The structure of $\omega$-regular semigroups


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THE STRUCTURE OF \( \omega \)-REGULAR SEMIGROUPS

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1. - Finding the complete structure of regular semigroups of a certain class has succeeded only when sufficiently strong conditions on idempotents and (or) ideals have been imposed. On the one hand, there is the theorem of REES [7], giving the structure of completely \( 0 \)-simple semigroups, and its successive generalizations to primitive regular semigroups [2], and \( 3 \)- and \( 3 \)-regular semigroups [4]. On the other hand, with very different restrictions, REILLY [8] has determined the structure of bisimple \( \omega \)-semigroups and, independently of each other, KOCHIN [1] of inverse simple \( \omega \)-semigroups, and MUNN [5] of inverse \( \omega \)-semigroups.

An \( \omega \)-chain with zero is a poset \( \{ e_i \mid i > 0 \} \cup \{ 0 \} \), with \( e_i > e_j \) if \( i < j \), and \( 0 < e_i \) for all \( i, j \). We call a regular semigroup \( S \) \( \omega \)-regular, if \( S \) has a zero, and the poset of its idempotents is an orthogonal sum \([2]\) of \( \omega \)-chains with zero. We announce here the complete determination of the structure of such semigroups, including various special cases thereof, and briefly mention their isomorphisms.

2. - An \( \omega \)-regular semigroup can be uniquely written as an orthogonal sum of \( \omega \)-regular prime (i.e., with \( 0 \) a prime ideal) semigroups. This reduces the problems of structure and isomorphism to \( \omega \)-regular prime semigroups. We distinguish three cases:

(i) \( 0 \)-simple,
(ii) Prime with a proper \( 0 \)-minimal ideal,
(iii) Prime without a \( 0 \)-minimal ideal.

Case (i) is the most difficult (and interesting), and includes a variety of special cases some of which reduce to those constructed by REILLY [8], KOCHIN [1], and MUNN [5], [6].
and define \([i, \alpha, j]\) to satisfy
\[
[i, \alpha, j]d = (i - j) - (\langle \alpha, i \rangle - \langle \alpha, j \rangle).
\]

\textbf{Construction 1.} - On the set
\[
S = \{ (\alpha, m, g, n, \beta) \mid \alpha, \beta \in A, \ m, n \geq 0, \ g \in V \} \cup \{0\},
\]
define a multiplication by, for \(g_i \in G_i, \ g_j \in G_j, \ v = n - s - [i, \beta, j],\)
\[(\alpha, m, g_i, n, \beta)(\beta, s, g_j, t, \gamma) = \begin{cases}
(\alpha, m - [i, \alpha, j] - v, (g_i \sigma^{-v})g_j, t, \gamma), & \text{if } v < 0, \\
0, & \text{or } v = 0, i < j, \\
(\alpha, m, g_i(g_j \sigma^v), t + [i, \gamma, j] + v, \gamma), & \text{if } v > 0, \\
0, & \text{or } v = 0, i > j,
\end{cases}
\]
and all other products are equal to 0. The set \(S\), with this multiplication, will be denoted by \(\mathcal{O}(A, w; V, \sigma)\).

\textbf{Construction 2.} - On the set
\[
S' = \{ (\alpha, m, g, n, \beta) \mid \alpha, \beta \in A, \ m - w = n - w = i \pmod{d}, \ g \in G_i \} \cup \{0\},
\]
define a multiplication by, for \(g_i \in G_i, \ g_j \in G_j, \ v = n' - s' - [i, \beta, j],\)
where \(n = n'd + n'', \ s = s'd + s'', \ C \leq n'', \ s'' < d,
\[(\alpha, m, g_i, n, \beta)(\beta, s, g_i, t, \gamma) = \begin{cases}
(\alpha, m + s - n, (g_i \sigma^{-v})g_j, t, \gamma), & \text{if } n < s, \\
(\alpha, m, g_i(g_j \sigma^v), t + n - s, \gamma), & \text{if } n > s,
\end{cases}
\]
and all other products are equal to 0. The set \(S'\), with this multiplication, will be denoted by \(\mathcal{O}[A, w; V, \sigma]\).

The following is our fundamental result.

\textbf{Theorem 1.} - For a groupoid \(S\), the following statements are equivalent:

(i) \(S\) is a 0-simple \(\omega\)-regular semigroup;
(ii) \(S\) is isomorphic to \(\mathcal{O}(A, w; V, \sigma)\);
(iii) \(S\) is isomorphic to \(\mathcal{O}[A, w; V, \sigma]\).

The proof of "(i) \(\Rightarrow\) (ii)" consists of "introducing coordinates" into various \(L\) and \(R\)-classes, and of constructing the homomorphism \(\sigma\); it is quite long, and
Theorem 2. — The following conditions on a 0-simple \( \omega \)-regular semigroup \( S \) are equivalent:

(i) \( S \) is balanced;
(ii) \( S \) admits a representation as in Theorem 1, with \( w_\alpha = 0 \) for all \( \alpha \in \Lambda \);
(iii) \( \mathcal{S}(S) \) is a Brandt semigroup;
(iv) \( S \) is isomorphic to a Rees matrix semigroup \( \mathcal{M}^0(K; A, A; \Lambda) \) over a simple inverse \( \omega \)-semigroup \( K \), \( \Lambda \) is the identity matrix.

The structure of the semigroup \( K \) in Theorem 2 was determined by Kochin [1] and Munn [5], the Rees matrix semigroups over bisimple inverse semigroups were studied in [3] (for the 0-simple case in the theorem, cf. [3], cor. 5.7, and [6], th. 4.2). Various other special cases include: 0-bisimple, combinatorial, balanced, and combinations thereof.

Construction 3. — Let \( Y \) be a tree semilattice satisfying one of the two conditions:

1. \( Y \) has a zero \( \zeta \), and all elements of \( Y \) are of finite height;
2. \( Y \) has no zero, and is of locally finite length.

To every non-zero element \( \alpha \) of \( Y \), associate a Brandt semigroup \( S_\alpha \), suppose that the family \( \{ S_\alpha \} \) is pairwise disjoint, and that a homomorphism \( \varphi_\alpha : S_\alpha \to S_\alpha \) is given, where \( \alpha \) is the unique element of \( Y \) covered by \( \alpha \), with the properties:

(i) \( S_\alpha \varphi_\alpha \cap S_\beta \varphi_\beta = 0 \), if \( \alpha = \beta \);
(ii) For every infinite ascending chain \( \alpha_1 < \alpha_2 < \ldots \) in \( Y \), and every \( \alpha \in S_\alpha \), there exists \( \alpha_k \) such that \( \alpha \notin S_{\alpha_k} \varphi_{\alpha_k} \varphi_{\alpha_{k-1}} \cdots \varphi_{\alpha_1} \).

Let \( \psi_{\alpha, \beta} \) be the identity mapping on \( S_\alpha \), and for \( \alpha > \beta \), let
\[ \psi_{\alpha, \beta} = \varphi_\alpha \varphi_{\alpha_1} \cdots \varphi_{\alpha_n}, \] where \( \alpha > \alpha_1 > \ldots > \alpha_n > \beta \).
Let
\[ S = \left( \bigcup_{\alpha \in \mathcal{Y} \cap \zeta} (S_\alpha \setminus 0_\alpha) \right) \cup 0 , \]
where \( \zeta \) is the zero of \( Y \) (if \( Y \) has one), and 0 is an element not contained in any \( S_\alpha \); and on \( S \) define the multiplication \( \ast \) by
\[ a \ast b = (a^{(\alpha, \alpha_\beta})(b^{(\alpha, \alpha_\beta)}) , \text{ if } \alpha \beta \neq \zeta \text{ and } (a^{(\alpha, \alpha_\beta)})(b^{(\alpha, \alpha_\beta)}) \neq 0_\alpha \text{ in } S_\alpha , \]
and all other products are equal to 0. The set \( S \), with this multiplication, will be called a Brandt tree, if \( Y \) has a zero and a rooted Brandt tree otherwise.

**THEOREM 3.** - A semigroup \( S \) is prime \( \omega \)-regular and has a proper 0-minimal ideal if, and only if, \( S \) is an ideal extension of a 0-simple \( \omega \)-regular semigroup \( I \) by a Brandt tree \( T \) determined by a 0-restricted homomorphism of \( T \) into the top of \( I \).

Such a homomorphism is completely determined by its restriction to the socle \( \mathcal{S}(T) \) of \( T \), so all such homomorphisms are given by 0-restricted homomorphisms of \( \mathcal{S}(T) \) into \( \mathcal{S}(I) \), both of which are primitive inverse semigroups, and are easy to find explicitly.

**THEOREM 4.** - A groupoid \( S \) is a prime \( \omega \)-regular semigroup without 0-minimal ideals if, and only if, \( S \) is a rooted Brandt tree.

5. - The semigroups \( \mathcal{O}(A, w ; V, c) \) and \( \mathcal{O}[A, w ; V, c] \) do not seem to admit a neat isomorphism theorem, except in special cases. In the balanced case, using theorem 2, ([3], 4.1) and ([1], theor.4), we derive a satisfactory isomorphism theorem. A direct proof does the same in the case these semigroups are combinatorial. Isomorphisms of the semigroups in construction 3 are similar to those in [4], théorème 3.1, while isomorphisms of the semigroups in theorem 3 can be expressed by isomorphisms of \( I \) and \( T \) satisfying a commutative diagram.

**REFERENCES**


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