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FINITE SEMIGROUPS ADMITTING RING STRUCTURE

by Rolando E. PEINADO

A semigroup S is said to admit a ring structure if S is isomorphic to the multiplicative structure of some (associative) ring R (in case S has no zero elements, we consider S^0), where R belongs to a class of rings $\mathcal{R}$, we call S an $\mathcal{R}$ -semigroup. It is clear that the class of $\mathcal{R}$ -semigroup is not empty since the multiplicative structure of any ring is such a semigroup. By considering left (right) zero semigroup with more than one element, it is clear that not all semigroups are $\mathcal{R}$ -semigroups. The problem of characterizing the class of all $\mathcal{R}$ -semigroups, where $\mathcal{R}$ is the class of all rings, was first formally stated by P. DUBREIL [1] and KOGALOVSKIJ in [2], shows that it is impossible to give an axiomatic characterization of such class.

Here we proposed to give a brief account of results obtained by several people on the specific case of finite $\mathcal{R}$ -semigroup. For a more complete survey of the general case (includes unique addition $\mathcal{R}$ -semigroups) and a more complete bibliography, see R. E. PEINADO [3], to appear in Semigroup Forum.

If S is a finite $\mathcal{R}$ -semigroup without zero (S^0 admits a ring structure), and suppose R is a ring whose multiplicative structure is isomorphic to S^0 . Then R has no zero divisors, and hence it is cancellative and being finite, it is a finite integral domain, and hence R a field, moreover it is a Galois field of order p^n . This implies that S has order $p^n - 1$ and is a cyclic group. The converse is obvious.

THEOREM 1. - A finite semigroup S without zero is an $\mathcal{R}$ -semigroup if, and only if, S is a cyclic group of order $p^n - 1$ for some prime p .

This theorem enables us to restrict our attention to finite semigroups with zero element and zero divisors. In considering finite $\mathcal{R}$ -semigroup, the following results are important.

THEOREM 2. - Let R be a finite ring of order n , where

$$n = \prod_{i=1}^{t_i} p_i^{t_i}, \quad i = 1, \dots, k, \quad t_i > 0.$$

Then R is isomorphic to the direct sum of p_i -rings R_i , $i = 1, \dots, k$ of order $p_i^{t_i}$.

THEOREM 3. - Let S be a finite semigroup of order n ,

$$n = \prod p_i^{t_i}, \quad i = 1, \dots, k.$$

S is an $\mathcal{R}$ -semigroup if, and only if, S is the direct sum of $\mathcal{R}$ -semigroups S_i , of order $p_i^{t_i}$, $i = 1, \dots, k$.

THEOREM 4. - Let S be a finite cyclic semigroup; then S is an $\mathcal{R}$ -semigroup if, and only if, S is of order 1 or 2.

If $n = p$, a prime number, then there exists two non-isomorphic rings of order p . The ring with trivial multiplication, all products equal to zero and the Galois field of order p .

THEOREM 5. - A finite semigroup S with zero of order p , a primer number, is an $\mathcal{R}$ -semigroup if, and only if, S is a cyclic group with zero or a zero semigroup

COROLLARY. - A finite semigroup S with zero, of square-free order is an $\mathcal{R}$ -semigroup if, and only if, S is a direct product of zero semigroups and/or cyclic groups with zero.

THEOREM 6. - A finitely generated commutative semigroup S is an $\mathcal{R}$ -semigroup if, and only if, S is finite.

THEOREM 7. - A finite $\mathcal{R}$ -semigroup of cube-free order is commutative.

For the integers modulo n , $\mathbb{Z}_n$, we can obtain the following interesting result.

THEOREM 8. - Let $\bar{\mathbb{Z}}_p^t = \mathbb{Z}_p^t - \{0\}$. Then the multiplicative semigroup $\bar{\mathbb{Z}}_p^t$ contains a subgroup $G(p^t)$ isomorphic to $\bar{\mathbb{Z}}_p$, given by $G(p^t) = \langle X_p^t \rangle$, where, for the prime $p \neq 2$ and $t > 1$,

$$X_p^t = X_p^{t-1} \quad (\text{modulo } p^{t-1})$$

and

$$(X_p)^{\frac{p-1}{2}} \equiv p^t - 1 \quad (\text{modulo } p^t),$$

for $p = 2$, $G(2^t) = \langle 1 \rangle$,

for $t = 1$, $G(p) = \bar{\mathbb{Z}}_p$.

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