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ALGEBRAIC INDEPENDENCE OF THE NUMBERS $\prod_{K=0}^{\infty}(1 - p^{-2^K})$ WITH p PRIME

by Kenneth K. KUBOTA

This is a description of some refinements of a transcendence method developed by K. MAHLER in [8], [9], [10] almost fifty years ago. More recently, MAHLER has written expository articles [11], [12] describing his method and some open problems, two of which will be dealt with below. Full details appear in [3], [4].

1. Transcendence.

Let $R : \mathbb{C} \dashrightarrow \mathbb{C}$ be the map $z \longmapsto z^{\rho}$, where $\rho \geqslant 2$ is an integer. It can be shown that the product

(1) $$f(z) = \prod_{K=0}^{\infty}(1 - z^{\rho^K}) , \quad |z| < 1$$

defines a transcendental function satisfying the functional equation

(2) $$f(z) = (1 - z)f(Rz) .$$

THEOREM 1 (MAHLER [8]). - *If z_0 is an algebraic number in the open punctured unit disc, then $f(z_0)$ is transcendental.*

Proof. - Suppose $K = \mathbb{Q}(z_0 , f(z_0))$ is an algebraic number field. For each positive integer p, one can construct a non-zero auxiliary function of the form

$$E_p(z) = \sum_{j=0}^{p} \alpha_j(z) f(z)^j = \sum_{i=0}^{\infty} b_i z^i \neq 0$$

where the $\alpha_j(z)$ are polynomials of degree at most p with coefficients algebraic integers in K and $b_i = 0$ for $i \leqslant p^2$. In fact, this amounts to solving $p^2 + 1$ linear equation in the $(p + 1)^2$ coefficients of the $\alpha_j(z)$. A non-zero solution yields a non-zero auxiliary function since $f(z)$ is a transcendental function.

If m is the least index for which $b_m \neq 0$, then $E(R^k z_0) \sim b_m z_0^{m\rho^k}$ as $k \dashrightarrow \infty$. In fact, by Cauchy's inequality, there are positive numbers A and B with $|b_i| < AB^i$ for all $i \geqslant 0$. Since $R^k z = z^{\rho^k}$, one has

$$|(E_p(R^k z_0)/b_m z_0^{m\rho^k}) - 1| = |\sum_{i=m+1}^{\infty}(b_i/b_m)z_0^{(i-m)\rho^k}|$$
$$\leqslant 1 + (|z_0|^{\rho^k}/|b_m|) \sum_{j=0}^{\infty} AB^{m+1}(B|z_0|^{\rho^k})^j \dashrightarrow 1$$

as $k \dashrightarrow \infty$ because $|z_0| < 1$. In particular, there is a $c_1 > 0$ independent of both p and k such that for all k larger than some function of p, we have

(3) $$0 \neq |E_p(R^k z_0)| < \exp(- c_1 m\rho^k) \leqslant \exp(- c_1 p^2 \rho^k) .$$

To obtain a lower bound on $|E_p(R^k z_0)|$, one uses the Liouville inequality.

Recall [5] that the size $s(\alpha)$ of $\alpha \in K$ is defined by

$$s(\alpha) = \max_j(\log \text{den } \alpha \text{ , } \log|\alpha^{(j)}|)$$

where the $\alpha^{(j)}$ are the conjugates of α , and den α is the denominator of α . The Liouville inequality [5] is just

(4) $$\log|\alpha| \geqslant - 2[K:\mathbb{Q}]s(\alpha) \quad \text{for} \quad \alpha \in K \setminus (0) \text{ .}$$

The functional equation (2) iterates to give

$$f(z) = \prod_{i=0}^{k-1}(1 - z^{\rho^i})\ f(R^k z) \text{ ,}$$

and so

$$E_p(R^k z_0) = \sum_{j=0}^{p} a_j(z_0^{\rho^k})\ f(R^k z_0)^j$$
$$= \sum_{j=0}^{p} a_j(z_0^{\rho^k})\{f(z_0) \prod_{i=0}^{k-1}(1 - z_0^{\rho^i})^{-1}\}^j \text{ .}$$

Since the a_j have degree at most p and coefficients dependent on p , it is straightforward to verify that

$$s(E_p(R^k z_0)) \leqslant c_2\ p\rho^k + c_3$$

where $c_3 > 0$ depends on p but not k , and $c_2 > 0$ is independent of both p and k . By taking k larger than some function of p , it follows that

$$s(E_p(R^k z_0)) \leqslant 2c_2\ p\rho^k \text{ ,}$$

and so by equation (4) one has a $c_4 > 0$ independent of p and k with

$$|E_p(R^k z_0)| \geqslant \exp(- c_4\ p\rho^k) \text{ .}$$

If p is chosen sufficiently large, this contradicts equation (3).

2. Algebraic independence.

The title result obviously follows from :

THEOREM 2. - *If $f(z)$ is as in equation (1), and $z_0 = (z_{01} , \dots , z_{0n})$ is an algebraic point with*

$$0 < |z_{0j}| < 1 \quad \textit{for} \quad j = 1 , \dots , n$$

and $|z_{01}| , \dots , |z_{0n}|$ are multiplicatively independent, then

$$f(z_{01}) \text{ , } \dots \text{ , } f(z_{0n})$$

are algebraically independent numbers.

Proof. - The proof given in section 1 will be generalized. Let R be the function $R : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ defined by $(z_1 , \dots , z_n) \longmapsto (z_1^{\rho} , \dots , z_n^{\rho})$. The functions $f_i(z) = f(z_i)$ where $z = (z_1 , \dots , z_n)$ are functions algebraically independent over $\mathbb{C}(z)$ and satisfying

$$f_i(z) = (1 - z_i)\ f_i(Rz) \text{ ,}$$

These functional equations iterate to give

(5) $$f_i(z) = f_i(R^k z) \prod_{j=0}^{k-1}(1 - z_i^{\rho^j}) .$$

If the $f_i(z_0)$ where $z_0 = (z_{01}, \dots, z_{0n})$ are algebraically dependent, then they satisfy an equation

$$\sum_{\alpha_1,\dots,\alpha_n} t_{\alpha_1\dots\alpha_n} f_1(z_0)^{\alpha_1} \dots f_n(z_0)^{\alpha_n} = 0$$

where the $\mathfrak{t} = (\mathfrak{t}_{\alpha_1\dots\alpha_n})$ are finitely many algebraic integers. Define

(6) $$F(z, t) = \sum_{\alpha_1,\dots,\alpha_n} t_{\alpha_1\dots\alpha_n} f_1(z)^{\alpha_1} \dots f_n(z)^{\alpha_n}$$

where $t = (t_{\alpha_1\dots\alpha_n})$ is a set of indeterminants. Note that substituting equation (5) into equation (6) gives

$$F(z, t) = F(R^k z, t^{(k)})$$

where $t^{(k)} = (t^{(k)}_{\alpha_1\dots\alpha_n})$ is defined by

$$t^{(k)}_{\alpha_1\dots\alpha_n} = t_{\alpha_1\dots\alpha_n} \prod_{i=1}^{n} \prod_{j=0}^{k-1}(1 - z_i^{\rho^j})^{\alpha_i} .$$

If $\mathfrak{t}^{(k)}$ is defined by substituting $(\mathfrak{t}, z_0)$ in place of (t, z), then it follows that

(7) $$F(R^k z_0, \mathfrak{t}^{(k)}) = F(z_0, \mathfrak{t}) = 0$$

for all $k \geq 0$.

Suppose that one has for large positive integers p an auxiliary function of the form

(8) $$E_p(z) = \sum_{i=0}^{p} a_i(z,t) F(z,t)^i = \sum_{L_1,\dots,L_n} E_{pL_1\dots L_n}(t) z_1^{L_1}\dots z_n^{L_n} \not\equiv 0$$

where the $a_i(z, t)$ are polynomials of degree at most p in each z_i and each $t_{\alpha_1\dots\alpha_n}$ and with coefficients algebraic integers in $K = \mathbb{Q}(z_0, \mathfrak{t})$. Observe that by equation (7), one has

$$E_p(R^k z_0) = a_0(R^k z_0, \mathfrak{t}^{(k)}) .$$

One can calculate the size of $a_0(R^k z_0, \mathfrak{t}^{(k)})$ and as before the Liouville inequality shows that there is a $c_1 > 0$ independent of p and k with

(9) $$|E_p(R^k z_0)| = |a_0(R^k z_0, \mathfrak{t}^{(k)})| \geq \exp(-c_1 p\rho^k)$$

whenever

(10) $$a_0(R^k z_0, \mathfrak{t}^{(k)}) \neq 0 .$$

For the lower bound, one has

(11) $$E_p(R^k z_0) = \sum_{L_1,\dots,L_n} E_{pL_1\dots L_n}(\mathfrak{t}^{(k)})(z_{01}^{L_1} \dots z_{0n}^{L_n})^{\rho^k} .$$

Since the $|z_{0j}|$ are multiplicatively independent and $0 < |z_{0j}| < 1$, the set of $|z_{01}^{L_1} \dots z_{0n}^{L_n}|$ for $(L_1 , \dots , L_n)$ with $E_{pL_1 \dots L_n}(t) \neq 0$ has a largest element, say that corresponding to the exponents $m = (m_1 , \dots , m_n)$. An argument similar to that given for equation (3) shows that the term

$$E_{pm_1 \dots m_n}(\mathfrak{t}^{(k)})(z_{01}^{m_1} \dots z_{0n}^{m_n})\rho^k$$

dominates in (11). Of course, the $\mathfrak{t}^{(k)}$ pose an added difficulty ; but they have negligible effect essentially because

$$\log|\, \mathfrak{t}^{(k)}_{\alpha_1 \dots \alpha_n} | \leqslant c_2 \, k$$

is small compared to ρ^k . In this way, one obtains for k larger than some function of p that

$$(12) \qquad |E_p(R^k z_0)| \leqslant \exp(- c_3(m_1 + \dots + m_n)\rho^k)$$

where $c_3 > 0$ is independent of p and k . Therefore, if one could choose E_p so that $\sum_i m_i \geqslant p^{1+n^{-1}}$ and such that equation (10) holds for infinitely many k , then equations (9), (12) for large enough p would be a contradiction.

Guaranteeing this last condition is the main difficulty in the proof. For this, if $A(z , t) \in K[z , t]$ is a polynomial, define $A(z , t) \sim 0(\mathfrak{t})$ to mean that

$$A(R^k z_0 , \mathfrak{t}^{(k)}) = 0$$

for all sufficiently large k . Then one can prove following lemma.

LEMMA. - <u>The set</u> $\mathfrak{q}(\mathfrak{t})$ <u>of polynomials</u> $A(z , t) \in K[z , t]$ <u>with</u> $A(z , t) \sim 0(\mathfrak{t})$ <u>is a prime ideal of</u> $K[z , t]$ <u>with basis in</u> $K[t]$.

In particular, it makes sense to define $B(z , t) \sim 0(\mathfrak{t})$, where

$$(13) \qquad B(z , t) = \sum_{L_1, \dots, L_n} B_{L_1 \dots L_n}(t) \, z_1^{L_1} \dots z_n^{L_n} \in K[t][[z]]$$

is a power series, to mean $B_{L_1 \dots L_n}(t) \sim 0(\mathfrak{t})$ for all $L_1 , \dots , L_n$.

For arbitrary power series as in equation (13), the index is defined to be the least m for which there are $(m_1 , \dots , m_n)$ with

$$B_{m_1 \dots m_n}(t) \notin \mathfrak{q}(\mathfrak{t}) \quad \text{and} \quad m_1 + \dots + m_n = m .$$

The assertion that $\mathfrak{q}(\mathfrak{t})$ is prime, translates as

$$\text{Index}\,(B_1 B_2) = \text{Index}\, B_1 + \text{Index}\, B_2 .$$

A counting argument together with linear algebra now allows one to construct an auxiliary function of the form (11) with $a_0(z , t) \notin \mathfrak{q}(\mathfrak{t})$ and with

$$\text{Index}\, E_p(z) \gg p^{1+n^{-1}} .$$

Replacing equality with $\sim 0(\mathfrak{t})$ in the argument of the last paragraph gives equation (12) and hence the desired contradiction. Although the proof of the lemma will not be given here, it should be mentioned that the main ingredient is a vanishing theorem of the kind described in the next section.

3. Generalization to functions of several variables.

3.1. Let $R = (\rho_{ij})$ be an $n \times n$ non-negative rational integer matrix. Then R defines a map $R : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ by $z \longmapsto Rz$ where $z = (z_1, \ldots, z_n)$ and $Rz = (\prod_{j=1}^n z_j^{\rho_{1j}}, \ldots, \prod_{j=1}^n z_j^{\rho_{nj}})$.

Let $z_0 = (z_{01}, \ldots, z_{0n}) \in (\overline{\mathbb{Q}}^{\times})^n$ be an algebraic point with the $|z_{0j}|$ multiplicatively independent and $0 < |z_{0j}| < 1$ for $j = 1, \ldots, n$. Can there be a complex non-zero convergent power series $f(z)$ with $f(R^k z) = 0$ for all sufficiently large k ?

In the cases where R is either scalar : $R = \rho I$ with $\rho \geq 2$ or where the characteristic polynomial of R is irreducible over $\mathbb{Q}$ and R has only one eigenvalue of maximal absolute value $\rho > 1$, the answer is no, and the result forms the basis of MAHLER's papers [8], [9], [10]. However, in general the answer is yes. In fact, examples show that there are such power series for every choice of z_0 provided that R is either singular or possesses a root of unity eigenvalue. In answer to a problem of MAHLER [11], one can prove :

THEOREM 3. - *With R and z_0 as above, suppose in addition that R is non-singular and possesses no root of unity eigenvalues. Then $R^k z_0 \longrightarrow 0$ as $k \longrightarrow \infty$, and the only complex convergent power series $f(z)$ satisfying $f(R^k z_0) = 0$ for all sufficiently large k is the identically zero power series.*

The proof of this result as well as the refinement needed for the generalization of theorem 2 is an easy application of the following :

(a) Baker's theorem [1] on linear independence of logarithms,

(b) Turan's third main theorem ([13], p. 53) on lower bounds for exponential sums,

(c) Skolem-Mahler-Lech theorem [6] on zeros of linear recurrences,

(d) Perron-Frobenius theory [2] of non-negative matrices.

3.2. Let R be an $n \times n$ non-singular non-negative integer matrix with no root of unity eigenvalues, and $f_1(z), \ldots, f_m(z)$ be convergent power series with algebraic coefficients and satisfying functional equations

$$f_i(z) = a_i(z)\, f_i(Rz) + b_i(z) \qquad (14)$$

where the $a_i(z)$, $b_i(z)$ are rational functions with $a_i(0) \neq 0$. Denote by $\mathfrak{U}$ the set of all algebraic points $z_0 = (z_{01}, \ldots, z_{0n})$ with non-zero coordinates such that

(a) $R^k z_0 \longrightarrow 0$ as $k \longrightarrow \infty$

(b) The numerator and denominator of each $a_i(z)$ as well as the denominator of each $b_i(z)$ are non-zero at each of the points $R^k z_0$, $k \geqslant 0$.

(c) For every $(L_1, \ldots, L_n) \in \underset{\sim}{Z}^n \smallsetminus (0, \ldots, 0)$, the linear recurrence defined by the matrix product

$$g(k) = (L_1, \ldots, L_n) R^k \begin{pmatrix} \log|z_{01}| \\ \vdots \\ \log|z_{0n}| \end{pmatrix}$$

has at most finitely many zeros.

Here condition (b) is used to make sure one can iterate the functional equations whereas conditions (a) and (c) are weaker than the hypotheses used before that $0 < |z_{0j}| < 1$ and that the $|z_{0j}|$ be multiplicatively independent. One might expect :

CONJECTURE. - <u>If the</u> $f_i(z)$ <u>are algebraically independent over</u> $\underset{\sim}{C}(z)$ <u>and</u> $z_0 \in \mathcal{U}$, <u>then the numbers</u> $f_1(z_0), \ldots, f_n(z_0)$ <u>are algebraically independent.</u>

However, counterexamples can be constructed with matrices R as simple as $R = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$. One needs an additional hypothesis which roughly stated says that the rows of R^k grow equally fast. To be precise, recall [7] that the Hamilton-Jacobi theorem implies that the coordinates $\rho_{ij}^{(k)}$ of $R^k = (\rho_{ij}^{(k)})$ define linear recurrences, and so one can write

$$\rho_{ij}^{(k)} = \sum_r p_r(k)\, \alpha_r^k$$

where the $\alpha_r \in \underset{\sim}{C} \smallsetminus (0)$ are distinct and the $p_r(k)$ are non-zero polynomials. The degree (d, δ) of $\rho_{ij}^{(k)}$ is defined by

$$\delta = \max_r |\alpha_r| \quad \text{and} \quad d = \max_{|\alpha_r| = \delta} \deg p_r(k) .$$

The additional hypothesis is

$$(15) \qquad \min_i \max_j \deg \rho_{ij}^{(k)} = \max_{i,j} \deg \rho_{ij}^{(k)} .$$

One can show for example that this hypothesis is satisfied if R is of the form $R = A \oplus \ldots \oplus A$ where A is an irreducible square matrix, i. e. no permutation applied to both the rows and columns puts it in the form $\begin{pmatrix} B & 0 \\ * & C \end{pmatrix}$ with B, C square.

THEOREM 4. - <u>If, in addition to the above hypotheses, one has equation</u> (15), <u>then the conjecture holds true.</u>

3.3. For applications of theorem 4 as well as for its proof, one needs a simple test for deciding when the $f_i(z)$ are algebraically independent over $\underset{\sim}{C}(z)$.

This can be done in a purely algebraic setting. Let $\Omega : M \longrightarrow M$ be an endomorphism of a field M of characteristic zero, K be the subfield of M consisting of elements left fixed by Ω , and L be an intermediate field mapped into itself by Ω . For example, if M is the field of Laurent series, $f(z) \overset{\Omega}{\longmapsto} f(Rz)$ is as in section 3.1, and $L = \mathbb{C}(z)$, then theorem 3 can be used to show that $K = \mathbb{C}$.

Let f_{ij} for $i = 1 , \dots , m$ and $j = 1 , \dots , n(i)$ be elements of M satisfying "functional equations"

$$f_{ij} = a_i f_{ij}^{\Omega} + b_{ij}$$

where $a_i , b_{ij} \in L$, $a_i \neq 0$, and

(16) $$a_{i_1}/a_{i_2} \notin \mathcal{G} = \{g^{\Omega}/g \; ; \; g \in L \setminus (0)\} ,$$

for all $i_1 \neq i_2$. Note that condition (16) is merely a normalization, for if $a_{i_1}/a_{i_2} = g^{\Omega}/g \in \mathcal{G}$, then one can replace the $f_{i_1 j}$ with $gf_{i_1 j}$ which satisfies

$$gf_{i_1 j} = a_{i_2}(gf_{i_1 j})^{\Omega} + b_{i_1 j}\, g$$

without affecting algebraic independence over L . The criterion can now be stated as the following theorem.

THEOREM 5. - <u>If the</u> f_{ij} <u>are algebraically dependent over</u> L , <u>then they satisfy a non-trivial relation of the form</u>

$$\prod_{i=1}^{m} \Big(\sum_{j=1}^{n(i)} c_{ij} f_{ij} + g_i\Big)^{m_i} = g$$

<u>where the</u> m_i <u>are integers</u>, $c_{ij} \in K$, <u>and</u> $g_i , g \in L$.

As an example, this yields a corrected version of a result of MAHLER [10] :

COROLLARY. - <u>Let the</u> $f_{ij}(z)$ <u>be holomorphic functions defined on a connected open neighborhood of the origin and satisfying</u>

$$f_{ij}(z) = a_i f_{ij}(Rz) + b_{ij}(z)$$

<u>where the</u> $a_i \in \mathbb{C}$ <u>are distinct</u>, $b_{ij} \in \mathbb{C}(z)$, <u>and</u> R <u>is a non-singular non-negative rational integer matrix with no roots of unity in its spectrum. Then a necessary and sufficient condition in order for the</u> f_{ij} <u>to be algebraically independent over</u> $\mathbb{C}(z)$ <u>is that for each</u> $i = 1 , \dots , n$, <u>the functions</u>

$$f_{i1} , \dots , f_{in(i)}$$

<u>be</u> $\mathbb{C}$<u>-linearly independent</u> modulo $\mathbb{C}(z)$.

4. Other problems.

The second of the three problems of [11] generalizes the functional equation (14) by allowing the matrix R to vary. This leads to a functional recurrence of the form

(17) $$f_i(z) = a_i(z)\, f_{i+1}(R_{i+1} z) + b_i(z)$$

where the f_i , R_i and a_i b_i are infinite sequences of holomorphic functions, matrices, and rational functions respectively. For example, instead of considering a single function

$$f(z) = \sum_{k=0}^{\infty} z^{\rho^k} , \quad \rho \geq 2$$

satisfying the functional equation

$$f(z) = f(Rz) + z , \quad R = (\rho) ,$$

one considers a sequence of functions

$$f_i(z) = \sum_{k=i}^{\infty} z^{\rho_{i+1}\cdots\rho_k} \quad \text{for} \quad i \geq 0$$

satisfying a functional recursion

$$f_i(z) = f_{i+1}(R_{i+1}\, z) + z , \quad R_{i+1} = (\rho_{i+1}) .$$

For this example, one can show :

THEOREM 6. - <u>If the infinite sequence</u> $\{\rho_i\}$ <u>contains but a finite number of distinct integers</u> (≥ 2) , <u>then</u> $f(z_0)$ <u>is transcendental for all algebraic</u> z_0 <u>in the punctured open unit disc</u>.

This is a special case of a result [4] treating functional recursions of the form (17) in which the R_i are scalar matrices ; the case of general matrices is still open. An algebraic independence result along these lines can also be proved.

Finally, we mention that the third problem of [11] is still open. Here one asks about the transcendence of values of holomorphic functions $f(z)$ satisfying a polynomial functional equation

$$P(f(z) , f(z^\rho) , z) = 0 , \quad \rho \geq 2 .$$

The most interesting example is

$$f(z) = j\left(\frac{\log z}{2\pi i}\right) - z^{-1}$$

where $j(z)$ is the modular function of level one. In this case, all I am able to show is the inequality

$$[\mathbb{Q}(j(2^k z_0)):\mathbb{Q}] \gg 2^k/k .$$

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