geq 1$ is arbitrary, but in this case

$m-|\alpha|+|\beta|-1 = 2m-|\alpha|-\frac{3}{p}-1 \geq m-\frac{3}{p}-1 > 0$ by assumption. Hence $\sigma > p \geq 1$ and setting $\rho = \frac{\sigma}{\sigma-1}$, we write

$$|gh|_p \leq |g|_\rho |h|_\sigma \leq c \|u\|_{m,p}^2.$$

Similarly if $m-|\alpha|+|\beta|-1 = \frac{3}{p}$, then $\sigma \geq 1$ is arbitrary but in this case

$m-|\beta| = 2m-|\alpha|-1-\frac{3}{p} \geq m-1-\frac{3}{p} > 0$. Hence $\rho > p \geq 1$, we choose $\sigma = \frac{\rho}{\rho-1}$ and we write

$$|gh|_p \leq |g|_\sigma |h|_\rho.$$

The last case to consider is the case where ρ and σ are finite and given by

$$\frac{1}{\rho} = \frac{1}{p} - \frac{m-|\beta|}{3}, \quad \frac{1}{\sigma} = \frac{1}{p} - \frac{m-|\alpha|+|\beta|-1}{3}.$$

By Holder inequality (1.17) is satisfied in this case provided that

$$\frac{1}{\rho} + \frac{1}{\sigma} \leq \frac{1}{p}$$

i.e.

$$\frac{3}{p} - 2m + |\alpha| - 1 \leq 0$$

and this is true as $|\alpha| \leq m$ and $m > 1 + \frac{3}{p}$.

2. THE EXISTENCE AND UNIQUENESS RESULT.

Theorem. Assume that Ω is a regular bounded open set of $\mathbb{R}^3$ ⁽¹⁾; let m and p be given, $p \geq 1$, $m > 1 + \frac{3}{p}$. Then for each u_0 and f ,

$$(2.1) \quad u_0 \in W^{m,p}(\Omega)^3, \quad \operatorname{div} u_0 = 0, \quad u_0 \cdot n = 0 \quad \text{on } \partial\Omega,$$

$$(2.2) \quad f \in L^1(0,T; W^{m,p}(\Omega)^3),$$

there exists a unique function u and π , defined on $(0, T_*)$,

⁽¹⁾ It is sufficient to assume that $\partial\Omega$ is a two dimensional manifold of class C^{m+2} and Ω is locally situated on one side of $\partial\Omega$.

$$(2.3) \quad u \in L^\infty(0, T_*; W^{m,p}(\Omega)^3)$$

$$(2.4) \quad \pi \in L^\infty(0, T_*; W^{m+1,p}(\Omega))$$

where $T_* < \inf (T, T_1)^{(1)}$, and satisfying (0.1)-(0.4) on $(0, T_*)$.

Remarks. (i) The Theorem^{is} also valid in higher dimensions, with the natural modification on the assumption on m ($m > 1 + \frac{N}{p}$);

(ii) Because of the boundary layer effects we can not expect to prove as in Kato [5] the existence on $(0, T_*)$, for each $\nu > 0$ of a solution of the Navier Stokes equations belonging to $H^m(\Omega)^3$

The proof of uniqueness is standard. We will just show the existence of u and π , considering successively the case $p = 2$ and $p \neq 2$.

Case $p = 2$.

We apply the Galerkin method with a special basis $\{w_k\}$ which we first describe

(i) For m fixed as before, we consider the space $X_m \subset H^m(\Omega)^3$, endowed with the Hilbert scalar product $((\cdot, \cdot))_m$, and the space X_0 which is a closed subspace of $L^2(\Omega)^3$. It is clear that $X_m \subset X_0$ and X_m is dense in X_0 . By the Lax-Milgram theorem, for each $g \in X_0$, there exists a unique $w \in X_m$ such that

$$(2.5) \quad ((w, v))_m = (g, v), \quad \forall v \in X_m.$$

The linear mapping $g \mapsto w(g)$ is a compact self adjoint operator in X_0 and it possesses an orthonormal complete family of eigenvectors w_k :

$$(2.6) \quad \begin{cases} w_k \in X_m \text{ and} \\ ((w_k, v))_m = \lambda_k (w_k, v), \quad \forall v \in X_m. \end{cases}$$

(ii) Let us use the Galerkin method with this basis. For $\mu > 0$ fixed we look for

$$(2.7) \quad u_\mu = \sum_{j=1}^{\mu} g_{j\mu}(t) w_j$$

⁽¹⁾ See (1.10) and (1.15).

satisfying

$$(2.8) \quad \frac{d}{dt}(u_\mu, w_k) + ((u_\mu \cdot \text{grad})u_\mu, w_k) = (f, w_k), \quad 1 \leq k \leq \mu,$$

$$(2.9) \quad u_\mu(0) = u_{0\mu} = P_\mu u_0,$$

P_μ = the orthogonal projection in X_0 (or as well in X_m) on the space spanned by $w_1, \dots, w_k$.

The equations (2.8), (2.9) are equivalent to a system of ordinary differential equations for the $g_{j\mu}$, and the existence of a solution on some interval $(0, T_\mu)$ is standard. The following a priori estimates on u_μ show that $T_\mu = T_*$ is independent of μ .

(iii) The first a priori estimate is obtained by multiplying (2.8) by $g_{k\mu}(t)$ and adding in k . It is well known (see also §.1) that

$$((u_\mu \cdot \text{grad})u_\mu, u_\mu) = 0$$

and there remains

$$\frac{1}{2} \left(\frac{d}{dt} \right) |u_\mu|^2 = (f, u_\mu) \leq |f| \cdot |u_\mu|.$$

This shows that $T_\mu = T$ and that

$$(2.10) \quad u_\mu \text{ remains bounded in } L^\infty(0, T; L^2(\Omega)^3) \text{ as } \mu \rightarrow \infty.$$

We can also write (2.8) as

$$(2.11) \quad \left(\frac{du}{dt}, w_k \right) + (P[(u_\mu \cdot \text{grad})u_\mu], w_k) = (Pf, w_k)$$

($w_k \in X_0$). Now $P[(u_\mu(t) \cdot \text{grad})u_\mu(t)] \in X_m$, $Pf(t) \in X_m$, $\forall t$, (see (1.11)) and we can use (2.6). We multiply (2.11) by $\lambda_k g_k$ and add in k , $k = 1, \dots, \mu$. We obtain

$$(2.12) \quad \frac{1}{2} \left(\frac{d}{dt} \right) \|u_\mu\|_m^2 = ((P(f - (u_\mu \cdot \text{grad})u_\mu)), u_\mu)_m.$$

We have simply

$$P[f - (u_\mu \cdot \text{grad})u_\mu] = f - (u_\mu \cdot \text{grad})u_\mu - \text{grad } \pi_\mu,$$

where π_μ is defined in term of u_μ and f by relations similar to (1.2), (1.3) (replacing u by u_μ). The relation similar to (1.5) is satisfied and we get exactly the same relation as (1.8)

$$\left(\frac{d}{dt}\right) \|u_\mu\|_m^2 \leq c_1' \|u_\mu\|_m^2 + c_2' \|f\|_m.$$

We recall also that

$$\|u_\mu(0)\|_m = \|u_{0\mu}\|_m \leq \|u_0\|_m.$$

Whence,

$$\|u_\mu(t)\|_m \leq y(t), \quad \forall t < \inf(T, T_0)$$

and

$$(2.13) \quad \text{As } \mu \rightarrow \infty, u_\mu \text{ remains bounded in } L^\infty(0, T_*, H^m(\Omega)^3) \\ \forall T_* < \inf(T, T_0)$$

(iv) In order to pass to the limit in the non linear term using a compactness theorem, we need an estimate on $\frac{du_\mu}{dt}$.

Since the w_k are orthogonal in X_0 , we deduce from (2.11) that

$$\frac{du_\mu}{dt} = P_\mu P(f - (u_\mu \cdot \text{grad})u_\mu).$$

Hence

$$\left| \frac{du_\mu}{dt}(t) \right| \leq |f(t) - (u_\mu(t) \cdot \text{grad})u_\mu(t)|$$

and with (2.13) it is easily found that

$$(2.14) \quad \frac{du_\mu}{dt} \text{ remains bounded in } L^\infty(0, T_*; L^2(\Omega)^3) \text{ as } \mu \rightarrow \infty.$$

(v) The passage to the limit using (2.13), (2.14) and a compactness theorem (as in Lions [7]) is standard. We obtain at the limit the existence of $u \in L^\infty(0, T_*; X_m)$ such that

$$(2.15) \quad \frac{d}{dt}(u(t), v) + ((u(t) \cdot \text{grad})u(t), v) = (f(t), v) \quad \forall v \in X_0, \quad 0 < t < T_*.$$

$$(2.16) \quad u(0) = u_0$$

u satisfies all the properties announced, i.e. (0.2)-(0.4) and (2.3). Because of (2.15) the existence of π such that (0.1) is satisfied is standard (see Ladyzhenskaya [6]).

Case $p \neq 2$.

We proceed by regularization. We approximate u_0 and f by $u_{0\epsilon}$ and f_ϵ ,

$$u_{0\epsilon} \in X_s$$

$$f_\epsilon \in L^1(0, T; H^s(\Omega)^3)$$

with s sufficiently large so that

$$H^s(\Omega) \subset W^{m,p}(\Omega)$$

and $X_s \subset X_{m,p}$. We solve (0.1)-(0.4) with u_0 and f replaced by $u_{0\epsilon}$ and f_ϵ . The estimate analog to (1.14) and an easy estimate on $\frac{\partial u_\epsilon}{\partial t}$ allow us to pass to the limit as $\epsilon \rightarrow 0$ and we obtain (0.1)-(0.4) on $(0, T_*)$.

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