JANET AULT
MARIO PETRICH

The structure of $\omega$-regular semigroups


<http://www.numdam.org/item?id=SD_1969-1970__23_2_A6_0>
THE STRUCTURE OF ω-REGULAR SEMIGROUPS

by Janet AULT and Mario PETRICH

1. - Finding the complete structure of regular semigroups of a certain class has succeeded only when sufficiently strong conditions on idempotents and (or) ideals have been imposed. On the one hand, there is the theorem of REES [7], giving the structure of completely 0-simple semigroups, and its successive generalizations to primitive regular semigroups [2], and ω- and ω₁-regular semigroups [4]. On the other hand, with very different restrictions, REILLY [8] has determined the structure of bisimple ω-semigroups and, independently of each other, KOCHIN [1] of inverse simple ω-semigroups, and MUNN [5] of inverse ω-semigroups.

An ω-chain with zero is a poset \( \{ e_i \mid i \geq 0 \} \cup \{ 0 \} \), with \( e_i \geq e_j \) if \( i < j \), and \( 0 < e_i \) for all \( i, j \). We call a regular semigroup \( S \) ω-regular, if \( S \) has a zero, and the poset of its idempotents is an orthogonal sum [2] of ω-chains with zero. We announce here the complete determination of the structure of such semigroups, including various special cases thereof, and briefly mention their isomorphisms.

2. - An ω-regular semigroup can be uniquely written as an orthogonal sum of ω-regular prime (i.e., with 0 a prime ideal) semigroups. This reduces the problems of structure and isomorphism to ω-regular prime semigroups. We distinguish three cases:

(i) 0-simple,
(ii) Prime with a proper 0-minimal ideal,
(iii) Prime without a 0-minimal ideal.

Case (i) is the most difficult (and interesting), and includes a variety of special cases some of which reduce to those constructed by REILLY [8], KOCHIN [1], and MUNN [5], [6].

3. - Let \( A \) be a nonempty set, \( d \) be a positive integer, \( V \) be a semigroup which is a chain of \( d \) groups \( G_0 > G_1 > \ldots > G_{d-1} \), and \( \sigma \) be a homomorphism of \( V \) into \( G_0 \). Let \( w : A \to \{ 0, 1, \ldots, d-1 \} \) be any function, denoted by \( w : \alpha \to w_\alpha \). For \( \alpha \in A \), \( 0 \leq i, j < d \), define \( \langle \alpha, i \rangle \) by

\[ \langle \alpha, i \rangle = w_\alpha + i \pmod{d}, \quad 0 \leq \langle \alpha, i \rangle < d, \]
and define \([i, \alpha, j]\) to satisfy
\[
[i, \alpha, j] d = (i - j) - (\langle \alpha, i \rangle - \langle \alpha, j \rangle).
\]

**Construction 1.** - On the set
\[
S = \{(\alpha, m, g, n, \beta) \mid \alpha, \beta \in A, m, n > 0, g \in V\} \cup O,
\]
define a multiplication by, for \(g_i \in G_i, g_j \in G_j, v = n - s - [i, \beta, j],\)
\[
(\alpha, m, g_i, n, \beta)(\beta, s, g_j, t, \gamma)
\]
\[
= \begin{cases}
(\alpha, m - [i, \alpha, j] - v, (g_i \sigma^v)g_j, t, \gamma), & \text{if } v < 0,
\text{or } v = 0, i \leq j,

(\alpha, m, g_i(g_j \sigma^v), t + [i, \gamma, j] + v, \gamma), & \text{if } v > 0,
\text{or } v = 0, i > j,
\end{cases}
\]
and all other products are equal to 0. The set \(S\), with this multiplication,
will be denoted by \(\mathcal{O}(A, w; V, \sigma)\).

**Construction 2.** - On the set
\[
S' = \{(\alpha, m, g, n, \beta) \mid \alpha, \beta \in A, m - w_{\alpha} = n - w_{\beta} = i \mod d, g \in G_i\} \cup O,
\]
define a multiplication by, for \(g_i \in G_i, g_j \in G_j, v = n' - s' - [i, \beta, j],\)
where \(n = n'd + n'', s = s'd + s'', c \leq n'', s'' < d,\)
\[
(\alpha, m, g_i, n, \beta)(\beta, s, g_j, t, \gamma)
\]
\[
= \begin{cases}
(\alpha, m + s - n, (g_i \sigma^v)g_j, t, \gamma), & \text{if } n < s,

(\alpha, m, g_i(g_j \sigma^v), t + n - s, \gamma), & \text{if } n > s,
\end{cases}
\]
and all other products are equal to 0. The set \(S'\), with this multiplication,
will be denoted by \(\mathcal{O}[A, w; V, \sigma]\).

The following is our fundamental result.

**Theorem 1.** - For a groupoid \(S\), the following statements are equivalent:

(i) \(S\) is a \(0\)-simple \(\omega\)-regular semigroup;
(ii) \(S\) is isomorphic to \(\mathcal{O}(A, w; V, \sigma)\);
(iii) \(S\) is isomorphic to \(\mathcal{O}[A, w; V, \sigma]\).

The proof of "(i) \(\implies\) (ii)" consists of "introducing coordinates" into various
\(L\)- and \(R\)-classes, and of constructing the homomorphism \(\sigma\); it is quite long, and
is broken into a sequence of lemmas. For "(ii) $\implies$ (iii)", one finds a suitable
isomorphism, while "(iii) $\implies$ (i)" consists of a verification of the defining
properties of a 0-simple $\omega$-regular semigroup.

Define the top of $S$ in the theorem by $\mathcal{S}(S) = \{a \in S \mid e \leq a, a \not\leq f \text{ for }
\text{some maximal idempotents } e, f\} \cup 0$. Then $\mathcal{S}(S)$ is a primitive inverse semi-
group. It follows from the proof that we can always suppose that $w_\alpha = 0$ for some
$\alpha \in A$. Call $S$ balanced, if any two maximal idempotents of $S$ are $\Omega$-equivalent.

**THEOREM 2.** The following conditions on a 0-simple $\omega$-regular semigroup $S$ are
equivalent:

(i) $S$ is balanced;

(ii) $S$ admits a representation as in theorem 1, with $w_\alpha = 0$ for all $\alpha \in A$;

(iii) $\mathcal{S}(S)$ is a Brandt semigroup;

(iv) $S$ is isomorphic to a Rees matrix semigroup $\mathbf{R}(K; A; A; A)$ over a sim-
ple inverse $\omega$-semigroup $K$, $\Lambda$ is the identity matrix.

The structure of the semigroup $K$ in theorem 2 was determined by KOCHIN [1] and
MUNN [5], the Rees matrix semigroups over bisimple inverse semigroups were studied
in [3] (for the 0-simple case in the theorem, cf. [3], cor. 5.7, and [6], th. 4.2).
Various other special cases include: 0-bisimple, combinatorial, balanced, and
combinations thereof.

---

For the remaining cases, we will need the following.

**Construction 3.** Let $Y$ be a tree semilattice satisfying one of the two condi-
tions:

(1) $Y$ has a zero $\zeta$, and all elements of $Y$ are of finite height;

(2) $Y$ has no zero, and is of locally finite length.

To every non-zero element $\alpha$ of $Y$, associate a Brandt semigroup $S_\alpha$, suppose
that the family $\{S_\alpha\}$ is pairwise disjoint, and that a homomorphism $\phi_\alpha : S_\alpha \to S_\overline{\alpha}$ is given, where $\overline{\alpha}$ is the unique element of $Y$ covered by $\alpha$, with the proper-
ties:

(i) $S_\alpha \circ \phi_\alpha \circ S_\beta \circ \phi_\beta = 0$, if $\overline{\alpha} = \overline{\beta}$;

(ii) For every infinite ascending chain $\alpha_1 < \alpha_2 < \ldots$ in $Y$, and every $a \in S_\alpha_1$,
there exists $\alpha_k$ such that $a \notin S_\alpha_k \circ \phi_\alpha_k \circ \phi_{\alpha_{k-1}} \cdots \phi_{\alpha_2}$.

Let $\psi_{\alpha, \beta}$ be the identity mapping on $S_\alpha$, and for $\alpha > \beta$, let

$\psi_{\alpha, \beta} = \phi_\alpha \circ \phi_{\alpha_1} \cdots \phi_{\alpha_n}$, where $\alpha > \alpha_1 > \ldots > \alpha_n > \beta$. 


Let
\[ S = \bigcup_{\alpha \in \bar{Y}} \left( \mathbb{O}_\alpha \setminus \mathbb{O} \right) \cup \mathbb{O} , \]
where \( \zeta \) is the zero of \( Y \) (if \( Y \) has one), and \( \mathbb{O} \) is an element not contained in any \( \mathbb{O}_\alpha \); and on \( S \) define the multiplication \( * \) by
\[ a * b = (a\alpha, \alpha \beta)(b\beta, \alpha \beta), \]
if \( \alpha \beta \neq \zeta \) and \( (a\alpha, \alpha \beta)(b\beta, \alpha \beta) \neq 0 \alpha \beta \) in \( S \alpha \beta \),
and all other products are equal to \( 0 \). The set \( S \), with this multiplication, will be called a Brandt tree, if \( Y \) has a zero and a rooted Brandt tree otherwise.

**THEOREM 3.** - A semigroup \( S \) is prime \( \omega \)-regular and has a proper \( 0 \)-minimal ideal if, and only if, \( S \) is an ideal extension of a \( 0 \)-simple \( \omega \)-regular semigroup \( I \) by a Brandt tree \( T \) determined by a \( 0 \)-restricted homomorphism of \( T \) into the top of \( I \).

Such a homomorphism is completely determined by its restriction to the socle \( S(T) \) of \( T \), so all such homomorphisms are given by \( 0 \)-restricted homomorphisms of \( S(T) \) into \( S(I) \), both of which are primitive inverse semigroups, and are easy to find explicitly.

**THEOREM 4.** - A groupoid \( S \) is a prime \( \omega \)-regular semigroup without \( 0 \)-minimal ideals if, and only if, \( S \) is a rooted Brandt tree.

5. - The semigroups \( \mathbb{O}(A, w; V, \sigma) \) and \( \mathbb{O}[A, w; V, \sigma] \) do not seem to admit a neat isomorphism theorem, except in special cases. In the balanced case, using theorem 2, ([3], 4.1) and ([1], thèor.4), we derive a satisfactory isomorphism theorem. A direct proof does the same in the case these semigroups are combinatorial. Isomorphisms of the semigroups in construction 3 are similar to those in [4], thèorème 3.1, while isomorphisms of the semigroups in theorem 3 can be expressed by isomorphisms of \( I \) and \( T \) satisfying a commutative diagram.

**REFERENCES**


(Texte reçu le 15 septembre 1970)

Janet AULT and Mario PETRICH
Pennsylvania State University
Department of Mathematics
UNIVERSITY PARK, Pa (Etats-Unis)