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SOME REMARKS ON ORDERED FIELDS

by Masayoshi NAGATA

Let K be a field. It is well known that one can give an order in K so that K becomes an ordered field if, and only if, K is formally real, namely, -1 cannot be the sum of the squares of any finite number of elements in K . It is also well known that a formally real field is of characteristic zero, hence it contains the rational number field $\mathbb{Q}$.

The purpose of the present note is to give some remarks on the structure of an order of an ordered field and they can be stated in two theorems :

THEOREM 1. - Let K be an ordered field. Set

$$V = \{x \in K ; -n < x < n \text{ for some natural number } n\}$$

and

$$P = \{x \in K ; -(1/n) < x < 1/n \text{ for every natural number } n\} .$$

Then (V, P) is a valuation ring (i. e., V is a valuation ring with maximal ideal P), and the residue class field V/P is naturally isomorphic to a subfield of the real number field $\mathbb{R}$.

Conversely,

THEOREM 2. - Let (V, P) be a valuation ring of a field K . Assume that there is an injection φ of the field V/P into the real number field $\mathbb{R}$. Then one can make K an ordered field so that (V, P) coincides with such a pair defined in Theorem 1 with respect to the given order.

Thus orders for a field K corresponds to valuation rings of K having residue class fields contained in $\mathbb{R}$. But the correspondence is not one-one, as we shall discuss at the end of the article.

Proof of Theorem 1. - Assume that $x \in K$ and $x \notin V$. Then either x or $-x$ is greater than any natural number n , which implies that $-(1/n) < x^{-1} < 1/n$ and therefore $x^{-1} \in P$. Thus we see that (V, P) is a valuation ring of K . For each x in V , we set

$$S_x = \{r \in \mathbb{Q} ; x < r\} .$$

Then $\inf S_x$ exists in the real number field $\mathbb{R}$. Let ψ be the mapping of V in $\mathbb{R}$ such that $\psi x = \inf S_x$. Then one sees easily the ψ is a homomorphism whose kernel coincides with P .

Before proving Theorem 2, we need

LEMMA 3. - Let (W, P') be a maximally complete valuation ring ⁽¹⁾ of a field L such that

- (i) W/P' is algebraically closed and
- (ii) the value group of the valuation is divisible. Then the field L is algebraically closed.

Proof follows immediately from the definition of maximal completeness and we omit the detail.

Q. E. D.

COROLLARY 4. - Let (V, P) be a maximally complete valuation ring of a field K such that

- (i) V/P is real closed, and
- (ii) the value group of the valuation is divisible. Then K is real closed ⁽²⁾.

By the way, we note that the following is immediate from the definition of maximal completeness.

LEMMA 5. - Let (W, P') be a maximally complete valuation ring of a field L such that

- (i) W/P' is algebraically closed, and
- (ii) the value group of the valuation is divisible. Then the field L is algebraically closed.

Proof of Theorem 2. - In order to prove the theorem, we may replace K with an extension field. Therefore, first of all, we may assume that V is maximally complete. Then V is henselian and contains $\mathbb{Q}$, and therefore every maximal subfield K^* of V forms a complete set of representatives for V/P . Therefore, extending residue class field, we may assume that $V/P = \mathbb{R}$. On the other hand, if the value group G is not divisible, for instance if there is a g in G for which h such that $ph = g$ does not exist (p being a prime number), then we may add g/h to G by adjoining, p -th root of an element whose value is g . Repeating such a process, we may assume that G is divisible. Then corollary 4 implies that K is

⁽¹⁾ It is well known that a field K is real closed if, and only if, (i) K itself is not algebraically closed, and (ii) the algebraic closure of K is of finite degree over K , or if, and only if, $K(\sqrt{-1})$ is algebraically closed besides the condition (i) above.

⁽²⁾ For the notion of maximal completeness (due to KAPLANSKY), see for instance; SCHILLING (O. F. G.). - The theory of valuations. - New York, American mathematical Society, 1950 (Mathematical Surveys, 4).

real closed. K^* may be identified with R . Furthermore, setting

$$S = \{x^2 ; 0 \neq x \in K\} ,$$

we know that K is the disjoint union of S , $-S$, $\{0\}$, and K has a unique structure as an ordered field. Under the order, $a > b$ if, and only if, $a-b \in S$.

(1) Assume that $a \in P \cap S$ and that $a > 1/n$ for a natural number n . Then $a - (1/n) = b^2$ with $b \in K$. This implies that $-(1/n) \equiv b^2$ modulo P , which is impossible because $V/P = R$. Thus $a \in P \cap S$ implies that $a < 1/n$ for every natural number n . Therefore $a' \in P$ implies that $-(1/n) < a' < 1/n$ for every natural number n .

(2) Assume that $b \in S$, $b \notin V$. Then $b^{-1} \in P \cap S$, and therefore $b > n$ for every natural number n by virtue of (1) above.

(3) Assume now that $c \in S \cap V$. Then there is a $c^* \in K^*$, such that $c - c^* \in P$. By (1) above, we see that $-(1/n) < c - c^* < 1/n$ for every natural number n , hence $c^* - (1/n) < c < c^* + (1/n)$.

In view of these (1) ~ (3), we see easily that the valuation ring defined by the order of K coincides with V .

Q. E. D.

Remarks on the correspondance. - Many different orders of field K may give the same valuation ring (V, P) . Roughly speaking, there are two kind of reasons for this.

One is by the injection φ of V/P in R . Namely let ψ be a homomorphism of V into R whose kernel is P . Then, if ψ can be changed, then we surely have a different order in K .

Thus, from now on, we fix ψ also. Then, another cause comes from the structure of the value group G of the valuation defined by V . Namely, let G' be

$$\{2g ; g \in G\} .$$

For each $g \in G$, let c_g be an element of K whose value is g ; here, if $g \in G'$, we choose c_g to be a square element. Let U be the unit group of V . If $g \in G$, then c_g must be a positive element, hence $c_g u$ ($u \in U$) is positive if and only if, ψu is positive. Thus

PROPOSITION 6. - If $G = G'$ (i. e., if G is 2-divisible), then the pair (V, ψ) defines an order of K uniquely.

If $G \neq G'$, then we surely have arbitrariness in adjoining square roots on our way to extend the value group G to a 2-divisible group, and the arbitrariness allows

us to alter positivity of certain elements.

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