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Séminaire Équations aux dérivées partielles (Polytechnique) (1972-1973), exp. no 17, p. 1-5

<http://www.numdam.org/item?id=SEDP_1972-1973___A18_0>
SEMINAIRE GOULAOUIC-SCHWARTZ 1972-1973

SOME RATIONALLY CONVEX SETS

by J. WERMER
We consider a compact Hausdorff space $X$ and on $X$ a uniform algebra $\mathcal{A}$. That means that $\mathcal{A}$ is an algebra of continuous complex-valued functions on $X$, closed under uniform convergence on $X$, separating the points of $X$, and containing the constants.

With norm

$$\|f\| = \max_X |f|,$$

$\mathcal{A}$ is then a commutative Banach algebra with unit. According to Gelfand, $\mathcal{A}$ possesses a spectrum $\mathbb{M}(\mathcal{A})$, i.e. the space of all non-trivial homomorphisms of $\mathcal{A} \to \mathbb{C}$. $\mathbb{M}(\mathcal{A})$ is a compact Hausdorff space, in Gelfand's topology.

There is a natural injection of $X$ into $\mathbb{M}(\mathcal{A})$, namely the map sending each point $x$ into the functional of evaluation at $x$. This injection may or may not be onto, i.e. we may have $\mathbb{M}(\mathcal{A}) = X$ or $\mathbb{M}(\mathcal{A})$ larger than $X$.

When $\mathcal{A} = C(X)$, one has $\mathbb{M}(C(X)) = X$. We have

**Problem**: Let $\mathcal{A}$ be a uniform algebra on $X$ such that $\mathbb{M}(\mathcal{A}) = X$. What additional condition assures that $\mathcal{A} = C(X)$?

Of course, one has the classical condition of Stone:

$$f \in \mathcal{A} \Rightarrow \overline{f} \in \mathcal{A}.$$  

But in problems of uniform approximation in the complex domain this condition is usually difficult to verify.

In 1959, E. Bishop in [1] introduced the notion of a peak point. Let $X$ now be metrizable, $\mathcal{A}$ a uniform algebra on $X$. Fix $x_0 \in X$.

$x_0$ is a peak point for $\mathcal{A}$ if $\exists f \in \mathcal{A}$ with $f(x_0) = 1$ and $|f| < 1$ on $X \setminus \{x_0\}$.
Evidently, when $\mathcal{A} = C(X)$ every point of $X$ is peak point. When $\mathcal{A}$ is the disk algebra of functions analytic in the open unit disk and continuous in $|z| \leq 1$, $\mathbb{D}(\mathcal{A})$ is the full disk while the peak points are exactly the points on the boundary. In general, the set of peak points does not coincide with the Silov boundary of $\mathcal{A}$, but in fact coincides with the Choquet boundary.

Let now $X$ be a compact subset of $\mathbb{C}$. We denote by

$$ R(X) $$

the uniform algebra on $X$ which is the closure on $X$ of the set of rational functions of $z$ which are holomorphic on $X$.

It was pointed out by Mergelyan that there exist sets $X$ without interior points such that $R(X) \neq C(X)$. In [1] Bishop proved the following

**Theorem**: $R(X) = C(X)$ if and only if each point of $X$ is a peak point for $R(X)$.

The question now arose to what extent this result was a general property of uniform algebras. It is not easy to find, among examples arising in a natural way, uniform algebras distinct from $C(X)$, yet such that the spectrum of the algebra consists entirely of peak points.

In 1968, in his Yale thesis Brian Cole gave a very general construction of uniform algebras $\mathcal{A}$ with the property that every element of $\mathcal{A}$ has a square root in $\mathcal{A}$, and used this construction to produce an example of an $\mathcal{A}$ with $\mathbb{D}(\mathcal{A}) = X$, every point of $X$ is a peak point, yet $\mathcal{A} \neq C(X)$. Later on, he modified his construction to obtain an example which is doubly generated.

It remained of interest, however, to exhibit concrete and simple examples of such algebras. I want to discuss such a construction, due to Richard Basener and contained in his thesis, Brown University (1971).
Let $X$ now be a compact set in $\mathbb{C}^n$. We define $R(X)$, in analogy with the case $n=1$, as the closure in $C(X)$ of the set of quotients $\frac{P}{Q}$ where $P$, $Q$ are polynomials in $z_1, \ldots, z_n$ and $Q \neq 0$ on $X$.

Fix $m \in \mathfrak{M}(R(X))$. Put

$$a = (m(z_1), \ldots, m(z_n)), \ a \in \mathbb{C}^n.$$  

We claim:

For every polynomial $Q$:

$$(*) \quad Q(a) = 0 \Rightarrow Q \text{ vanishes somewhere on } X.$$  

For if not, $f'Q, Q(a) = 0, \frac{1}{Q} \in R(X)$. Then

$$1 = m\left(\frac{1}{Q} \cdot Q\right) = m\left(\frac{1}{Q}\right) m(Q) = 0,$$

since $m(Q) = Q(a)$. So $(*)$ holds.

**Definition**: $h_r(X) = \{a \in \mathbb{C}^n \mid (*) \text{ holds}\}$.

$h_r(X)$ is called the rationally convex envelop of $X$. To each $m \in \mathfrak{M}(R(X))$ there corresponds, as we have just seen, a point $a \in h_r(X)$. The map is easily seen to be bijective, and we may identify $\mathfrak{M}(R(X))$ with $h_r(X)$. We note that when $n = 1$, $h_r(X)$ evidently coincides with $X$. For $n > 1$, $h_r(X)$ may be larger than $X$.

Fix now a closed subset $S$ of the open disk $|z| < 1$ in the $z$-plane. Denote by $B$ the ball : $|z|^2 + |w|^2 \leq 1$ in $\mathbb{C}^2$ and by $\partial B$ its boundary. Put

$$X_S = \{(z,w) \in \partial B \mid z \in S\}.$$  

Thus $X_S$ is the set of those points on $\partial B$ which project into $S$.

Note that if $z \in S$, the entire circle

$$\Gamma_z = \{(z,\sqrt{1-|z|^2} \cdot e^{i\theta}) \mid 0 \leq \theta < 2\pi\}$$

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$$\Gamma_z = \{(z,\sqrt{1-|z|^2} \cdot e^{i\theta}) \mid 0 \leq \theta < 2\pi\}$$
lies in $X_S$. Thus $X_S$ is, in a sense, a fibrespace with base $S$ and fiber a circle.

Basener's result is the following:

**Theorem**: There is $S$ such that the algebra $R(X_S)$ has the properties:

(a) $R(X_S) \neq C(X_S)$.
(b) $h_r(X_S) = X_S$.
(c) Each point of $X_S$ is a peak point for $R(X_S)$.

The proof of (c) is trivial.

Let $(z_0, w_0) \in \partial B$. Put

$$P(z, w) = \frac{1}{2} \{z \overline{z}_0 + w \overline{w}_0 + 1\}.$$ 

Then $P(z_0, w_0)$ and $|P| < 1$ on the rest of $\partial B$. So (c) holds.

To obtain (a) we only need $S$ such that $R(S) \neq C(S)$. For then there is a complex measure $\mu$ on $S$, $\mu \neq 0$, with $\mu \downarrow R(S)$. For each $F \in C(X_S)$, put

$$L(F) = \int_S \mu(z) \left( \int_{\Gamma_z} F \, dm_z \right),$$

where $m_z$ is normalized Lebesgue measure on $\Gamma_z$. The $L$ is a bounded linear functional on $C(X_S)$, and $\neq 0$.

If $F$ is holomorphic in some neighborhood of $X_S$, it is easily verified that $\int_{\Gamma_z} F \, dm_z$ is holomorphic in $Z$ in a neighborhood of $S$, and so $\in R(S)$. Hence $L(F) = 0$. It follows that $L$ vanishes on $R(X_S)$, and so (a) holds.

To obtain (b) we must restrict $S$ rather severely, and we do not give the details here. They are given in Basener's forthcoming paper [2], and also in [3], pp. 202-203. The crucial point in the proof of (b) is the notion of a Jensen measure.

Let $\mathcal{O}$ be a uniform algebra on a space $X$ and $m \in M(\mathcal{O})$. A Jensen measure $\mu_m$ for $m$ is a probability measure on $X$ such that Jensen's...
inequality

$$\log |\hat{\mathcal{f}}(m)| \leq \int_{\mathcal{X}} \log |f| \, d\mu_{m}$$

holds for all $f \in \mathcal{A}$. Concerning Jensen measures, see [3] or [4].

Cole's work, discussed above, also is treated in [3] and [4].

REFERENCES


