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ON ENFLO'S EXAMPLE OF A BANACH SPACE WITHOUT THE APPROXIMATION PROPERTY

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In this note we give an example of a Banach space without the approximation property. This answers negatively a problem posed by S. Banach near forty years ago. The first such example is due to Enflo [1]. Here it is presented with significient modifications which make the construction much more simpler, although "the main idea" is exactly the same as in the original work of Enflo. The modifications are due mainly to Figiel, but also to Pelczynski and to the author.

Let E, F be Banach spaces. By L(E,F) we shall denote the class of all continous linear operators from E into F, by L(E) the class L(E,E) and by $L_0(E)$ those operators from L(E) which are finite dimensional.

We say that Banach space E has the approximation property if

$$\forall \underset{C\text{-compact}}{C \subset E} \; \exists\, u \in L_0(E) \; \forall\, x \in C \; \|u(x)-x\| \leq 1$$

and we say that E has the bounded approximation property if

$$\exists\, M \; \forall \underset{C\text{-compact}}{C \subset E} \; \underset{\|u\| \leq M}{\exists u \in L_0(E)} \; \forall x \in C \; \|u(x)-x\| \leq 1$$

It was proved by A. Grothendieck [2] that if the Banach space E is reflexive then the approximation property and the bounded approximation property are equivalent.

Let E be a Banach space and E' its dual space. A family $\{e_n, e_n'\}_{n\in N}$ is called K-biorthogonal system if $e_n \in E$, $e_n \in E'$ $\|e_n\|$, $\|e_n'\| \leq K$ for $n \in N$ and $\langle e_n, e_m' \rangle = 0$ if $n \neq m$ and $\langle e_n, e_n' \rangle = 1$. For any $A \subset N$ let $E^A = \overline{\text{span}\,\{e_n | n \in A\}}$. For any $u \in L(E^A, E)$ let us define

$$tr_A u = \frac{1}{|A|} \sum_{n \in A} \langle u(e_n), e_n' \rangle \text{ where } |A| = \text{card}(A).$$

The following properties are an easy consequence of the definition :

1°) $tr_A u$ is a linear functional on L(E)

2°) $|tr_A u| \leq K \sup_{n \in A} \|u(e_n)\| \leq K^2 \|u\|$

3°) $tr_A I_d = 1$ where I_d is the identity operator in E.

Moreover there holds :

4°) if $\{(e_n),(e'_n)\}_{n\in N}$ is complete in E, e.g. $\overline{\text{span}\ \{e_n | n\in N\}} = E$ and $N_i \subset N$ for i = 1,2... is a sequence of subsets of N such that $\lim_{i\to\infty} |N_i| = +\infty$ then for each $u \in L_o(E)$, $\lim_{i\to\infty} tr_{N_i} u = 0$.

<u>Proof</u> : if $u = e_m \otimes x'$ $m \in N$, $x' \in E'$ then

$$tr_{N_i} u = \frac{1}{|N_i|} \sum_{n\in N_i} \langle e_m, e'_n\rangle\langle e_n, x'\rangle = \frac{1}{|N_i|} \sum_{n\in N_i} \delta^n_m \langle e_n, x'\rangle$$

and thus $|tr_{N_i} u| \leq K \frac{\|x'\|}{|N_i|}$ and hence $\lim_{i\to\infty} tr_{N_i} u = 0$.

Since $(e_n)_{n\in N}$ is linearly dense in E, the operators $e_m \otimes x'$, $m \in N$, $x' \in E'$ are linearly dense in the operator-norm topology in $L_o(E)$. Therefore by 1°) and 2°) we obtain that $\lim_{i\to\infty} tr_{N_i} u = 0$ for each $u \in L_o(E)$.

<u>Proposition 1</u> : Let $\{e_n, e'_n\}_{n\in N}$ be a K-biorthogonal, complete system in E. Let us assume that there exists a sequence of positive numbers (α_i) such that $\sum_{i=1}^{\infty} \alpha_i < \infty$ and a sequence (N_i) of finite subsets of N such that $\lim_{i\to\infty} |N_i| = +\infty$ and for each $u \in L(E)$ there holds

$$|tr_{N_{i+1}} u - tr_{N_i} u| \leq \alpha_i \|u\| \quad \text{for } i = 1, 2, \ldots$$

then E has not the bounded approximation property.

<u>Proof</u> : On the contrary let us assume that E has the bounded approximation property and that M is a constant as in the definition. Let i_o be such that $\sum_{i\geq i_o} \alpha_i \leq \frac{1}{4M}$ and let $u \in L_o(E)$ be such that $\|u(2K e_n) - 2K e_n\| \leq 1$ for each $n \in N_{i_o}$ and such that $\|u\| \leq M$.

Then $\|(u - I_d)(e_n)\| \leq \frac{1}{2K}$ for $n \in N_{i_o}$ and by 2°) we get $|tr_{N_{i_o}} u - tr_{N_{i_o}} I_d| \leq \frac{1}{2}$,

hence by 3°) $|tr_{N_{i_o}} u| \geq \frac{1}{2}$.

For each $i > i_o$ there holds

$$|tr_{N_i} u| = |tr_{N_{i_o}} u + \sum_{K=i_o}^{i-1} (tr_{N_{K+1}} u - tr_{N_K} u)| \geq$$

$$|tr_{N_{i_o}} u| - \sum_{K=i_o}^{i-1} |tr_{N_{K+1}} u - tr_{N_K} u| \geq \frac{1}{2} - \sum_{K=i_o}^{i-1} \alpha_K M \geq \frac{1}{4}$$

and thus $\lim_{i\to\infty} tr_{N_i} u \neq 0$ and this is a contradiction with 4^o.

Let T denote the unite circle $\{z \mid |z|=1\}$ in $\mathbb{C}^1$ and $d\theta$ the Haar measure on T.

By L_p, $1 \leq p < \infty$, we shall mean the Banach space of all measurable mappings $f: T \to \mathbb{C}^1$ such that

$$\|f\|_p = \left(\int_\Gamma |f(\theta)|^p d\theta\right)^{\frac{1}{p}} < \infty$$

The dual space L_p is isometric with L_q, where $\frac{1}{p} + \frac{1}{q} = 1$ and the duality is given by

$$\langle f,g\rangle = \int_\Gamma f\bar{g}d\theta \quad \text{for } f \in L_p,\ g \in L_q.$$

5°) The sequence $\{z^n, z^n\}_{n\in\mathbb{Z}}$ is an 1-biorthogonal complet system in L_p.

In the sequel C_p^1, $C_{p,\lambda}^2$, C_p^3,... will denote universal constant depending only on p and λ.

We shall need the following two properties of this system

6°) $\|\sum_{i=1}^n z^i\|_p \leq C_p^1\, n^{1-\frac{1}{p}}$ for each natural number n

7°) If $I = \{n_1, n_2, \ldots, n_k\}$ and for some $\lambda > 1$ $\frac{n_{i+1}}{n_i} \geq \lambda$ for $i = 1,\ldots k-1$ then

$$\|\sum_{i\in I} \alpha_i z^i\|_p \leq C_{p,\lambda}^2 \left(\sum_{i\in I} |\alpha_i|^2\right)^{\frac{1}{2}}$$

(6°) is obtained by easy computations and 7°) is the property of the "lacunary series" and it may be found in [3]).

<u>Lemma 1</u> : Let $1 \leq p < \infty$ and let I be a finite subset of Z such that $|I|$ is an even number. Then there exists $A \subset I$ such that $|A| = \frac{|I|}{2}$ and

$$\|\sum_{i\in A} z^i - \sum_{i\in I\setminus A} z^i\|_p \leq C_p^3 \ |I|^{\frac{1}{2}}$$

<u>Proof</u> : Let $(\epsilon_i)_{i\in I}$ be a sequence of independent random variables such that ϵ_i is didtributed by the rule

$$P(\epsilon_i=1) = P(\epsilon_i=-1) = \frac{1}{2}$$

Then by the Kintchine inequality :

$$E\|\sum_{i\in I} \epsilon_i \ z^i\|_p^p = E\int_T |\sum_{i\in I} \epsilon_i z^i|^p d\theta \leq C_p^4 \ |I|^{\frac{p}{2}}$$

Moreover $E|\sum_{i\in I} \epsilon_i|^2 = I$. Thus by the Chebyschev inequality we get

$$P(|\sum_{i\in I} \epsilon_i| \leq \sqrt{2} \ |I|^{\frac{1}{2}}) \geq \frac{1}{2}$$

This implies that there exists a sequence of signs $(\epsilon_i^o)_{i\in I}$ such that $\|\sum_{i\in I} \epsilon_i^o z^i\|_p \leq (2C_p^4)^{\frac{1}{p}} \ |I|^{\frac{1}{2}}$ and $|\sum_{i\in I} \epsilon_i^o| \leq \sqrt{2} \ |I|^{\frac{1}{2}}$.

Let A' be the set of those $i \in I$ for which $\epsilon_i^o = +1$ then

$$\|\sum_{i\in A'} z^i - \sum_{i\in I\setminus A'} z^i\|_p \leq (2C_p^4)^{\frac{1}{p}} \ |I|^{\frac{1}{2}} \text{ and}$$

$$|\ |A'| - |I\setminus A'|\ | \leq \sqrt{2} \ |I|^{\frac{1}{2}}$$

Therefore we can transfere from A' to its complement in I, or conversly , not more then $|I|^{\frac{1}{2}}$ arbitrary elements in such way that we obtain a set A such that $|A| = \frac{|I|}{2}$ and

$$\|\sum_{i\in A} z^i - \sum_{i\in I\setminus A} z^i\|_p \leq ((2C_p^4)^{\frac{1}{p}} + 2) \ |I|^{\frac{1}{2}}$$

This completes the proof.

Lemma 2 : Let $1 \le p < \infty$, and let $I = [1, 2^n]$ $n \ge 2$. Then there exist $A, B \subset I$ such that

$$|A| = \frac{|I|}{2}, \quad |B| = \frac{|I|}{4}, \quad A \cap B = \emptyset$$

and for each $u \in L(L_p^{A \cup B}, L_p)$ there holds

$$|tr_A u - tr_B u| \le C_p^5 \, 2^{n(\frac{1}{p} - \frac{1}{2})} \|u\|$$

Proof : Let A be such as in lemma 1, e.g. $|A| = \frac{|I|}{2}$ and

$\|\sum_{i \in A} z^i - \sum_{i \in I \setminus A} z^i\|_p \le C_p^3 (2^n)^{\frac{1}{2}}$. Let us apply once again lemma 1 to $I \setminus A$ and let B be such that $B \subset I \setminus A$ $|B| = \frac{|I|}{4}$ and such that $\|\sum_{i \in B} z^i - \sum_{i \in (I \setminus A) \setminus B} z^i\|_p \le C_p^3 (2^n)^{\frac{1}{2}}$. Then

$$* \ \|\sum_{i \in A} z^i - 2 \sum_{i \in B} z^i\|_p \le \|\sum_{i \in A} z^i - \sum_{i \in I \setminus A} z^i\|_p + \|\sum_{i \in (I \setminus A) \setminus B} z^i - \sum_{i \in B} z^i\|_p \le 2C_p^4 (2^n)^{\frac{1}{2}}$$

Let $u \in L(L_p^{A \cup B}, L_p)$ then let us put

$\bar{u} = \int_T \rho_\theta^{-1} u \, \rho_\theta \, d\theta$ where ρ_θ is an operator given by $\rho_\theta(f)(z) = f(\theta z)$.

It is easy to see that $\bar{u}$ is diagonal e.g.

$$\bar{u} = \sum_{i \in I} \lambda_i z^i \otimes z^i \quad \text{and} \quad \|\bar{u}\| \le \|u\|$$

Moreover $tr_A \bar{u} = tr_A u$ and $tr_B \bar{u} = tr_B u$. Let $g = \sum_{i \in I} z^i$ then

$$|tr_A u - tr_B u| = |tr_A \bar{u} - tr_B \bar{u}| = \frac{2}{|I|} |\langle u(\sum_{i \in A} z^i - 2 \sum_{i \in B} z^i), g\rangle| \le$$

$$\frac{2}{|I|} \|\bar{u}\| \, \|g\|_q \, \|\sum_{i \in A} z^i - 2 \sum_{i \in B} z^i\|_p \le \frac{2}{2^n} \|u\| C_q^1 (2^n)^{1 - \frac{1}{q}} \, 2C_p^4 (2^n)^{\frac{1}{2}} \le$$

$4C_q^1 \, C_p^4 \, (2^n)^{\frac{1}{p} - \frac{1}{2}}$. This completes the proof.

For $n \geq 3$ let A_n and B_n be subsets of $[1, 2^{[\frac{n+1}{2}]}]$ as in lemma 2 e.g.

$$A_n \cap B_n = \emptyset, \ |A_n| = 2^{[\frac{n+1}{2}]-1}, |B_n| = 2^{[\frac{n+1}{2}]-2}$$

and such that for each $u \in L(L_p^{A_n \cup B_n}, L_p)$

$$|tr_{A_n} u - tr_{B_n} u| \leq C_p^5 \|u\| \, 2^{[\frac{n+1}{2}](\frac{1}{p} - \frac{1}{2})}$$

Let us enumerate the elements of A_n and B_n by

$$A_n = \{a_1^n, a_2^n, \ldots . a^n_{2^{[\frac{n+1}{2}]-1}}\}, \ B_n = \{b_1^n, b_2^n, \ldots b^n_{2^{[\frac{n+1}{2}]-2}}\}$$

and let us denote $c_j^n = 2^{2^n+j}$ and
$N = \{(n,i,j) | n,i,j$ positive integers, $n \geq 3$, $1 \leq i \leq 2^{[\frac{n+1}{2}]-1}$, $1 \leq j \leq 2^{[\frac{n}{2}]-1}\}$

For $(n,i,j) \in N$ let $e_{(n,i,j)} = \frac{1}{\sqrt{2}}(z^{a_i^n + c_j^n} + z^{b_j^{n+1} + c_i^{n+1}})$

Let $E_p = \overline{\text{span}\,\{e_{(n,i,j)} | (n,i,j) \in N\}}$ in L_p

8°) $\{e_{(n,i,j)}, e_{(n,i,j)}\}_{(n,i,j) \in N}$ is a $\sqrt{2}$-biorthogonal complete system in E_p.

9°) for each $x \in E_p$
$$\langle x, e_{(n,i,j)}\rangle = \sqrt{2}\langle x, z^{a_i^n + c_j^n}\rangle = \sqrt{2}\langle x, z^{b_j^{n+1} + c_i^{n+1}}\rangle .$$

Let $N_m = \{(n,i,j) | (n,i,j) \in N, n = m\}$

<u>Lemma 3</u> : Let $2 \leq p < \infty$. For each $u \in L(E_p)$

$$|tr_{N_n} u - tr_{N_{n-1}} u| \leq C_p^6 \, 2^{\frac{n}{2}(\frac{1}{p} - \frac{1}{2})} \|u\|$$

<u>Proof</u> : For any $u \in L(E_p)$ it is

$$\mathrm{tr}_{N_n} u - \mathrm{tr}_{N_{n-1}} u = \frac{1}{2^{[\frac{n+1}{2}]-1}} \cdot \frac{1}{2^{[\frac{n}{2}]-1}} \sum_{i=1}^{2^{[\frac{n+1}{2}]-1}} \sum_{j=1}^{2^{[\frac{n}{2}]-1}} \langle u(e_{n,i,j}), e_{n,i,j}\rangle -$$

$$- \frac{1}{2^{[\frac{n}{2}]-1}} \frac{1}{2^{[\frac{n-1}{2}]-1}} \sum_{i=1}^{2^{[\frac{n}{2}]-1}} \sum_{j=1}^{2^{[\frac{n-1}{2}]-1}} \langle u(e_{n-1,i,j}), e_{n-1,i,j}\rangle =$$

$$\frac{1}{2^{[\frac{n}{2}]-1}} \sum_{j=1}^{2^{[\frac{n}{2}]-1}} \left(\frac{1}{2^{[\frac{n+1}{2}]-1}} \sum_{i=1}^{2^{[\frac{n+1}{2}]-1}} \langle u(e_{n,i,j}), e_{n,i,j}\rangle - \frac{1}{2^{[\frac{n-1}{2}]-1}} \sum_{i=1}^{2^{[\frac{n-1}{2}]-1}} \langle u(e_{n-1,j,i}), e_{n-1,j,i}\rangle \right)$$

Let us fix j and let $s \in L(L_p^{A_n \cup B_n}, E_p)$ be given by

$$s(z^{a_i^n}) = e_{n,i,j} \quad i = 1,2\ldots 2^{[\frac{n+1}{2}]-1}$$

$$s(z^{b_i^n}) = e_{n-1,j,i} \quad i = 1,2,\ldots 2^{[\frac{n+1}{2}]-1}$$

and let $r \in L(E_p, L_p)$ be given by $r(f)(z) = z^{-c_j^n} f(z)$.

Then by 9°) :

$$\left| \left(\frac{1}{2^{[\frac{n+1}{2}]-1}} \sum_{i=1}^{2^{[\frac{n+1}{2}]-1}} \langle u(e_{n,i,j}), e_{n,i,j}\rangle - \frac{1}{2^{[\frac{n-1}{2}]-1}} \sum_{i=1}^{2^{[\frac{n-1}{2}]-1}} \langle u(e_{n-1,j,i}), e_{n-1,j,i}\rangle \right) \right| = \sqrt{2}\,|\mathrm{tr}_{A_n} rus - \mathrm{tr}_{B_n} rus| \le C_p^5\, 2^{\frac{n}{2}(\frac{1}{p}-\frac{1}{2})} \|r\,u\,s\| \le$$

$C_p^5\, 2^{\frac{n}{2}(\frac{1}{p}-\frac{1}{2})} \|r\|\, \|u\|\, \|s\|$. But $\|r\| = 1$. Let us compute the norm of s :

Let $x = \sum_{i=1}^{2^{[\frac{n+1}{2}]-2}} \mu_i z^{a_i^n} + \sum_{i=1}^{2^{[\frac{n+1}{2}]-2}} \lambda_i z^{b_i^n}$. Then :

$$\|s(x)\|_p = \| \sum_{i=1}^{2^{[\frac{n+1}{2}]-1}} \mu_i e_{n,i,j} + \sum_{i=1}^{2^{[\frac{n+1}{2}]-2}} \lambda_i e_{n-1,j,i} \|_p =$$

$$\| \sum_{i=1}^{2^{[\frac{n+1}{2}]-1}} \mu_i (z^{a_i^n + c_j^n} + z^{b_j^{n+1} + c_i^{n+1}}) + \sum_{i=1}^{2^{[\frac{n+1}{2}]-2}} \lambda_i (z^{a_j^{n-1} + c_i^{n-1}} + z^{b_i^n + c_j^n}) \|_p \leq$$

$$\| \sum_{i=1}^{2^{[\frac{n+1}{2}]-1}} \mu_i z^{a_i^n + c_j^n} + \sum_{i=1}^{2^{[\frac{n+1}{2}]-2}} \lambda_i z^{b_i^n + c_j^n} \|_p + \| \sum_{i=1}^{2^{[\frac{n+1}{2}]-1}} \mu_i z^{b_j^{n+1} + c_i^{n+1}} \|_p +$$

$$\| \sum_{i=1}^{2^{[\frac{n+1}{2}]-2}} \lambda_i z^{a_j^{n-1} + c_i^{n-1}} \|_p = \|x\|_p + \| \sum_{i=1}^{2^{[\frac{n+1}{2}]-1}} \mu_i z^{c_i^{n+1}} \|_p +$$

$$\| \sum_{i=1}^{2^{[\frac{n+1}{2}]-2}} \lambda_i z^{c_i^{n-1}} \|_p \leq \|x\|_p + c_{p,2}^2 \Big(\sum_{i=1}^{2^{[\frac{n+1}{2}]-1}} |\mu_i|^2 \Big)^{\frac{1}{2}} + c_{p,2}^2 \Big(\sum_{i=1}^{2^{[\frac{n+1}{2}]-2}} |\lambda_i|^2 \Big)^{\frac{1}{2}}$$

(because (c_i^{n+1}) and (c_i^{n-1}) are lacunary with $\lambda = 2$)

$\leq (1 + 2C_{p,2}^2) \|x\|_p$ because

$$\Big(\sum_{i=1}^{2^{[\frac{n+1}{2}]-1}} |\mu_i|^2 + \sum_{i=1}^{2^{[\frac{n+1}{2}]-2}} |\lambda_i|^2 \Big)^{\frac{1}{2}} = \|x\|_2 \leq \|x\|_p .$$

Thus $\|s\| \leq (1 + 2\,C_{p,2}^2)$ and hence

$|tr_{N_n} r u s - tr_{N_{n-1}} r u s| \leq (1 + 2\,C_{p,2}^2)\, C_p^5\, 2^{\frac{n}{2}(\frac{1}{p} - \frac{1}{2})}$ and this proves lemma 3.

Since $\sum_{n=1}^{\infty} 2^{\frac{n}{2}(\frac{1}{p} - \frac{1}{2})} < \infty$ for $2 < p < \infty$ we get

<u>Corollary</u> : The Banach space E_p for $2 < p < \infty$ has not the approximation property.

<u>REFERENCES</u>

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