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SOME CONSEQUENCES OF A KIND OF HAHN-BANACH'S THEOREM

by Richard BECKER

Abstract. - The aim of this work is to give some consequences of a theorem of H. DINGES used by M. F. SAINTE-BEUVE.

Preliminaries

1. THEOREM. - Let X be an ordered vector space, and p an extended sub-linear functional on X , such that $p(x) \in \mathbb{R} \cup (+\infty)$ for each $x \in X$, and $p(x) \leq 0$ for each $x \leq 0$. Let Y a linear subspace of X , and f a linear form on Y majorized by p . There exists a linear form on $Z = \{x; \exists x_1, x_2 \in Y \text{ with } x_1 \leq x \leq x_2\}$ which extends f and is majorized by p on Z ([5], [11]).

What is needed concerning conical measures can be found in [3] (§ 30, 38, 40). Notation not included in [3]. In this paper, $\mathcal{C}$ will be the class of weakly complete convex cones, not necessarily proper.

2. Summary. - Part I is devoted to conical measures. We generalize specially (proposition 12) the theorem of Cartier-Fell-Meyer ([10] p. 112) concerning dilations of measures on a metrizable convex compact set. Positive measures on a metrizable convex compact set can be considered as conical measures on a proper convex closed cone of $\mathbb{R}^N$. Here, we will consider arbitrary conical measures on $\mathbb{R}^N$.

Part II (A) extends a result of STRASSEN ([8], p. 300-301), from which the theorem of Cartier-Fell-Meyer can be derived. We weaken, here, a condition of compactness (proposition 21). Part II (B) extends some results about "theory of balayage" ([8], p. 294, 297). This theory studies cones of continuous functions on a compact set containing a strictly positive function. We weaken this condition.

Part I : The case of conical measures.

I (A). Conical measures on an arbitrary weak space.

Recall the following proposition which enlightens the definition of the order $<$.

3. PROPOSITION. - Let E a complete weak space, and $\Gamma \subset E$ a convex cone of $\mathcal{C}$. For each $f \in h(E)$, such that $f|_{\Gamma}$ is sub-linear, there exist $h_1, h_2, \dots, h_n \in E'$, such that we have on Γ , $f = \text{lub}(h_1, h_2, \dots, h_n)$.

Proof. - We can suppose E of finite dimension.

There exist $u_1, \dots, u_p \in E'$ such that, for each $x \in E$, $f(x)$ is equal to one of the $u_p(x)$. Hence, for each pair $x, y \in \Gamma$, there exist $p_{x,y}$, an integer $\leq p$, such that

$$f(x) = u_{p_{x,y}}(x), \text{ and } f(y) \geq u_{p_{x,y}}(y).$$

For each $x \in \Gamma$, let $v_x = \text{glb}_{y \in \Gamma} (u_{p_{x,y}})$. The family $(v_x)_{x \in \Gamma}$, is finite, and we have on Γ $f = \text{lub}_{x \in \Gamma} (v_x)$; as $(-v_x) \in S(E)$, we can conclude with the help of the elementary form of the theorem of Hahn-Banach because dimension of $E < \infty$.

4. PROPOSITION. - If E is a complete weak space, and $\mu \in M^+(E)$, then, for each $\lambda \in E'$ with $\lambda \neq 0$, the two following properties are equivalent.

1° $\forall f \in h^+(E)$, we have $\mu(f) = \lim(\mu(f \wedge n\lambda^+))$ when $n \rightarrow \infty$.

2° $\exists m$, σ -additive and positive functional on the tribe on $e = \lambda^{-1}(1)$ generated by $h(E)|_e$, such that $\mu(f) = m(f|_e)$, for each $f \in h(E)$.

If μ satisfies to 1° and 2°, then each $\lambda \in M^+(E)$, with $\lambda < \mu$, satisfies also to 1° and 2°.

Proof. 1° and 2° are equivalent on account of ([3], 38.13).

Proof that λ satisfies to 1°. Note that $h(E) = S^+(E) - S^+(E)$. Let $f \in S^+(E)$, we have

$$0 \leq \lambda(f - f \wedge n\lambda^+) \leq \lambda((f - n\lambda)^+) \leq \mu((f - n\lambda)^+) \rightarrow 0 \text{ when } n \rightarrow \infty,$$

hence

$$\lambda(f) = \lim(\lambda(f \wedge n\lambda^+)) \text{ when } n \rightarrow \infty.$$

5. PROPOSITION. - Suppose E is a weak space, and $\lambda, \mu \in M^+(E)$. If $\lambda < \mu$, then, for each sequence $\lambda_1, \lambda_2, \dots, \lambda_n$ of $M^+(E)$ such that $\lambda = \sum_1^n \lambda_i$, there exists a sequence $\mu_1, \mu_2, \dots, \mu_n$ of $M^+(E)$ such that $\mu = \sum_1^n \mu_i$, and $\lambda_i < \mu_i$ for $i = 1, 2, \dots, n$.

Proof. - For each $f \in h(E)$, let $\hat{f}$ such that :

1° $\hat{f} = \text{glb}(\lambda; \lambda \in E' \text{ and } \lambda \geq f)$ if f is majorized by an element of E' . (In fact, on account of [1] (chap. II, § 7, exercice 24), we have $-\hat{f} \in S(E)$.)

2° Otherwise, $\hat{f} \equiv +\infty$ on E .

For each $v \in M^+(E)$, let p_v such that :

1° If $\hat{f} \neq \infty$, $p_v(f) = \text{glb}(v(g); -g \in S(E), g \geq f)$. We have $p_v(f) \in \mathbb{R}$.

2° If $\hat{f} \equiv +\infty$, $p_v(\hat{f}) = +\infty$.

For $i = 1, 2, \dots, n$, let $p_i = p_{v_i}$.

On the space $(h(E))^n$, let us consider the functional p , such that

$$(f_i)_{1 \leq i \leq n} \mapsto p((f_i)) = \sum_1^n p_i(f_i).$$

p is sub-linear with values in $R \cup (+\infty)$, and

$$(f_i \leq 0, \text{ for } i = 1, 2, \dots, n) \implies (p((f_i)) \leq 0).$$

Let $\bar{\mu}$ the linear form on the diagonal of $(h(E))^n$, such that

$$\bar{\mu}((f, f, \dots, f)) = \mu(f).$$

$\bar{\mu}$ is majorized by p . As each element of $(h(E))^n$ is majorized by an element of the diagonal, we can apply the version of the theorem of Hahn-Banach recalled in 1. $\bar{\mu}$ has an extension $\tilde{\mu} \in (h(E)^n)_+^*$ with $\tilde{\mu} \leq p$. We can write $\tilde{\mu} = (\mu_i)_{1 \leq i \leq n}$ with $\mu_i \in h(E)_+^*$, for $i = 1, 2, \dots, n$.

The μ_i are convenient.

6. PROPOSITION. - Suppose E is a complete weak space, and $\lambda, \mu \in M^+(E)$. The two following properties are equivalent.

$$1^\circ \quad \lambda < \mu.$$

2° There exists a conical measure $\pi \in M^+(M^+(E) \times M^+(E))$ carried by the cone $B = \{(\varepsilon_x, \nu) ; x \in E \text{ and } \varepsilon_x < \nu\}$, such that $r(\pi) = (\lambda, \mu)$.

Proof. - For simplification, we will write sometimes M instead of $M(E)$ and M^+ instead of $M^+(E)$.

$1^\circ \implies 2^\circ$: For each sequence $\lambda_1, \lambda_2, \dots, \lambda_n$ satisfying the hypothesis of proposition 5, let us choose a sequence $\mu_1, \mu_2, \dots, \mu_n$ satisfying the conclusion of 5.

We say that a sequence $s' = \lambda'_1, \lambda'_2, \dots, \lambda'_m$ is finer than a sequence $s = \lambda_1, \lambda_2, \dots, \lambda_n$ if, and only if, there exists a partition of $\{1, 2, \dots, m\}$ into n subsets $p_1, p_2, \dots, p_n$, such that $\lambda_i = \sum_{j \in p_i} \lambda'_j$ for $i = 1, 2, \dots, n$. Let U_s the set consisting of all the (finite) sequences finer than s . The family of sets U_s is a filter basis over $U_{(\lambda)}$ where (λ) means the sequence λ . Let φ be the application

$$s \longmapsto \varphi(s) = \pi_s = \sum_1^n \varepsilon_{r(\lambda_i)}, \mu_i,$$

we have $\pi_s \in M^+(M^+ \times M^+)$. The family of sets $\varphi(U_s)$ is a filter basis over $M^+(M^+ \times M^+)$. We have $r(\pi_s) = \sum_1^n (\varepsilon_{r(\lambda_i)}, \mu_i)$.

Each element of $h(E)^+$ is majorized by an element of $S(E)^+$, and for each $f \in S(E)^+$, we have

$$\sum_1^n (\varepsilon_{r(\lambda_i)})(f) = \sum_1^n f(r(\lambda_i)) \leq \sum_1^n \lambda_i(f) = \lambda(f).$$

Hence the filter basis $\varphi(U_s)$ has at least a cluster point, let π . The element π answers the question, since each π_s is carried by B , and we have

$$r(\pi) = \lim(r(\pi_s)) = (\lambda, \mu).$$

$2^\circ \implies 1^\circ$: If $\pi \in M^+(M^+ \times M^+)$ with $r(\pi) = (\lambda, \mu)$, and if π is carried by B , then for each $f \in S(E)$, we have $\pi((-f, f)) \geq 0$, since the element $(-f, f)$

of $h(E) \times h(E)$ is ≥ 0 on B . Hence we have $\lambda < \mu$.

7. Remark. - We can prove 6 with the method of [10] (p. 108) (and without the theorem of § 1) by looking at the convex closure of the set

$$\{(\varepsilon_x, v) ; x \in E \text{ and } r(v) = x\}$$

in $M^+ \cdot M^+$. Then § 5 can be obtained for R^n as in [10] (p. 112) and in the general case by a projective limit argument.

8. Definition (of a pure pair and a pure measure). - Suppose $\lambda, \mu \in M^+(E)$. We say the pair (λ, μ) is pure if, and only if,

$$(\mu' \in M^+(E) \text{ and } \mu' \leq \mu, \lambda < \mu') \text{ involves } (\mu' = \mu).$$

Suppose $\lambda \in M^+(E)$. We say that λ is pure, when the two following equivalent condition are fulfilled.

1° $(\varepsilon_{r(\lambda)}, \lambda)$ is a pure pair.

2° K_λ admits 0 as an extremal point.

Proof.

1° $\implies$ 2°: Suppose 2° is false. Let $\lambda_1 \leq \lambda$, and $\lambda_2 \leq \lambda$ with $r(\lambda_1) = -r(\lambda_2) \neq 0$. If $\lambda_0 = \lambda - (\lambda_1 + \lambda_2)/2$, we have $0 \leq \lambda_0 \leq \lambda$, $\lambda_0 \neq \lambda$, and $r(\lambda_0) = r(\lambda)$, then $(\varepsilon_{r(\lambda)}, \lambda)$ is not a pure pair.

2° $\implies$ 1°: Suppose $\mu \leq \lambda$ with $\mu \geq 0$ and $r(\mu) = 0$. Let $\mu = \mu_1 + \mu_2 + \dots + \mu_n$ be any decomposition of μ where $\mu_i \geq 0$. We have $\mu_i \in K_\lambda$, and $\sum_{1 \leq i \leq n} r(\mu_i) = 0$, hence $r(\mu_i) = 0$ for $i = 1, 2, \dots, n$. Then $\mu = 0$.

9. Example. - In the cartesian product R^2 , let a, b, c, d be the consecutive vertices of a square of center 0. If

$$\lambda = \varepsilon_a + \varepsilon_b + \varepsilon_c + \varepsilon_d + \varepsilon_{-(c+d)},$$

$$\lambda_1 = \varepsilon_a + \varepsilon_b + \varepsilon_c + \varepsilon_d,$$

$$\lambda_2 = \varepsilon_c + \varepsilon_d + \varepsilon_{-(c+d)},$$

we have $r(\lambda_1) = r(\lambda_2) = 0$, and $(\lambda - \lambda_1), (\lambda - \lambda_2)$ are pure.

10. PROPOSITION. - Suppose E is a complete weak space, and $\lambda, \mu \in M^+(E)$ with $\lambda < \mu$. Then, the three following properties are equivalent.

1° The pair (λ, μ) is pure.

2° For each $\pi \in M^+(M^+ \times M^+)$, representing (λ, μ) according to § 6 and carried by the cone B , then the restriction π_0 of π to the cone $A = \{(0, v) ; v \in M^+(E) \text{ and } r(v) = 0\}$ is equal to zero.

3° Each $\pi \in M^+(M^+ \times M^+)$ representing (λ, μ) , and carried by the cone B , is carried by the cone $B_p = \{(\varepsilon_x, v) ; x \in E, r(v) = x, v \text{ is pure}\}$.

Proof.

$1^\circ \implies 2^\circ$: Suppose 2° is false. If π represents (λ, μ) with $\pi_0 \neq 0$ (for the definition of π_0 , see [3], 30.8), we have $r(\pi_0) = (0, \nu)$ with $\nu \neq 0$ and $r(\nu) = 0$.

For each $f \in S(E)$, we have $(-f, f) \geq 0$ on B . Hence

$$\nu(f) = \pi_0((-f, f)) \leq \pi((-f, f)) = -\lambda(f) + \mu(f)$$

Therefore $\lambda < \mu - \nu$, and (λ, μ) is not pure.

$2^\circ \implies 3^\circ$: We can write $\pi = \lim_u \sum \varepsilon(\varepsilon_x, \nu)$ with $(\varepsilon_x, \nu) \in B$ where u is an ultrafilter.

For each $\nu \in M^+(E)$, let us choose $p_\nu \in M^+(E)$ such that :

- (a) p_ν is pure ,
- (b) $p_\nu \leq \nu$,
- (c) $k \cdot p_\nu = p_{k \cdot \nu}$ for any $k \geq 0$.

We will prove that $\pi = \lim_u (\sum \varepsilon(\varepsilon_x, p_\nu))$.

We have

$$(\lambda, \mu) = (\lambda, \lim_u (\sum p_\nu)) + (0, \lim (\nu - p_\nu)) .$$

On account of the hypothesis, we have $\lim_u \sum (\nu - p_\nu) = 0$, hence

$\lim_u (\sum \varepsilon(0, \nu - p_\nu)) = 0$. For each $f \in S(M \times M)$, we have

$$f(\varepsilon_x, p_\nu) - f(0, p_\nu - \nu) \leq f(\varepsilon_x, \nu) \leq f(0, \nu - p_\nu) + f(\varepsilon_x, p_\nu) .$$

As we have $\lim_u (\sum \varepsilon(0, \nu - p_\nu)) = 0$, then $\pi(f) = \lim_u (\sum f(\varepsilon_x, p_\nu))$. Therefore π is carried by B_p .

$3^\circ \implies 1^\circ$: Suppose 1° is false. We have $(\lambda, \pi) = r(\varepsilon_{(\lambda, \alpha)} + \varepsilon_{(0, \beta)})$ with (λ, α) pure, and $(0, \beta) \in A$ with $\beta \neq 0$. Therefore $\varepsilon_{(0, \beta)}$ is not carried by B_p .

11. Example (G. CHOQUET). - In R^2 suppose C_1 and C_ρ are the circles (for the classical distance) of center 0 with radius 1 and $\rho > 1$. For each $x \in C_1$, let $x_1, x_2 \in C_\rho$ so that (x_1, x_2) is tangent to C_1 at x . Let dx be the Haar measure on C_1 . We have

$$\int_{C_1} (\varepsilon_{x_1} + \varepsilon_{x_2}) dx = \rho' \int_{C_1} \varepsilon_x dx \quad (\text{with } \rho' > 1)$$

as conical measures.

The pair $(\varepsilon_x, \varepsilon_{x_1} + \varepsilon_{x_2})$ is pure for each $x \in C_1$, but the resultant of $\int_{C_1} \varepsilon(\varepsilon_x, \varepsilon_{x_1} + \varepsilon_{x_2}) dx$ is the pair $(\int_{C_1} \varepsilon_x dx, \rho' \int_{C_1} \varepsilon_x dx)$ which is not pure since $\rho' > 1$.

I (B). Conical measures on R^N or R^n .

12. PROPOSITION. - Suppose $\lambda, \mu \in M^+(R^N)$ with $\lambda < \mu$, and the pair (λ, μ) is pure. Then, there exist :

1° a K_σ of $(\mathbb{R}^N \setminus 0)$, let X such that each half-line issued from 0 intersects X into at most one point,

2° a Radon measure Λ on X ,

3° a Borel application $x \mapsto \mu_x$ defined on X where μ_x is a Radon measure on X such that $r(\mu_x) = x$.

And we have :

(a) Λ is a localization of λ (Note that Λ is unique when X is given).

(b) $\mu = \int_X \mu_x d\Lambda(x)$.

Proof (with the notations of the proof of § 6). - We had

$$\pi_s = \sum_1^n \varepsilon_{(\varepsilon_{r(\lambda_i)}, \mu_i)}.$$

For each $n \in \mathbb{N}$, let x_n be the function n -th coordinate on $\mathbb{R}^N$. We have

$$\pi_s((|x_p|, |x_p|)) \leq \lambda(|x_p|) + \mu(|x_p|) \leq 2\mu(|x_p|).$$

Let ℓ be the affine l. s. c. function defined on $M^+ \times M^+$ by

$$\ell(\alpha, \beta) = \sum_p ((\alpha + \beta)|x_p|)/2^{p+1} \mu(|x_p|).$$

We have

$$\ell(r(\pi_s)) = \sum_p \pi_s(|x_p|, |x_p|)/2^{p+1} \mu(|x_p|) \leq \sum 1/2^p \leq 1.$$

π has a localization by a Radon measure m on a cap K of $M^+ \times M^+$, with $K = \{(\alpha, \beta); \alpha, \beta \in M^+, \text{ and } \ell(\alpha, \beta) \leq 1\}$. Moreover m can be assumed to be carried by the cone B .

Let Ψ be the l. s. c. function defined on $\mathbb{R}^N$ by $\Psi(x) = \ell(\varepsilon_x, \varepsilon_x)$. For each $n \in \mathbb{N}$, let $K_n = \{(\varepsilon_x, \alpha); (\varepsilon_x, \alpha) \in K, \text{ and } 1/(n+1) < \Psi(x) \leq 1/n\}$. Let m_n be the restriction of m to K_n . We have $m = \sum m_n$ on account of § 10, since (λ, μ) is a pure pair. Let π_n be the conical measure on $M^+ \times M^+$ localized by m_n .

Let m'_n be the Radon measure on $(n+1)K$ such that, for each continuous function f on $(n+1)K$, we have

$$m'_n(f) = \int_K \Psi(x) f(\varepsilon_x/\Psi(x), \alpha/\Psi(x)) dm_n(\varepsilon_x, \alpha),$$

then m'_n localizes π_n .

Suppose p is the projection on the first factor of the product $M^+ \times M^+$, then $p(m'_n)$ is carried by $\tilde{K} = \{\varepsilon_x; x \in \mathbb{R}^N \text{ with } \Psi(x) = 1\}$. $\tilde{K}$ is a Borel set because Ψ is l. s. c., moreover $\tilde{K}$ intersects each half-line issued from 0 in at most one point.

Suppose $x \mapsto m_x^n$ is a disintegration of m'_n with respect to p ([2], p. 58). Then each m_x^n has a resultant which is a conical measure ν_x^n on $\mathbb{R}^N$, and we have $\mu(\nu_x^n) = x$.

Now $\Lambda = \sum_n p(m'_n)$ can be seen as a Radon measure on a K_σ subset X_Λ of $\tilde{K}$. We

can write, for each $n \in \mathbb{N}$, $p(m_n^1) = u_n \Lambda$ where u_n is a Borel function on X_λ . We have $\sum_n u_n = 1$, Λ -a.e.

Recall that Λ represents λ , and that $\mu = \sum_n \int v_x^n d(p(m_n^1))$ (equality of conical measures), then we have $\mu = \int (\sum_n u_n v_x^n) d\Lambda$. Therefore $\mu_x = \sum_n u_n v_x^n$ exists as a conical measure Λ -a.e., and we have $r(\mu_x) = x$.

On account of [3] (38.8), there exists a compact subset H of R^N with $H = \prod_1^\infty (-k_n, k_n)$ where $k_n > 0$, such that μ is localizable on H by a Radon measure.

For simplification we shall use the same notation for μ , and its unique ([7], prop. 2.13) localization on the set $E(H) = \{x; x \in H, \forall k > 1, kx \notin H\}$. As μ_x is a Daniell integral on $h(R^N)$ ([3] 38.13), and since

$$E(H) = \{x; \text{lub}(|x_n|/k_n) = 1\},$$

then ([9] prop. II.7.1) μ_x can be extended to a σ -additive measure, called also μ_x for simplification, on the tribe $\mathcal{C}$ of $E(H)$ generated by the closed half-spaces containing 0. Recall we know that, for each $f \in h(R^N)$, the map $x \mapsto \mu_x(f)$ is Borel-measurable. Then, for each $e \in \mathcal{C}$, we have $\mu(e) = \int \mu_x(e) d\Lambda$. Let X_μ be a K_σ subset of $E(H)$ which bears μ . In order to show that μ_x lives on X_μ for Λ -a.e. x , it is sufficient to prove the following lemma.

13. LEMMA. - Each compact subset A of $E(H)$ is a member of $\mathcal{C}$.

Proof. - Let us suppose the sequence $(\omega_n)_{n \in \mathbb{N}}$ is a basis of open subsets of R^N . Let Σ be the subset of $\mathbb{N}$ such that $n \in \Sigma$ if, and only if, there exists $h \in h^+(E)$, with $h = 0$ on A , and $h > 0$ on ω_n . For each $n \in \Sigma$, we choose $h_n \in h^+(E)$, with $h_n = 0$ on A , and $h_n > 0$ on ω_n . Let us show that, for each $x \notin R^+ A$, we have $h_n(x) > 0$ for at least one $n \in \Sigma$. For each $y \in A$, there exists $h_y \in E'$ with $h_y(x) > 0$, and $h_y(y) < 0$. By compactity, there exists $h_x \in h^+(E)$, with $h_x(x) > 0$, and $h_x = 0$ on A . As the set $\{z; h_x(z) > 0\}$ is open, then there exists $n \in \mathbb{N}$ such that $x \in \omega_n$ and $h_x > 0$ on ω_n . Therefore, we have $n \in \Sigma$ and $h_n(x) > 0$. Now, if we let $h = \text{lub}_{n \in \Sigma} (h_n)$, then we have $h = 0$ on A , and $h(z) > 0$ for each $z \notin R^+ A$. Hence $A \in \mathcal{C}$.

Now, it is easy to complete the proof of § 12 by a mixture of X_λ and X_μ .

14. Remark (M. F. SAINTE BEUVE [11], theorem 3). - In the case of R^n , we can take the unit sphere of R^n (for the usual distance) for X .

15. Example (Answer to a question of G. CHOQUET). - Let $\mathcal{M}$ be the set of Radon measures on $[0, 1]$, and $\mathcal{M}_1^+$ the subset of probability measures.

Let E the vector subspace of $\mathcal{M}$ generated by the Dirac probabilities, E is equipped with the weak*-topology.

Suppose μ is the maximal measure on $\mathcal{M}_1^+$ which represents the element $dx \in \mathcal{M}_1^+$.

The measure μ and dx induce, in a canonical way, elements of $M^+(E)$, $\tilde{\mu}$ and ε_{dx} , since $E \cap M_1^+$ is dense in M_1^+ .

Let φ be the canonical injection from $[0, 1]$ into M , and $X = \varphi([0, 1])$. We have $\varepsilon_{dx} < \tilde{\mu}$ (in fact, $\varepsilon_{dx} = \varepsilon_r(\tilde{\mu})$ in the weak completion of E), however $\tilde{\mu}$ has a localization on the compact subset X of E , while ε_{dx} does not have such a localization.

Part II : Extension of a result of STRASSEN and "theory of balayage".

II (A). Extension of a result of STRASSEN.

16. Notations and definitions. - Suppose X and Y are two compacts (HAUSDORFF) spaces and $x \mapsto M_x$ is a mapping of X in the set of closed convex subsets of $M^+(Y)$ (positive Radon measures on Y).

For each $f \in C(Y)$ (continuous real functions on Y), we let

$$\forall x \in X, \hat{f}(x) = \text{lub}_{\nu \in M_x} (\nu(f)),$$

we have $\hat{f}(x) \in \bar{R}$, and $(\hat{f}(x) = -\infty) \iff (M_x = \emptyset)$.

The map $f \mapsto \hat{f}$ has been previously considered by P.-A. MEYER ([8], p. 301).

Suppose $\lambda \in M^+(X)$. For each function φ on X with values in $\bar{R}$, we let

$$\lambda^*(\varphi) = \text{glb}(\lambda(u); u \geq \varphi, u \text{ l. s. c. on } X, \text{ with values in } R \cup (+\infty)).$$

We have $\lambda^*(\varphi) \in \bar{R}$.

If $\lambda \in M^+(X)$ and $\mu \in M^+(Y)$, we write $\lambda < \mu$ if, and only if, for each $f \in C(Y)$, we have $\mu(f) \leq \lambda^*(\hat{f})$. We let $p_\lambda(f) = \lambda^*(\hat{f})$.

17. PROPOSITION. - Suppose $\lambda \in M^+(X)$ and $\mu \in M^+(Y)$ with $\lambda < \mu$. For each sequence $\lambda_1, \dots, \lambda_n$, such that $\lambda = \lambda_1 + \dots + \lambda_n$ with $\lambda_i \geq 0$, there exists a sequence $\mu_1, \dots, \mu_n$ with $\mu_i \geq 0$, such that $\mu = \mu_1 + \dots + \mu_n$, and $\lambda_i < \mu_i$ for $i = 1, 2, \dots, n$.

Proof. - Let 1 be the constant function equal to 1 on Y .

We have $\lambda^*(\widehat{-1}) \geq \mu(-1) > -\infty$. Hence, for each $f \in C(Y)$, we have

$$\lambda_i(\hat{f}) \in R \cup (+\infty) \text{ for } i = 1, 2, \dots, n,$$

then we can use the same proof than in proposition 5.

18. PROPOSITION. - We let $H = \{(\varepsilon_x, \nu); x \in X, \nu \in M_x\}$. If $\lambda \in M^+(X)$ and $\mu \in M^+(Y)$, the two following properties are equivalent

1° $(\lambda, \mu) \in \text{conv}(R^+ H)$ in $M^+(X).M^+(Y)$ equipped with the weak-* topology.

2° For each $f \in C(Y)$, we have

$$\mu(f) \leq \text{glb}(\lambda(g); g \in C(X) \text{ and } \hat{f} \leq g)$$

Proof. - We apply the theorem of Hahn-Banach.

Suppose $g \in \mathcal{C}(X)$ and $f \in \mathcal{C}(Y)$. Then $(g, -f)$ is in the polar of H if, and only if, $\hat{f} \leq g$.

$1^\circ \implies 2^\circ$: If $f \in \mathcal{C}(Y)$, we have $\lambda(g) \geq \mu(f)$ for each $g \in \mathcal{C}(X)$ with $\hat{f} \leq g$, hence 2° is fulfilled.

$2^\circ \implies 1^\circ$: For each $g \in \mathcal{C}(X)$ and each $f \in \mathcal{C}(Y)$ with $\hat{f} \leq g$, we have, on account of 2° , $\mu(f) \leq \lambda(g)$. Hence 1° is fulfilled on account of the bipolar theorem.

19. Definition of the relation $\ll$. - If $\lambda \in \mathcal{M}^+(X)$, proposition 18 invites us to let, for each $f \in \mathcal{C}(Y)$

$$q_\lambda(f) = \text{glb}(\lambda(g) ; g \in \mathcal{C}(X) \text{ and } g \geq \hat{f}).$$

Note we have $p_\lambda \leq q_\lambda$. Moreover, if H is a closed subset of $\mathcal{M}^+(X) \times \mathcal{M}^+(Y)$, then we have $p_\lambda(-1(y)) = q_\lambda(-1(y))$ because $-\hat{1}(y)$ is negative, and u. s. c.

If $\mu \in \mathcal{M}^+(Y)$, we write $\lambda \ll \mu$ if, and only if, $\mu < q_\lambda$. We have

$$(\lambda < \mu) \implies (\lambda \ll \mu).$$

Of course, we can prove the analogous of proposition 17 for the relation $\ll$. Note, in the case, study by P.-A. MEYER ([8] p. 302) (i. e. H is compact), $\hat{f}$ is u. s. c. so that $\hat{f} = \text{glb}(g ; g \in \mathcal{C}(X), g \geq \hat{f})$. Hence $p_\lambda = q_\lambda$.

20. PROPOSITION. - Suppose $\bar{K}$ is the closure, in $\mathcal{M}^+(X) \times \mathcal{M}^+(Y)$, equipped with the weak-* topology, of the set

$$K = \{(\varepsilon_x / (1 + v_x(1)), v_x / (1 + v_x(1))) ; x \in X, v_x \in M_x\}$$

If $\lambda \in \mathcal{M}^+(X)$ and $\mu \in \mathcal{M}^+(Y)$, the two following properties are equivalent :

$1^\circ \lambda \ll \mu$,

2° There exists a positive Radon measure π on the compact set $\bar{K}$ such that $r(\pi) = (\lambda, \mu)$.

Proof.

$1^\circ \implies 2^\circ$: Each element u of $\text{conv}(R^+ H)$ can be written $u = \sum_{x \in X} k_x^u (\varepsilon_x, v_x^u)$ where the k_x^u are unique, positive, and equal to 0 except for a finite number of $x \in X$. We have $v_x^u \in M_x$.

On account of § 18, there exists an ultrafilter $\mathcal{U}$ on $\text{conv}(R^+ H)$ such that $\lim_{\mathcal{U}}(u) = (\lambda, \mu)$.

u is the resultant of the following conical measure π_u on $\mathcal{M}^+(X) \times \mathcal{M}^+(Y)$ with $\pi_u = \sum_{x \in X} a_x^u \varepsilon((b_x^u, c_x^u))$ where

$$a_x^u = (1 + v_x^u(1))k_x^u, \quad b_x^u = \varepsilon_x / (1 + v_x^u(1))$$

and

$$c_x^u = v_x^u / (1 + v_x^u(1)).$$

π_u can be also seen as a positive Radon measure on $\bar{K}$.

We have $\lim_u (\pi_u(1)) = \lambda(1) + \mu(1)$. Hence $\lim_u (\pi_u)$ exists as a positive Radon measure π on $\bar{K}$ and $r(\pi) = (\lambda, \mu)$.

2° $\implies$ 1°: Each discrete positive Radon measure on K can be written

$$m = \sum_{p \in K} a_p^m \varepsilon((b_p^m, c_p^m)) \quad \text{where } (b_p^m, c_p^m) \in K, \quad a_p^m \geq 0 \quad \text{and} \quad a_p^m = 0,$$

except for a finite number of $p \in K$.

There exists an ultrafilter u on the discrete positive measures on K such that $\lim_u (m) = \pi$.

If $g \in C(X)$ and $f \in C(Y)$ with $g \geq \hat{f}$, we have

$$\lambda(g) = \lim_u \left(\sum_{p \in K} a_p^m b_p^m(1) g(b_p^m/b_p^m(1)) \right)$$

and

$$\mu(f) = \lim_u \left(\sum_{p \in K} a_p^m c_p^m(f) \right).$$

As $g \geq \hat{f}$, we have $b_p^m(1) g(b_p^m/b_p^m(1)) \geq c_p^m(f)$, hence $\lambda(g) \geq \mu(f)$.

21. PROPOSITION (Extension of a result of STRASSEN [8], p. 302). - Suppose moreover that X and Y are metrizable, and that H is a closed subset of $\mathcal{M}^+(X) \times \mathcal{M}^+(Y)$ equipped with the weak-* topology. If $\lambda \in \mathcal{M}^+(X)$ and $\mu \in \mathcal{M}^+(Y)$ with $\lambda \ll \mu$, then, there exists a Borel mapping $x \mapsto v_x$ defined on X such that $v_x \in M_x$ λ -a.e., and $\mu \geq \int v_x d\lambda(x)$, and $0 \ll \mu - \int v_x d\lambda(x)$.

Proof. - Note that $\{x; M_x = \emptyset\}$ is a G_δ λ -null subset of X since $\lambda(-\hat{1}(y)) \geq \mu(-1(y)) > -\infty$. We shall use the notations of the proof of § 20. Suppose v is the projection of $\mathcal{M}^+(X) \times \mathcal{M}^+(Y)$ on $\mathcal{M}^+(X)$. We have $v(\pi) = \lambda$ as conical measures on $\mathcal{M}^+(X)$. Let $A_0 = \{(0, \beta); \beta \in \mathcal{M}_1^+(Y)\}$, and π_0 the part of π carried by A_0 .

Let $\pi' = \pi - \pi_0$.

Suppose $\pi' = \pi'_1 + \dots + \pi'_n + \dots$ is a decomposition of π' such that, for each n , π'_n lives on $A_n = \{(\alpha, \beta); \alpha \in \mathcal{M}^+(X), \beta \in \mathcal{M}^+(Y) \text{ and } \alpha(1) \geq 1/n\}$.

Let π''_n the Radon measure on $(1, n)\bar{K}$ such that, for each $f \in C((1, n)\bar{K})$, we have

$$\pi''_n(f) = \int \alpha(1) f(\alpha/\alpha(1), \beta/\alpha(1)) d\pi'_n(\alpha, \beta).$$

π'_n and π''_n induce the same conical measure on $\mathcal{M}^+(X) \times \mathcal{M}^+(Y)$. Then $v(\pi''_n)$ is carried by $\{\varepsilon_x, x \in X\}$; the image of λ by the map $x \mapsto \varepsilon_x$ of X into $\mathcal{M}^+(X)$ is $\sum_n v(\pi''_n)$. If $\varepsilon_x \mapsto v_x^n$ is a disintegration ([2], p. 58) of π''_n with respect to v , then, for each n , we have $v(\pi''_n)$ -a.e., that v_x^n lives on the set $\{(\varepsilon_x, v); v \in M_x \text{ and } v(1) \leq n\}$, let $v_x^n \in M_x$ such that

$$r(\varepsilon_{\varepsilon_w} \otimes \pi_x^n) = (\varepsilon_x, v_x^n).$$

For each $x \in X$, we identify x and ε_x . Let $\mu_0 \in \mathcal{M}^+(Y)$ be such that $(0, \mu_0) = r(\pi_0)$. We have $\lambda = \sum_n v(\pi''_n)$ and $\mu - \mu_0 = \sum_n \int v_x^n dv(\pi''_n)$.

If we let $\nu_x = \sum_n \nu_x^n (d\nu(\pi_n)/d\lambda)$, then, we have $\nu_x \in M_x$, λ -a.e. and $\mu - \mu_0 = \int \nu_x d\lambda(x)$. As π_0 is carried by A_0 , we have $0 \ll \mu_0$.

22. Remark. - Strictly speaking, in [8] (chap. 11), Strassen theorem is T51 which admits T52 as a consequence, but T51 can be also derived from T52. We sketch a proof, with the notations of [8]. Suppose E'_1 is the unit ball of E' equipped with the weak*-topology. For each $\omega \in \Omega$, let P_ω be the set

$$\{y; y \in E', y \leq p_\omega\}.$$

We suppose $P_\omega \subset E'_1$. Let $M_\omega = \{\nu; \nu \in \mathcal{M}_1^+(E'_1), r(\nu) \in P_\omega\}$. Now, suppose (x_n) is a sequence of E everywhere norm-dense in the unit ball E_1 of E . Let φ be the map $\Omega \rightarrow [-1, 1]^N$ such that $(\varphi(\omega))_n = p_\omega(x_n)$. We let $X = \overline{\varphi(\Omega)}$ and $\Lambda = \varphi(\lambda)$, which is a regular Borel measure on X ([9] prop. II7.2). For each $t = (t_n)$ in X , because of [8] (p. 300 footnote), there exists a sublinear form p_t on E such that $p_t(x_n) = t_n$, for each n , and $p_t(E_1) \in [-1, 1]$. Then the definition of P_t and M_t (given for $t = p_\omega$) are meaningful, and the set $\{(t, \nu); t \in X, \nu \in M_t\}$ is a compact subset of $X \times \mathcal{M}_1^+(E'_1)$. Now it is sufficient to apply T52 to X and measure Λ , with $Y = E'_1$ using the map $X \rightarrow \mathcal{P}(\mathcal{M}_1^+(Y))$ defined by $t \mapsto M_t$, and taking for μ an extension of x' to $\mathcal{C}(Y)$ such that, for each $f \in \mathcal{C}(Y)$, $\mu(f) \leq \Lambda(\hat{f})$, then T51 follows since in X , $\varphi(\Omega)$ is of Λ -outer measure equal to $\Lambda(1)$.

II (B). Theory of balayage.

23. Notations. - Suppose X is a compact (HAUSDORFF) space, Γ a convex subcone of $\mathcal{C}(X)$ which is an inf-lattice (i. e. if $f, g \in \Gamma$, then $\inf(f, g) \in \Gamma$), and Γ^0 is the polar of Γ in $\mathcal{M}(X)$.

Using the previous notations, we take $Y = X$,

$$M_x = (\varepsilon_x - \Gamma^0) \cap \mathcal{M}^+(X) = \{\mu; \mu \in \mathcal{M}^+(X) \text{ and } \mu|_\Gamma \leq \varepsilon_x|_\Gamma\}.$$

Note that we do not suppose as in [8] (p. 294-297) that Γ contains a strictly positive function.

24. Definition (of f_Γ and r_λ). - For each $f \in \mathcal{C}(X)$, we let

$$f_\Gamma = \inf\{g; g \in \Gamma \text{ and } g \geq f\}$$

and for each $\lambda \in \mathcal{M}^+(X)$, we let $r_\lambda(f) = \lambda(f_\Gamma)$. r_λ is a sublinear functional on $\mathcal{C}(X)$, with values in $\mathbb{R} \cup (+\infty)$, and we have $p_\lambda \leq q_\lambda \leq r_\lambda$.

25. PROPOSITION (Extension of a balayage formula of MOKOBODZKI [8] chap. 11 T45):
For each $f \in \mathcal{C}(X)$ with $f < 0$, we have $f_\Gamma = \hat{f}$.
Moreover the following properties are equivalent :

1° There is no element > 0 in Γ ,

2° $\hat{1} \equiv +\infty$ everywhere on X ,

3° $\hat{1}$ is equal to infinity in at least one point of X ,

4° $\hat{1}$ is unbounded on X .

Proof. - Let us prove that $f_\Gamma = \hat{f}$ for each $f < 0$ of $\mathcal{C}(X)$.

If $\lambda \in \mathcal{M}^+(X)$, because of the theorem of Hahn-Banach recalled in § 1, for each $k \in]-r_\lambda(-f), r_\lambda(f)]$, there exists $\mu_k \in \mathcal{M}^+(X)$ with $\mu_k(f) = k$ and $\mu_k \leq r_\lambda$. It suffices now to take $\lambda = \varepsilon_x$ and $k = f_\Gamma(x) = r_{\varepsilon_x}(f)$. Now $1^\circ \implies 2^\circ$ can be proved in the same way, and we see that $4^\circ \implies 1^\circ$.

26. PROPOSITION.

(a) Suppose f is an u. s. c. function < 0 on X . We have $f_\Gamma = \hat{f}$ (the definition of $\hat{f}$ is as in § 16 and that of f_Γ as in § 24).

(b) If (f_i) is a family of u. s. c. functions < 0 on X , directed downward, having a limit f , we have $(f_i) \rightarrow f_\Gamma$.

Proof.

(a) can be proved as in [7] (prop. 5.6) because it is enough to work, for each $x \in X$, on a compact subset of M_x .

(b) can be proved as in [7] (prop. 5.6).

Proposition 25 enables us to give a balayage proof of the following result of CHOQUET-DENY [4].

27. PROPOSITION. - Suppose Γ is a closed convex subcone of $\mathcal{C}^-(X)$ which is an inf-lattice and contains -1 . If we let

$$\hat{\Gamma} = \{f; f \in \mathcal{C}^-(X) \text{ with } m(f) \leq f(x), \\ \forall x \in X, \forall m \in \mathcal{M}^+(X) \text{ with } m|_\Gamma \leq \varepsilon_x|_\Gamma\},$$

then we have $\Gamma = \hat{\Gamma}$.

Proof. - $\hat{\Gamma}$ is a closed convex subcone of $\mathcal{C}^-(X)$ which is an inf-lattice and $\Gamma \subset \hat{\Gamma}$. For each $f \in \mathcal{C}(X)$ such that $f < 0$, we have, because of 25, $f_\Gamma(x) = \text{lub}_{\nu \in M_x}(\nu(f))$, and we see that $f_{\hat{\Gamma}}(x) = \text{lub}_{\nu \in M_x}(\nu(f))$, hence $f_\Gamma = f_{\hat{\Gamma}}$. Therefore, by Dini lemma, we have $f = f_\Gamma$ if, and only if, $f \in \Gamma$ and $f = f_{\hat{\Gamma}}$ if, and only if, $f \in \hat{\Gamma}$, hence $\Gamma \cap \{f < 0\} = \hat{\Gamma} \cap \{f < 0\}$. Then $\Gamma = \hat{\Gamma}$, since Γ and $\hat{\Gamma}$ are the closure of $\Gamma \cap \{f < 0\}$ and $\hat{\Gamma} \cap \{f < 0\}$.

28. Remark. - Suppose Γ is separating. Then we can apply to Γ the theorem 48 of [8] (chap. 11) about the Silov compacts. It is enough to apply [8] (chap. 11, th. 48) to the cone $\Gamma_1 = \{f; f = g + a \text{ with } g \in \Gamma \text{ and } a \geq 0\}$ which is an inf-lattice.

REFERENCES

- [1] BOURBAKI (N.). - Espaces vectoriels topologiques. Chap. 1 et 2. - Paris, Hermann, 1953 (Act. scient. et ind., 1189 ; Bourbaki, 15).
- [2] BOURBAKI (N.). - Intégration. Chap. 6. - Paris, Hermann, 1959 (Act. scient. et ind., 1281 ; Bourbaki, 25).
- [3] CHOQUET (G.). - Lectures on analysis. 3 volumes. - New York, W. A. Benjamin, 1969 (Mathematics Lecture Note Series).
- [4] CHOQUET (G.) and DENY (J.). - Ensembles semi-réticulés et ensembles réticulés de fonctions continues, J. Math. pures et appl., 9e série, t. 36, 1957, p. 179-189.
- [5] DINGES (H.). - Decomposition in ordered semi-groups, J. funct. Analysis, t. 5, 1970, p. 436-483.
- [6] GOULLET de RUGY (A.). - Géométrie des simplexes. - Paris, C.D.U. et S.E.D.E.S., 1968.
- [7] GOULLET de RUGY (A.). - La théorie des cônes biréticulés, Ann. Inst. Fourier, Grenoble, t. 21, 1971, fasc. 4, p. 1-64.
- [8] MEYER (P.-A.). - Probabilités et potentiel. - Paris, Hermann, 1966 (Act. scient. et ind., 1318 ; Publ. Inst. Math. Univ. Strasbourg, 14).
- [9] NEVEU (J.). - Bases mathématiques du calcul des probabilités. - Paris, Masson, 1964.
- [10] PHELPS (R. R.). - Lectures on Choquet's theorem. - Princeton, D. Van Nostrand Company, 1966 (Van Nostrand mathematical Studies, 7).
- [11] SAINTE-BEUVE (M. F.). - Sur une relation d'ordre entre mesures positives sur S^n , Séminaire d'Analyse convexe, Montpellier, 1976, n° 4.

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