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THE STRUCTURE OF \( \omega \)-REGULAR SEMIGROUPS

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1. Finding the complete structure of regular semigroups of a certain class has succeeded only when sufficiently strong conditions on idempotents and (or) ideals have been imposed. On the one hand, there is the theorem of REES [7], giving the structure of completely \( 0 \)-simple semigroups, and its successive generalizations to primitive regular semigroups [2], and \( 3 \)- and \( 3_i \)-regular semigroups [4]. On the other hand, with very different restrictions, REILLY [8] has determined the structure of bisimple \( \omega \)-semigroups and, independently of each other, KOCHIN [1] of inverse simple \( \omega \)-semigroups, and MUNN [5] of inverse \( \omega \)-semigroups.

An \( \omega \)-chain with zero is a poset \( \{e_i \mid i > 0\} \cup \{0\} \), with \( e_i > e_j \) if \( i < j \), and \( 0 < e_i \) for all \( i, j \). We call a regular semigroup \( S \) \( \omega \)-regular, if \( S \) has a zero, and the poset of its idempotents is an orthogonal sum [2] of \( \omega \)-chains with zero. We announce here the complete determination of the structure of such semigroups, including various special cases thereof, and briefly mention their isomorphisms.

2. An \( \omega \)-regular semigroup can be uniquely written as an orthogonal sum of \( \omega \)-regular prime (i.e., with \( 0 \) a prime ideal) semigroups. This reduces the problems of structure and isomorphism to \( \omega \)-regular prime semigroups. We distinguish three cases:

(i) \( 0 \)-simple,
(ii) Prime with a proper \( 0 \)-minimal ideal,
(iii) Prime without a \( 0 \)-minimal ideal.

Case (i) is the most difficult (and interesting), and includes a variety of special cases some of which reduce to those constructed by REILLY [8], KOCHIN [1], and MUNN [5], [6].
and define \([i, \alpha, j] \) to satisfy
\[
[i, \alpha, j] d = (i - j) - (\langle \alpha, i \rangle - \langle \alpha, j \rangle).
\]

**Construction 1.** - On the set
\[ S = \{ (\alpha, m, g, n, \beta) \mid \alpha, \beta \in A, \ m, n > 0, \ g \in V \} \cup O, \]
define a multiplication by, for \( g_i \in G_i, \ g_j \in G_j, \ v = n - s - [i, \beta, j], \)
\[
(\alpha, m, g_i, n, \beta)(\beta, s, g_j, t, \gamma)
\]

\[
= \begin{cases} 
\alpha, m - [i, \alpha, j] - v, (g_i \sigma^{-v})g_j, t, \gamma, & \text{if } v < 0, \\
(\alpha, m, g_i(g_j \sigma^{-v}), t + [i, \gamma, j] + v, \gamma), & \text{if } v > 0,
\end{cases}
\]

and all other products are equal to 0. The set \( S \), with this multiplication, will be denoted by \( \mathcal{O}(A, w; V, \sigma) \).

**Construction 2.** - On the set
\[ S' = \{ (\alpha, m, g, n, \beta) \mid \alpha, \beta \in A, \ m - w_\alpha = n - w_\beta = i \pmod{d}, \ g \in G_i \} \cup O, \]
define a multiplication by, for \( g_i \in G_i, \ g_j \in G_j, \ v = n' - s' - [i, \beta, j], \)
where \( n = n'd + n'', \ s = s'd + s'', \ c \leq n'', \ s'' < d, \)
\[
(\alpha, m, g_i, n, \beta)(\beta, s, g_j, t, \gamma)
\]

\[
= \begin{cases} 
(\alpha, m + s - n, (g_i \sigma^{-v})g_j, t, \gamma), & \text{if } n \leq s, \\
(\alpha, m, g_i(g_j \sigma^{-v}), t + n - s, \gamma), & \text{if } n > s,
\end{cases}
\]

and all other products are equal to 0. The set \( S' \), with this multiplication, will be denoted by \( \mathcal{O}[A, w; V, \sigma] \).

The following is our fundamental result.

**THEOREM 1.** - For a groupoid \( S \), the following statements are equivalent:

(i) \( S \) is a 0-simple \( \omega \)-regular semigroup;
(ii) \( S \) is isomorphic to \( \mathcal{O}(A, w; V, \sigma) \);
(iii) \( S \) is isomorphic to \( \mathcal{O}[A, w; V, \sigma] \).

The proof of "(i) \( \implies \) (ii)" consists of "introducing coordinates" into various \( L \) - and \( R \) -classes, and of constructing the homomorphism \( \sigma \); it is quite long, and
is broken into a sequence of lemmas. For "(ii) $\Rightarrow$ (iii)"$, one finds a suitable isomorphism, while "(iii) $\Rightarrow$ (i)" consists of a verification of the defining properties of a 0-simple $\omega$-regular semigroup.

Define the top of $S$ in the theorem by $3(S) = \{a \in S \mid e \leq a, a \not\leq f \text{ for some maximal idempotents } e, f \} \cup 0$. Then $3(S)$ is a primitive inverse semigroup. It follows from the proof that we can always suppose that $w_\alpha = 0$ for some $\alpha \in \Lambda$. Call $S$ balanced, if any two maximal idempotents of $S$ are $\Omega$-equivalent.

THEOREM 2. - The following conditions on a 0-simple $\omega$-regular semigroup $S$ are equivalent:

(i) $S$ is balanced;
(ii) $S$ admits a representation as in theorem 1, with $w_\alpha = 0$ for all $\alpha \in \Lambda$;
(iii) $3(S)$ is a Brandt semigroup;
(iv) $S$ is isomorphic to a Rees matrix semigroup $\mathcal{R}(K; \Lambda, \Lambda; \Lambda)$ over a simple inverse $\omega$-semigroup $K$, $\Lambda$ is the identity matrix.

The structure of the semigroup $K$ in theorem 2 was determined by KOCHIN [1] and MUNN [5], the Rees matrix semigroups over bisimple inverse semigroups were studied in [3] (for the 0-simple case in the theorem, cf. [3], cor. 5.7, and [6], th. 4.2). Various other special cases include: 0-bisimple, combinatorial, balanced, and combinations thereof.

\section{Construction 3.} - Let $Y$ be a tree semilattice satisfying one of the two conditions:

(1) $Y$ has a zero $\zeta$, and all elements of $Y$ are of finite height;
(2) $Y$ has no zero, and is of locally finite length.

To every non-zero element $\alpha$ of $Y$, associate a Brandt semigroup $S_\alpha$, suppose that the family $\{S_\alpha\}$ is pairwise disjoint, and that a homomorphism $\varphi_\alpha : S_\alpha \to S_\beta$ is given, where $\alpha$ is the unique element of $Y$ covered by $\alpha$, with the properties:

(i) $S_\alpha \varphi_\alpha \cap S_\beta \varphi_\beta = 0$, if $\alpha = \beta$;
(ii) For every infinite ascending chain $\alpha_1 < \alpha_2 < \cdots$ in $Y$, and every $\alpha \in S_\alpha_1$, there exists $\alpha_k$ such that $a \notin S_\alpha_k \varphi_\alpha_k \cdots \varphi_\alpha_1, \alpha_2$.

Let $\psi_{\alpha, \beta}$ be the identity mapping on $S_\alpha$, and for $\alpha > \beta$, let

$$\psi_{\alpha, \beta} = \varphi_\alpha \varphi_\alpha_1 \cdots \varphi_\alpha_n$$

where $\alpha > \alpha_1 > \cdots > \alpha_n > \beta$. 

where $\xi$ is the zero of $Y$ (if $Y$ has one), and $0$ is an element not contained in any $S_\alpha$; and on $S$ define the multiplication $\ast$ by

$$a \ast b = (a^{\alpha}, a^\beta)(b^{\beta}, a^\beta), \quad \text{if } \alpha \beta \neq \xi \quad \text{and} \quad (a^{\alpha}, a^\beta)(b^{\beta}, a^\beta) \neq 0^\beta \text{ in } S_\alpha^\beta;$$

and all other products are equal to 0. The set $S$, with this multiplication, will be called a Brandt tree, if $Y$ has a zero and a rooted Brandt tree otherwise.

**Theorem 3.** - A semigroup $S$ is prime $\omega$-regular and has a proper $0$-minimal ideal if, and only if, $S$ is an ideal extension of a $0$-simple $\omega$-regular semigroup $I$ by a Brandt tree $T$ determined by a $0$-restricted homomorphism of $T$ into the top of $I$.

Such a homomorphism is completely determined by its restriction to the socle $S(T)$ of $T$, so all such homomorphisms are given by $0$-restricted homomorphisms of $S(T)$ into $3(I)$, both of which are primitive inverse semigroups, and are easy to find explicitly.

**Theorem 4.** - A groupoid $S$ is a prime $\omega$-regular semigroup without $0$-minimal ideals if, and only if, $S$ is a rooted Brandt tree.

5. - The semigroups $G(A, w; V, \sigma)$ and $G[A, w; V, \sigma]$ do not seem to admit a neat isomorphism theorem, except in special cases. In the balanced case, using theorem 2, ([3], 4.1) and ([1], theor.4), we derive a satisfactory isomorphism theorem. A direct proof does the same in the case these semigroups are combinatorial. Isomorphisms of the semigroups in construction 3 are similar to those in [4], théorème 3.1, while isomorphisms of the semigroups in theorem 3 can be expressed by isomorphisms of $I$ and $T$ satisfying a commutative diagram.

**References**


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