M. Sato

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MICROLOCAL STRUCTURE OF A SINGLE LINEAR PSEUDODIFFERENTIAL EQUATION

by M. SATO

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§ 1. Let \( P(x, D)u = 0 \) be a single pseudodifferential equation of finite order \( m \) defined in a neighborhood of \( (x_0, i\eta_0 \infty) \), a point in the cosphere bundle \( \sqrt{-1} S M \) of a real analytic manifold \( M \) of dimension \( n \), and denote with \( V \) and \( \overline{V} \) its characteristic variety and the complex conjugate thereof, namely the complex hypersurfaces in a complex neighborhood \( U \) of \( (x_0, i\eta_0 \infty) \) defined by \( P_m(z, \zeta) = 0 \) and \( \overline{P_m(z, \zeta)} = 0 \), respectively, \( P_m \) denoting the principal symbol of \( P \). If \( f(z, \zeta) = 0 \) be a reduced local equation for \( V \), one can write \( P_m(z, \zeta) = a(z, \zeta)(f(z, \zeta))^1 \) with some integer \( 1 > 0 \) and non vanishing factor \( a(z, \zeta) \).

**Assumption 1**: \((x_0, i\eta_0 \infty)\) is a non singular point of \( V \) as well as of \( V \cap \overline{V} \).

**Assumption 2**: The restriction onto \( V \cap \overline{V} \) of the canonical 1-form \( \omega = \zeta_1 dz_1 + \ldots + \zeta_n dz_n \) does not vanish at \((x_0, i\eta_0)\).

The codimension of \( V \cap \overline{V} \) in \( U \) is either 1 or 2 according as \( V = \overline{V} \) (the "real characteristics" case) or not. In the latter case, the degree of osculation of \( V \) and \( \overline{V} \) is a constant integer, say \( k(\geq 1) \), along \( V \cap \overline{V} \) in a neighborhood of \((x_0, i\eta_0 \infty)\). This case we classify further into two, according as \( V \cap \overline{V} \) is involutory or not. Here \( V \cap \overline{V} \) is said to be involutory if, together with the (reduced) local defining equations \( f_1 = f_2 = 0 \) of \( V \cap \overline{V} \), their Poisson bracket \([f_1, f_2]\) vanishes on \( V \cap \overline{V} \). (Of course, similar definition applies to a subvariety of an arbitrary codimension). In the opposite case of non-involutory \( V \cap \overline{V} \), \((x_0, i\eta_0 \infty)\) is a non degenerate point if \([f_1, f_2] (x_0, i\eta_0) \neq 0 \).

**Assumption 3**: In the case of non real \( V \) and non involutory \( V \cap \overline{V} \), our \((x_0, i\eta_0 \infty)\) is a non degenerate point of \( V \cap \overline{V} \).

Note that in this case assumption 3 plus the first part of Assumption 1 implies Assumption 2 and the second part of Assumption 1.

**Theorem 1**: Under the Assumptions 1, 2 (and 3, in the case (iii) below), the equation \( P(x, D)u = 0 \) is microlocally equivalent to one of the following equations, considered at \( x = 0, \eta = (1, 0, 0, \ldots, 0) \). (Note that our assumptions implies \( n \geq 2 \) in the cases (i), (iii) and \( n \geq 3 \) in the case (ii).)
(i) (The real characteristics case)

\[ D_2^1 u = 0 \quad \text{or} \quad x_2^1 u = 0, \text{if one prefers}, \]

(ii) (The non real characteristics case, with involutory \( V \cap \bar{V} \))

\[
(D_1^{k-1} D_2 + iD_3^{k-1})^1 u = 0
\begin{align*}
\text{or} & \quad (D_2 + ix_3^k)^1 u = 0 \\
\text{or} & \quad (x_2 + ix_3^k)^1 u = 0
\end{align*}
\]

(iii) (The non real characteristics case, with non involutory \( V \cap \bar{V} \))

\[ (D_2^+ ix_2^k D_1)^1 u = 0. \]

By virtue of the principles of microlocal analysis developed in [1], this theorem is readily reduced to the corresponding geometrical statement, namely to the following.

Theorem 2: By a real contact transformation any hypersurface \( V \) satisfying assumptions 1, 2, 3 reduces microlocally to one of the following

(i) \( \zeta_2 = 0 \quad \text{or} \quad z_2 = 0 \),

(ii) \( \zeta_1^{k-1} \zeta_2 + i\zeta_3^k = 0 \quad \text{or} \quad \zeta_2 + i\zeta_3^k = 0 \) \quad \text{or} \quad z_2 + i\zeta_3^k = 0),

(iii) \( \zeta_2 + i\zeta_2^{k+1} = 0 \).

The case (i) is a classical result since Lagrange-Hamilton-Jacobi (see [1]). The case (ii), by slightly modifying the proof of theorem 2.2.1 of [1] (which says that an involutory manifold \( V \) of an arbitrary codimension \( r \) which intersects transversally with its complex conjugate \( \bar{V} \) at an involutory submanifold (of codimension \( 2r \)) and satisfies the Assumptions 1 and 2 above at \( (x_0, i\eta_0) \approx ) \), can always be contact-transformed microlocally to \( \zeta_2 + i\zeta_3 = 0, \ldots, \zeta_2r + i\zeta_{2r+1} = 0 \) considered at \( x = 0, \eta = (1, 0, \ldots, 0) \). We always have \( 2r+1 \leq n \).
Namely, we first prove Lemma 3 below, and thence our statement above (as well as theorem 2.2.1 of [1] cited above) will follow.

Let $V$ denote an involutory submanifold of codimension $r$ in $U$, and $V_o$ a submanifold of codimension 1 in $V$, both of them passing through $(x_o, i\eta_o)$. Their local defining equations will be given by $f_1 = \ldots = f_r = 0$ and $f_1 = \ldots = f_r = q = 0$, respectively. (Hence $q = 0$ defines a non singular hypersurface $U_o$ in $U$ passing through $(x_o, i\eta_o)$ which intersects transversally with $V$ at $V_o$.) Here and in what follows, all functions to be considered on $U$ are holomorphic functions in $(z, \zeta) = (z_1, \ldots, z_n; \zeta_1, \ldots, \zeta_n)$ which are homogeneous in variables $\zeta_j$.

Let $\Lambda$ denote an open set in $\mathbb{C}^r$ containing the origin whose point we denote by $\lambda = (\lambda_1, \ldots, \lambda_r)$. Let $g(\lambda) = g(z, \zeta; \lambda)$ and $\psi(\lambda) = \psi(z, \zeta; \lambda)$ be holomorphic functions in $U \times \Lambda$ which vanish on $V \times \Lambda$.

Hence we can write

$$g(\lambda) = g_1(\lambda)f_1 + \ldots + g_r(\lambda)f_r, \quad \psi(\lambda) = \psi_1(\lambda)f_1 + \ldots + \psi_r(\lambda)f_r$$

with $g_j(\lambda)$ and $\psi_j(\lambda)$ holomorphic in a neighborhood of $(x_o, i\eta_o, 0)$ in $U \times \Lambda$. Finally, we denote with $\Delta(\lambda)$ the determinant of the following $r \times r$-matrix

$$\{q, \psi(\lambda)\} \left( \frac{\partial \psi_j(\lambda)}{\partial \lambda_k} \right)_{j,k=1,\ldots,r} = \{q, g(\lambda)\} \left( \frac{\partial g_j(\lambda)}{\partial \lambda_k} \right)_{j,k=1,\ldots,r}.$$

We note that the equation $\Delta(\lambda) = 0$ as well as the condition that $\Delta(\lambda)$ should be non vanishing for a generic vector $\lambda$, depends only on $V, V_o$, $g(\lambda)$ and $\psi(\lambda)$ and is not affected by the ambiguity of the choice of $f_j$, $q$, $g_j(\lambda)$ and $\psi_j(\lambda)$. We now state.

**Lemma 3**: Let holomorphic functions $h_{01}, \ldots, h_{or}$ which vanish at $(x_o, i\eta_o)$ be given on $U_o$ so that $\Delta(h_{01}, \ldots, h_{or}) \neq 0$ on $V_o$. Then they can be prolonged to holomorphic functions $h_1, \ldots, h_r$ in a neighborhood of $U_o$ in $U$ so that $\{\psi(h_1, \ldots, h_r), g(h_1, \ldots, h_r)\} = 0$ holds identically.
And indeed, one can construct such \( h_1, \ldots, h_r \) by solving a Kowalewskian system of (non-linear) first order partial differential equations, as will be seen in the below.

We remark that, if \( h_j^*(z, \zeta) \) denote any holomorphic extension of \( h_j \) into a neighborhood of \( U_0 \) in \( U \), the restriction onto \( V \{ q, \hat{\psi}(h^*) \} \) coincides with \( \{ q, \hat{\psi}(\lambda) \} \) because one has

\[
\{ q, \hat{\psi}(h^*) \} = \{ q, \hat{\psi}(\lambda) \} \mid_{\lambda \to h^*} + \sum_j \{ q, h_j^* \} \frac{\partial \hat{\psi}(\lambda)}{\partial \lambda_j} \mid_{\lambda \to h^*}
\]

and \( \frac{\partial \hat{\psi}(\lambda)}{\partial \lambda_j} \equiv 0 \) mod. \( f_1, \ldots, f_r \).

**Proof of Lemma 3:** Along with the ordinary Poisson bracket

\[
\{ \psi, \hat{\psi} \} = \sum_k \left( \frac{\partial \psi}{\partial \psi_k} \frac{\partial \hat{\psi}}{\partial z_k} - \frac{\partial \psi}{\partial z_k} \frac{\partial \hat{\psi}}{\partial \psi_k} \right)
\]

we have the following "prolonged" expression for the bracket of \( \hat{\psi}(w) = \hat{\psi}(z, \zeta; w) \) and \( \psi(w) = \psi(z, \zeta; w) \) involving functions \( w_j = w_j(z, \zeta) \):

\[
\{ \psi(w), \hat{\psi}(w) \} = \sum_k \left( \frac{\partial \psi}{\partial \psi_k} + \sum_{j,k} \frac{\partial \psi}{\partial w_j} \frac{\partial \hat{\psi}}{\partial w_k} + \sum_{j,k} \frac{\partial \psi}{\partial \zeta_j} \frac{\partial \hat{\psi}}{\partial \zeta_k} \right)
\]

with \( (w_z)_j \) and \( (w_\zeta)_j \) denoting \( \frac{\partial w_j}{\partial z_k} \) and \( \frac{\partial w_j}{\partial \zeta_k} \), respectively. The right-hand side expression will be denoted by \( \Theta(w, w_z, w_\zeta) = \Theta(z, \zeta; w, w_z, w_\zeta) \). Since \( V \) is involutory, there exist holomorphic functions \( \Theta_{ijj}^j(\lambda) \) in a neighborhood of \( (x_0, i\eta_0; 0) \) in \( U \times \Lambda \) so that we have

\[
\{ \psi(\lambda), \hat{\psi}(\lambda) \} = \sum_k \left( \frac{\partial \psi(\lambda)}{\partial \psi_k} \frac{\partial \hat{\psi}(\lambda)}{\partial z_k} - \frac{\partial \psi(\lambda)}{\partial z_k} \frac{\partial \hat{\psi}(\lambda)}{\partial \psi_k} \right) = \Theta_{01}(\lambda) f_1 + \ldots + \Theta_{0r}(\lambda) f_r
\]

whence we obtain

\[
\Theta(w, w_z, w_\zeta) = \Theta_{01}(w, w_z, w_\zeta) f_1 + \ldots + \Theta_{0r}(w, w_z, w_\zeta) f_r
\]

by setting
Let us further consider the case where \( w_j = w_j(t) = w_j(z, \zeta; t) \) involve a parameter \( t \) and are holomorphic in \((z, \zeta; t) \in U \times \mathbb{C} \) in a neighborhood of \((x_0, i\pi_0; 0)\). Of course we have

\[
[\psi(w(t)), \xi(w(t))] = \Theta(w(t), w_z(t), w_{\zeta}(t)) = \Theta(w(t), w_z(t), w_{\zeta}(t)) \text{ as long as } t \text{ is an independent parameter, while we obtain, when } t \text{ is substituted by } q(z, \zeta), \text{ the following identity:}
\]

\[
[\psi(q(w))), \xi(w(q))] = \Theta(w(t), w_z(t), w_{\zeta}(q(t))) + \frac{\partial \psi(w(t))}{\partial t} \{q, \xi(w(t))\} - \frac{\partial \xi(w(t))}{\partial t} \{q, \psi(w(t))\} = 0.
\]

The expression inside the bracket on the right hand side is again a linear form of \( f_1, \ldots, f_r \), and, by equating to 0 each of the coefficients we form a system of equations.

\[
\Theta (z, \zeta; w, w_z, w_{\zeta}) + \frac{\partial \psi(w)}{\partial t} \{q, \xi(w)\} - \frac{\partial \xi(w)}{\partial t} \{q, \psi(w)\} = 0,
\]

or equivalently

\[
\Theta (w, w_z, w_{\zeta}) + \sum_k \{q, \xi(w)\} \frac{\partial \psi_j(w)}{\partial w_k} - \{q, \psi(w)\} \frac{\partial \xi_j(w)}{\partial w_k} \frac{\partial w_k}{\partial t} = 0, \quad (j = 1, \ldots, r)
\]

This is a determined system of first order differential equations for unknown functions \( w_1, \ldots, w_r \) in \((z, \zeta; t)\), and, under the assumptions of the lemma, one has a well-posed Cauchy problem if one assigns to \( w_j(t) \) initial data at \( t = 0 \) such that \( \Delta(w(0)) \neq 0 \). Therefore, existence of prolongations \( h_j \) of \( h_{oj} \) with the properties claimed in the lemma is implied if one first chooses an arbitrary holomorphic extension \( h_j^* \) of \( h_{oj} \) to a neighborhood of \( U_0 \) in \( \bar{U} \), then solves the above system of equations by assigning \( h_j^* \) as initial data (see the remark following the lemma) to obtain the local solutions \( w_j(z, \zeta; t) \) and finally, defines \( h_j \) by \( h_j(z, \zeta) = w_j(z, \zeta; q(z, \zeta)) \). Note that \( h_j \) and \( h_j^* \) coincide on \( U_0 \) because we
Remark 1: If $\xi_j$, $\varphi_j$, $h_{0j}$ are all of real coefficients (i.e. $\tilde{\xi}_j(z,\zeta;\lambda) = \xi_j(z,\zeta;\lambda)$, etc.) $h_j$ can also be chosen real-coefficiented.

Remark 2: If $W$ is another involutory submanifold of codimension $s(\leq r)$ in $U$ containing $V$ as submanifold (i.e. $V \subset W \subset U$), if our defining equation $f_1 = 0,\ldots,f_r = 0$ of $V$ is so chosen that the first $s$ equations define $W$, and if $\bar{\varphi}(\lambda)$ vanishes on $W \times \Lambda$ so that it has the form

$$\bar{\varphi}(\lambda) = \bar{\varphi}_1(\lambda)f_1 + \ldots + \bar{\varphi}_s(\lambda)f_s,$$

then we have

$$\psi(w(t)) \big|_W = \psi(w(0)) \big|_W$$

and hence, $\psi(h) \big|_W = \psi(h^*) \big|_W$,

provided that $\{q,\bar{\varphi}(w(0))\} \neq 0$ at $(x_0, i\eta_0)$. In particular, if initial data $h^*$ are so chosen that $[f_j, \psi(h^*)] \big|_W = 0$ holds for $j = 1,\ldots,s$, then one has $[f_j, \psi(h)] \big|_W = 0$ for $j = 1,\ldots,s$, because for a holomorphic function $g$ on $U$, $[f_j, g] \big|_W$, $j = 1,\ldots,s$ is completely determined by $g \big|_W$ (and hence one can naturally talk about $[f_j, g_0] \big|_W$ for a holomorphic function $g_0$ on $U$).

Proof: Combining the equations

$$\{\psi(w), \bar{\varphi}(w)\} = \Theta (w, w_z, w_\zeta),$$

where

$$\Theta (w, w_z, w_\zeta) + \frac{\partial \psi(w)}{\partial t} < q, \bar{\varphi}(w) > - \frac{\partial \bar{\varphi}(w)}{\partial t} [q, \psi(w)] = 0$$

and taking into account the congruence $\bar{\varphi}(w) \equiv 0 \pmod{f_1,\ldots,f_s}$ we have

$$\{q, \bar{\varphi}(w)\} \frac{\partial \psi(w)}{\partial t} + [\psi(w), \bar{\varphi}(w)] \equiv 0 \pmod{f_1,\ldots,f_s}$$

and this we regard as a differential equation on $W$, satisfied by an unknown function $\psi(w) = \psi(z,\zeta; w(z,\zeta; t))$ of $(z,\zeta; t)$ modulo $f_1,\ldots,f_s$.

($\bar{\varphi}$ is regarded as known). Then the given $\psi(w(t))$ as well as $t$ independent $\psi(w(0))$ both constitute holomorphic solutions to this equation corresponding to the same initial data $\psi(w(0)) \pmod{f_1,\ldots,f_s}$. Therefore by uniqueness of holomorphic solutions they coincide. (q.e.d.)
§ 3. Proof of theorem 2

We can assume without loss of generality that the reduced principal symbol \( f(z, \zeta) \) be of the form \( f = f_1 + if_2 \) (cf. [2]). The involutory \( V \cap \overline{V} \) is defined by \( f_1 = f_2 = 0 \). Letting a homogeneous polynomial \( A \) of \( u, v \) be given by

\[
(u + v)^k = u^k + A(u, v)v \quad \text{(i.e. } A(u, v) = \sum_{\nu=1}^{k} \nu^{k-\nu} u^{k-\nu} v^{\nu-1})
\]

we define \( \xi, \phi, \psi, \psi_j \) as follows:

\[
\xi(\lambda) = \xi(z, \zeta; \lambda) = \lambda_1^k f_1 - A(\lambda_1 f_2, \lambda_2 f_1) \lambda_2 f_2
\]

\[
\phi_1(\lambda) = \lambda_1^k, \quad \phi_2(\lambda) = -A(\lambda_1 f_2, \lambda_2 f_1) \lambda_2 f_2^{k-1}
\]

\[
\psi(\lambda) = \lambda_1 f_2 + \lambda_2 f_1, \quad \psi_1(\lambda) = \lambda_2, \quad \psi_2(\lambda) = \lambda_1,
\]

so that we have

\[
(\lambda_1^k + iA(\lambda_1 f_2, \lambda_2 f_1) \lambda_2)(f_1 + if_2) = \xi(\lambda) + i(\psi(\lambda))^k,
\]

\[
\phi(\lambda) = \phi_1(\lambda)f_1 + \phi_2(\lambda)f_2, \quad \psi(\lambda) = \psi_1(\lambda)f_1 + \psi_2(\lambda)f_2,
\]

and apply lemma 3 to it. The matrix \( \partial \psi_j / \partial \lambda_k \) is equal to \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \)

while \( \partial \phi_j / \partial \lambda_k \) is congruent to \( \begin{pmatrix} k\lambda_k^{-1} & 0 \\ 0 & 0 \end{pmatrix} \) (resp. to \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \)) modulo \( f_1 \) and \( f_2 \) if \( k \geq 2 \) (resp. \( k = 1 \)). Also we have \( \{q, \xi(\lambda)\} \equiv \lambda_1^k \{q, f_1\} \) (mod. \( f_1, f_2 \)).

Hence \( \Delta(\lambda) \big|_V \), which is the determinant of

\[
\{q, \psi(\lambda)\} \begin{pmatrix} k\lambda_k^{-1} & 0 \\ 0 & 0 \end{pmatrix} - \{q, \xi(\lambda)\} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

is given by \(- \lambda_1^k [q, f_1]^2 \) for \( k \geq 2 \). (Similarly we have \( \Delta(\lambda) = - (\lambda_1^2 + \lambda_2^2 ([q, f_1]^2 + [q, f_2]^2) \) for \( k = 1 \)).
So, in the case of \( k \geq 2 \), by choosing a real-coefficiented \( q(z,\zeta) \) such that \( q(x_0, i\eta_0) = 0 \), \( \{q, f_1\}(x_0, i\eta_0) \neq 0 \) which of course exists, and initial data \( h_{0j}, j = 1, 2 \), such that \( h_{01}(x_0, i\eta_0) \neq 0 \) (e.g. \( h_{01} = 1 \), \( h_{02} = 0 \)), the condition \( \Delta (h_{01}, h_{02}) \neq 0 \) holds at \( (x_0, i\eta_0) \) and \( h_{0j} \) are prolonged to such \( h_j \) that satisfy \( \{\psi(h_1, h_2), \psi(h_1, h_2)\} = 0 \). The homogeneous degree of \( \psi(h_1, h_2) \), and \( \psi(h_1, h_2) \) in \( \zeta \)-variables can be adjusted (to 0, for example) by a corresponding adjustment to the initial data \( h_{0j} \). The property that \( h_{01} \neq 0 \) at \( (x_0, i\eta_0) \) also implies that \( \psi(h_1, h_2) + i(\psi(h_1, h_2))^k = 0 \) is equivalent to \( f_1 + if_2 = 0 \) as a reduced defining equation of \( V \), and \( \psi(h_1, h_2) = \psi(h_1, h_2) = 0 \) to \( f_1 = f_2 = 0 \) as reduced defining equations of \( V \cap \overline{V} \). Consequently \( d\psi \), \( d\psi \) and \( w \) are linearly independent at \( (x_0, i\eta_0) \). The classical Jacobi theory now tells that \( \psi(h_1, h_2) \) and \( \psi(h_1, h_2) \) go to \( z_2 \) and \( z_3 \) by a suitable contact transformation which is real coefficiented and sends \( (x_0, i\eta_0) \) to \( (0, i(1, 0, ..., 0)) \). Then the defining equation of \( V \) assumes the form \( z_2 + iz_3^k = 0 \) and our theorem is proved. In place of \( (z_2, z_3) \) one may as well choose \( (z_2/z_1, z_3) \) or \( (z_2/z_1, z_3/z_1) \) to result \( z_2 + iz_3^k z_1 = 0 \) or \( z_1 z_2 + iz_3^k = 0 \) as the standard form of defining equation of \( V \). (q. e. d.)

Finally we show how the key Lemma 2.2.2 to the theorem 2.2.1 of \([1]\) is derived from lemma 3. Let again \( V \) be an involutory manifold of codimension \( s \) whose local defining equations \( f_1 = \ldots = f_s = 0 \) have the property that \( df_1, \ldots, df_s, df_1^c, \ldots, df_s^c \) are linearly independent in the neighborhood of \( (x_0, i\eta_0) \). (Whence \( V \) intersects with its complex conjugate transversally), and assume \( V \cap \overline{V} \) is also involutory (of codimension \( 2s \)). Here \( f_j^c \) is defined by \( f_j^c(z, \zeta) = f_j(z, \overline{\zeta}) \).

Choose first a \( G(z, \zeta) \) such that \( \{G, f_j\}|_V = 0 \) (i.e. \( \{G, f_j\} \equiv 0 \mod. f_1, \ldots, f_s \) for \( j = 1, \ldots, s \) and such that \( dG, df_1, \ldots, df_s, \) \( w \) are linearly independent at \( (x_0, i\eta_0) \). Choose then a real coefficiented function \( q(z, \zeta) \) so that \( q(x_0, i\eta_0) = 0 \) and \( \{G, q\}(x_0, i\eta_0) \neq 0 \) hold. Define \( \xi(\lambda) \) and \( \xi^c(\lambda) \) by \( \xi(\lambda) = \lambda f_1 + \ldots + \lambda f_s \) and \( \xi^c(\lambda) = \overline{\lambda} f_1 + \ldots + \overline{\lambda} f_s \), respectively. This means in particular that \( V, r, \lambda = (\lambda_1, \ldots, \lambda_r) \), \( f = (f_1, \ldots, f_r) \) and \( (\xi, \psi) \) in lemma 3 are now replaced by \( V \cap \overline{V}, 2s \).
...(λ, \bar{λ}) = (λ_1, ..., λ_s; \bar{λ}_1, ..., \bar{λ}_s), (f^c, f^c) = (f_1, ..., f_s; f^c_1, ..., f^c_s) and
(\xi, \xi^c), respectively. Under these circumstances \( \Delta(\lambda) \) in lemma 3, as the
determined of the matrix

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
0 & 0 & \cdots & 0
\end{bmatrix} - \delta \xi(\lambda) = \begin{bmatrix} 0 & \cdots & 0 & 1 \\
0 & \cdots & 1 & 0
\end{bmatrix},
\]

takes the form \( \Delta(\lambda, \bar{λ}) = (-[q, \xi(\lambda)]q, \xi^c(\lambda))]^S = (-1)^S |[q, \xi(\lambda)]|^2S \).
Hence, by lemma 3 and remark 2 to lemma 3, we can conclude that by a sui-
table choice of \( h_j(t) \) we have

\[
\{\xi^c(h^c(q)), \xi(h(q))\} = 0, \text{ and } \{\xi^c(h^c(q)), f_j\} \equiv 0 \pmod{f_1, ..., f_s}
\]
while \( d\xi(h(q)), d\xi^c(h^c(q)) \) and \( \omega \) are linearly independent at \((x_0, i\eta_0)\).
This is lemma 2.2.2 of [1].

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