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GINZBURG-LANDAU VORTICES: THE STATIC MODEL

by **Tristan RIVIÈRE**

1. INTRODUCTION

1.1. Physical origin of the problem

One of the first explanatory models for superconductivity (which refers to the existence of permanent currents in certain substances, with no energy dissipation) has been proposed during the fifties by V. Ginzburg and L. Landau, from the Landau theory of phase transitions. Following this model, the degree of superconductivity of a body occupying a domain Ω of $\mathbb{R}^3$, is characterized by a "wave function" $\Omega \rightarrow \mathbb{C}$ referred to as the order parameter. In the quantum theory of J. Bardeen, L.N. Cooper and J. Schrieffer (BCS theory), which came in 1957 to justify the Ginzburg-Landau phenomenological model, the square of the modulus of this order parameter $|u|^2$ represents the local electron pair (Cooper pairs) density, responsible for the superconductivity. For $|u| = 1$ this density is maximum and minimum for $|u| = 0$.

The energy functional for a superconductor proposed by Ginzburg and Landau is

$$\mathcal{J}(u, A) = \int_\Omega |\kappa^{-1} du - iAu|^2 + \frac{1}{2}|1 - |u|^2|^2 + |dA|^2 - 2\int_\Omega dA.h_e$$

where A is the 1-form vector potential associated to the induced field dA in the superconductor ($du - iAu$ is thus a 1-form taking its values in $\mathbb{C}$). h_e is the 2-form representing the external field applied to the superconductor. This is one of the parameters of the problem together with the constant κ, known as the coupling constant, which depends on the sample considered, and which plays an essential part in the theory, as we shall see in the following. As a ratio of two lengths, $\kappa = \frac{\lambda}{\xi}$, where λ is the penetration depth of the external field h_e in the sample (see the following) and ξ is the characteristic size of a vortex (see section 2), is a dimensional constant. Note that this functional is also the Yang-Mills-Higgs action in the abelian gauge theory modeling the interaction between a classical magnetic field and a Higgs particle.

Schematically, the observed phenomena are as follows. When the applied field is zero, the superconductor is said to be in the *pure state*:

$$\begin{cases} |u| = 1 & \text{in } \Omega \\ dA = 0 & \text{in } \Omega. \end{cases}$$

The density of Cooper pairs is maximum and the induced field is zero. When the applied field is sufficiently strong (sample dependent) the superconductivity disappears:

$$\begin{cases} |u| = 0 & \text{in } \Omega \\ dA = h_e & \text{in } \Omega. \end{cases}$$

The density of Cooper pairs is then minimal and the induced field coincides with the applied field. The superconductor is in the *normal state*.

The nature of the transition from the *pure state* to the *normal state* depends on the composite one and in particular on the value of κ. One observes that for $\kappa < 1/\sqrt{2}$ (type I superconductor), this transition is sharp and happens for a certain strength of the applied field which is independent of κ. Instead, for $\kappa > 1/\sqrt{2}$ (type II superconductor), as the external field increases, to go from the *pure state* to the *normal state*, we pass through a different phase known as a *mixed state*, where more and more regions of normal state contained in tubes (vorticity filaments) around which the phase of u makes one or several circular turns, appears. When the sample is homogeneous and the external field uniform, these tubes line up in the direction of the field, to form periodic Abrikosov lattices, named after the physician who first showed their existence. It is observed that this lattice is triangular in the fundamental state. We pass from the *pure state* to the *mixed state*, for an applied field known as the "first critical field" $H_{c_1} \simeq O\left(\frac{\log \kappa}{\kappa}\right)$, and we leave the *mixed state* to go into the *normal state* for an applied field known as the "second critical field" $H_{c_2} \simeq O(\kappa)$. The phase diagram (figure 1) summarizes the observations mentioned above. For a more complete account of the physics of superconductors the reader can refer to: [dG], [SST], [Ti]...

1.2. The mathematical questions underlying superconductivity

There are numerous difficulties that arise when one wants to give a mathematically rigorous sense to the previous observations, starting from the Ginzburg-Landau model. A first reduction is to consider a 1 or 2-dimensional version of the model ($\Omega \subset \mathbb{R}^3$ and h_e then have the symmetries corresponding to those reductions: space comprised between two parallel planes or infinite cylinder, in uniform magnetic fields...). In this talk, we shall not consider the studies in 1-dimension which are however extensive and which enable very often a more refined analysis of the phase diagram ([BH1], [BH2], [Af]... for a complete presentation of these results, see [AT]). We shall consider here only the 2-dimensional case, which is the minimal dimension to observe vortices (dimension 3 and higher dimensions are treated in [Ri2], [LR], [LR2] and also in [BBM]).

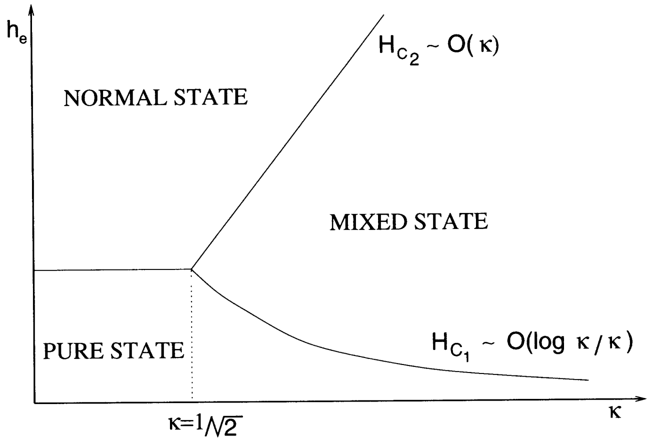

FIGURE 1. Phases diagram

Ω is thus an infinite cylinder and h_e is a uniform field parallel to the direction of the axis of the cylinder. Ω then denotes the 2-dimensional section of this cylinder and h_e being a 2-form which is constant on this section is often confused with the number giving its intensity. The aim is thus to understand the nature of fundamental states of the functional $\mathcal{J}$, and also of the critical points in general, as a function of the different values of (κ, h_e) in the phase diagram represented in figure 1. By "understanding the nature of the fundamental states of $\mathcal{J}$", we mean essentially identifying the zero set of the order parameter u of a solution minimizing $\mathcal{J}$, which corresponds to the 2-dimensional section of the vortex lattice expected in the mixed phase.

To simplify the analysis we consider the change of variables $A \longrightarrow \kappa A$ in the original model, which then leads us to the functional

$$\mathcal{G}_\kappa(u, A) = \int_\Omega |du - iAu|^2 + \frac{\kappa^2}{2}|1 - |u|^2|^2 + |dA|^2 + 2h_e \int_\Omega dA.$$

This new functional verifies the gauge invariance $\mathcal{G}_\kappa(u, A) = \mathcal{G}_\kappa(e^{i\phi}u, A + d\phi)$ for any function ϕ on Ω. It is then possible to extend the model to any domain Ω which is any 2-dimensional manifold. (u, A) are then respectively the sections and connections of a complex line bundle E on Ω on which we fix a hermitian product whose real part is $(\ ,\ )$ or $|\ |^2$ for the quadratic form. $du - iAu$ is replaced by the covariant derivative $d_A u$ of u with respect to A and dA is the curvature of the connection A. In the following, we note $h = *dA$.

Section 2 is devoted to the study of the Ginzburg-Landau free energy $\mathcal{F}$ without interaction with the external field (i.e $h_e = 0$). In 2.1, we present the work of Jaffe and Taubes on the integrable or non-interacting case $\kappa = 1/\sqrt{2}$. We state their conjectures for the cases $\kappa < 1/\sqrt{2}$ and $\kappa > 1/\sqrt{2}$. In 2.2, we describe the BBH asymptotic analysis (F. Bethuel, H. Brezis and F. Hélein [BBH]) in the London limit: $\kappa \to +\infty$ which corresponds to the strongly repulsive case. In this limit, to which we shall restrict in the following, the vorticity phenomena appear more clearly; this case is also close to many of the type II superconductors we have in practice, for which the a-dimensional parameter κ is very large. In section 2.3, we revisit the part played by the renormalized energy $\mathcal{W}$ coming from the BBH asymptotic analysis used to describe the critical points of $\mathcal{F}$. Finally, we give answers to the conjectures of Jaffe and Taubes in the London limit and we extend them to more general cases. The third part is devoted to the complete study of the functional $\mathcal{G}$ comprising the interaction term with the external field. The vorticity is then no longer a fixed parameter as in the previous section but becomes a variable of the problem. The contents of this section covers part of S. Serfaty's PhD thesis, and the work she did in collaboration with E. Sandier.

2. STUDY OF THE FREE ENERGY FUNCTIONAL $\mathcal{F}$

2.1. The integrable or non-interacting case $\kappa = 1/\sqrt{2}$

In [JT], A. Jaffe and C. Taubes study the critical points on $\mathbb{R}^2$ of the free energy functional

$$\mathcal{F}_\kappa(u, A) = \int_{\mathbb{R}^2} |d_A u|^2 + \frac{\kappa^2}{2}|1 - |u|^2|^2 + |dA|^2$$

and which are solutions to the Euler equations

$$(1) \qquad \begin{cases} d_A^* d_A u = \kappa^2 u(1 - |u|^2) \\ \\ d^* dA = (iu, d_A u) \end{cases}$$

where d_A^* is the operator acting on the 1-forms η such as $d_A^*\eta = d^*\eta + iA\wedge\eta$. Supposing that the intrinsic quantities $|d_A u|$, $|1 - |u||$ and dA are decreasing (polynomially), the renormalized magnetic field is an integer N

$$N = \frac{1}{2\pi}\int_{\mathbb{R}^2} dA$$

which corresponds to the degree of $u/|u|$ on circles of sufficiently large radii. This is known as the homotopy class of the couple (u, A). For a given N, say $N \geq 0$, it has been observed by E.B. Bogomol'nyi [Bog] that, for the particular value $\kappa = 1/\sqrt{2}$, the functional $\mathcal{F}$ can be rewritten under the following form

$$\mathcal{F}_\kappa(u, A) \int_{\mathbb{R}^2} \quad |\Re(d_{A_1} u) - \Im(d_{A_2} u)|^2 + |\Re(d_{A_2} u) - \Im(d_{A_1} u)|^2$$

$$+ | * dA + \tfrac{1}{2}(|u|^2 - 1)|^2 + 2\pi N$$

($\Re$ denotes the real part and $\Im$ the imaginary part of a complex number). One of the principal results of [JT] is then the following:

THEOREM 2.1 ([JT]). — *Any critical point* (u, A) *of* $\mathcal{F}_{1/\sqrt{2}}$ *of finite energy has a defined homotopy class* N *and verifies*

$$\mathcal{F}_{1/\sqrt{2}}(u, A) = 2\pi\ |N|. \tag{2}$$

In particular, it minimizes $\mathcal{F}_{1/\sqrt{2}}$ *in its homotopy class.*

The proof of this result can be understood as follows. Consider a critical point (u, A) of $\mathcal{F}_k$. From the Euler equations (1), we can deduce the following elliptic equations verified by the intrinsic quantities $1 - |u|^2$ and $h = *dA$

$$\left\{\begin{array}{l} -\Delta\dfrac{(1-|u|^2)}{2} + 2\kappa^2|u|^2\dfrac{(1-|u|^2)}{2} = |d_A u|^2 \\[2ex] -\Delta h + |u|^2 h = (d_A u; i * d_A u) \end{array}\right. \tag{3}$$

where $-\Delta = d^*d$ and

$$(d_A u; i * d_A u) = -<\Re(d_A u), \Im(*d_A u)> + <\Im(d_A u), \Re(*d_A u)>$$

($<\ ,\ >$ is the scalar product on the 1-forms). Using (3) we get from the maximum principle, on the one hand that $|u| < 1$ (unless $|u| \equiv 1$ on $\mathbb{R}^2$), and on the other hand that the intrinsic quantities $|h|$, $|d_A u|$ and $|1 - |u||$ decrease exponentially fast to infinity, so that one has in particular a well defined homotopy class for (u, A). These exponential decreases have an important physical interpretation linked to the mass of the Higgs particle.

Another important ingredient to prove theorem 2.1 is the conservation law of the energy-momentum tensor, which we shall use extensively throughout this talk. The energy-momentum tensor $(T_{ij})_{i,j\in\{1,2\}}$ is given by

$$T_{i,j} = 2\delta_{ij}|dA|^2 + 2(d_{A_i}u, d_{A_j}u) - \delta_{ij} f_\kappa(u, A), \tag{4}$$

where $f_k(u, A)$ is the free energy density $f_\kappa(u, A) = |d_A u|^2 + \frac{\kappa^2}{2}|1 - |u|^2|^2 + |dA|^2$. The conservation law of this tensor results from the fact that (u, A) is on the one hand a critical point of $\mathcal{F}_k$, and on the other hand C^∞, which can be deduced from the Euler equations (1); it is thus also a critical point for variations of the domain. Using Noether theorem, the translational invariance of the domain then gives rise to divergence free quantities, which constitute this conservation law

$$\forall j = 1, 2 \qquad \frac{\partial}{\partial x_i} T_{ij} = 0. \tag{5}$$

From this law, we can deduce the following Pohozaev identity, which is a consequence of the fact that (u, A) is a critical point of the infinitesimal action of dilations $r\frac{\partial}{\partial r}$ (i.e we multiply (5) by x_j, sum on j and integrate over $\mathbb{R}^2$)

$$\int_{\mathbb{R}^2} \frac{\kappa^2}{2}|1-|u|^2|^2 = \int_{\mathbb{R}^2} |dA|^2. \tag{6}$$

Let us restrict ourselves to the case $\kappa = 1/\sqrt{2}$. One can easily see that for that particular value of the parameter, by summing or subtracting the two equations of (3), the maximum principle leads to

$$|dA| = |h| \leq \frac{|1-|u|^2|}{2}. \tag{7}$$

Moreover, for $\kappa = 1/\sqrt{2}$, the Pohozaev identity becomes

$$\int_{\mathbb{R}^2} \left(\frac{1-|u|^2}{2} - *dA\right)\left(\frac{1-|u|^2}{2} + *dA\right) = 0. \tag{8}$$

Excluding the simple case where $|u| \equiv 1$ ($N = 0$) and thus $dA \equiv 0$ from (6), we have $|u| < 1$ and combining (7) and (8) gives either $*dA = \frac{1-|u|^2}{2}$ (for $N > 0$), or $*dA = -\frac{1-|u|^2}{2}$ (for $N < 0$). Bootstrapping for example $*dA = \frac{1-|u|^2}{2}$ in the equations, one easily obtains that $\Re(d_{A_1}u) = \Im(d_{A_2}u)$ and that $\Re(d_{A_2}u) = \Im(d_{A_1}u)$, which together with the observation of Bogomol'nyi proves the theorem.

Theorem 1 tells us that solving the solutions of the second order equations, comes up in the case $\kappa = 1/\sqrt{2}$ and for the class of configurations (u, A) of finite energy, to the study of the solutions of the first order equations (for $N > 0$)

$$*dA = \frac{1-|u|^2}{2} \tag{9}$$

$$\Re(d_{A_2}u) = \Im(d_{A_1}u) \tag{10}$$

$$\Re(d_{A_1}u) = \Im(d_{A_2}u), \tag{11}$$

which justifies the fact that this case is called integrable. The qualitative study of the solutions to these equations gives quite easily that the set of zeros of $|u|$ is constituted of a finite number of points where the index of u is strictly positive. We can thus represent it by exactly N points, not necessarily distinct from one another, $\{x_1 ... x_N\}$ each of multiplicity 1. We can then verify that $v = \log|u|^2$ is a solution of

$$-\Delta v + e^v - 1 = -4\pi \sum_{k=1}^{N} \delta_{x_k} \qquad \text{in } \mathcal{D}'(\mathbb{R}^2). \tag{12}$$

A convexity argument shows that the solution v to equation (12) is unique for any configuration of points $\{x_1 ... x_N\}$. We easily bootstrap this uniqueness of $|u|$ in equations (9), (10) and (11) to deduce the uniqueness (up to the gauge action) of the configuration (u, A). We can then go the other way round, and taking N arbitrary points in the plane, exhibit a solution to (9)...(11). We have then proven the theorem enunciated below.

THEOREM 2.2 ([JT]). — *The space of the solutions of finite energy of the abelian Yang-Mills-Higgs (1) in the plane, in the integrable case $\kappa = 1/\sqrt{2}$, is nothing but (up to the action of the gauge group) the space of configurations of points in the plane, having integer multiplicities, all of the same sign.*

One of the striking points of the integrable case $\kappa = 1/\sqrt{2}$, is that not only can the vortices (zeros of u) be anywhere in the plane, but that the energy of the solutions is independent of the relative positions of the vortices, and is equal to 2π times the number of vortices. This is why the integrable case is also known as the non-interacting case. By handwaving arguments, A. Jaffe and C. Taubes conjecture that $\kappa = 1/\sqrt{2}$ is the limiting case between two opposite behaviors of vortices amongst themselves; for $\kappa < 1/\sqrt{2}$, whatever be their signs, the vortices have a tendency to attract one another, which justifies in some way the absence of a mixed phase, while in the case $\kappa > 1/\sqrt{2}$ the vortices with same signs should repel one another, but since their presence is imposed by the energy input due to the external field (see section 3), this leads to the possibility of a mixed state (see figure 1). Precisely the conjecture of Jaffe and Taubes for the repulsive case ($\kappa > 1/\sqrt{2}$) is as follows.

CONJECTURE 2.3 ([JT]). — *For $\kappa > 1/\sqrt{2}$, there exist stable solutions of YMH (1) on all $\mathbb{R}^2$ if and only if $|N| = 1, 0$; moreover they have an axial symmetry (up to the gauge action).*

In both of the next sub-sections, we shall rigorously account for these expected behaviors of the vortices amongst themselves in the strongly repulsive case ($\kappa \to +\infty$) and give a partial answer to the conjecture 2.3 (see theorem 2.10).

2.2. The strongly repulsive case or "London limit" $\kappa \to +\infty$: the BBH asymptotic analysis

From now on, we shall study the behavior of the vortices in the strongly repulsive case ($\kappa \to +\infty$). To prevent the vortices from separating from one another to infinity, we study $\mathcal{F}$ on a compact 2-dimensional manifold M without boundary, in order not to have to take artificial boundary conditions. For clarity, we shall restrict ourselves to the case where M is a flat torus. In fact the metric on M, as well as its topology, does not modify the qualitative aspects of the results. We also take a hermitian complex line bundle on M whose Euler class $e(E)$ verifies $\int_M e(E) = N > 0$, which implies that any section u intersecting transversely the zero section, does it algebraically N times. We thus fix the total vorticity of the problem, which becomes a parameter of the problem. In section 3 of this talk, we shall take into account the influence of the external field, so that the total vorticity will be once again a variable of the problem as in the original model. The existence of a couple section-connection (u_k, A_k) minimizing $\mathcal{F}_k$ in the Sobolev spaces $W^{1,2}$ of sections and connections (the expression in a trivialization of the bundle of (u, A) gives the functions and 1-forms $W^{1,2}$) is now a classical problem which requires the use of Coulomb gauges in order to render the functional coercitive (see [Uh]). We thus propose to study the behavior of such minimizing couples (u_k, A_k) when κ tends to infinity. The difficulty of such an analysis comes from the fact that we do not dispose of estimates *a priori* sufficient, independent of κ, in any functional space, to prove any weak convergence towards something. In particular, we can verify that $\mathcal{F}_\kappa(u_k, A_\kappa) \to +\infty$ and more precisely we have

$$\mathcal{F}_\kappa(u_\kappa, A_\kappa) \simeq 2\pi N \log \kappa. \tag{13}$$

F. Bethuel, H. Brezis and F. Hélein give a complete description of this asymptotic in [BBH] for the case $A = 0$ on a domain of $\mathbb{R}^2$ (this is completed in [St], adapted to the gauge invariant model in [BR1] and later in [Qin], and followed the study of the symmetric case on $\mathbb{R}^2$ in [BC]). They then establish the following result:

THEOREM 2.4 ([BBH]). — *Given a sequence of configurations (u_κ, A_κ) minimizing $\mathcal{F}_\kappa$ for a sequence of κ tending towards infinity, there exist a sub-sequence (u'_κ, A'_κ) and N distinct points $\{p_1...p_N\}$ of M such that*

$$(u_{\kappa'}, A_{\kappa'}) \longrightarrow (u_\star, A_\star) \qquad in\ \ C^k_{loc}(\tilde{M}) \tag{14}$$

where $\tilde{M} = M \setminus \{p_1...p_N\}$, $(u_\star, A_\star)$ is a couple unitary section-connection of E over $\tilde{M}$, which is a critical point of the functional

$$\mathcal{F}_\star(u_\star, A_\star) = \int_{\tilde{M}} |d_{A_\star} u_\star|^2 + |dA_\star|^2. \tag{15}$$

Moreover the index of the singular $A_\star$-harmonic section $u_\star$ at each p_j is $+1$, which gives in particular that the limiting curvature $h_\star = *dA_\star$ verifies the "London equation"

$$d^* dh_\star + h_\star = 2\pi \sum_{k=1}^{N} \delta_{p_k} \qquad in\ \mathcal{D}'(M). \tag{16}$$

Remark 1. — The positions of the limiting vortices $p_1...p_N$ determine uniquely the couple $(u_\star, A_\star)$ (up to the gauge invariance).

We sketch the main points of the proof below—we omit the index κ.

Again, the maximum principle applied to the first equation of (3) gives $|u| \leq 1$. Combining this L^∞ bounding of $|u|$ and the bounding of the configuration energy (u, A) given by (13), by means of classical elliptic estimates (in the spirit of interpolation inequalities of the Gagliardo-Nirenberg type [BBH0]), we get the following control over the L^∞ norm of the covariant derivative of u (which is actually optimal)

$$\|d|u|\|_{L^\infty(M)} \leq \|d_A u\|_{L^\infty(M)} = O(\kappa). \tag{17}$$

The general strategy will consist in identifying and covering in the best possible way the zero set of $|u|$, which coincides with the region of loss of compactness of the sequence of configurations (u, A) in $W^{1,2}$. This region is called "bad set"; more precisely.

it refers to the following set:

$$\mathcal{M} = \{x \in M \quad \text{such that} \quad |u|(x) < 1/2\} \tag{18}$$

(the constant 1/2 is in fact chosen arbitrarily between 0 and 1). We then have the following "quantization" result.

There exists $\delta > 0$ independent of κ such that, for all x_0 in M,

$$\kappa^2 \int_{B_{\kappa^{-1}}(x_0)} |1 - |u|^2|^2 \le \delta \qquad \Longrightarrow \qquad x_0 \notin \mathcal{M}. \tag{19}$$

This result is an immediate consequence of (17). It implies that a point of $\mathcal{M}$ contributes, on a ball of radius κ^{-1}, to a "finite" part of the Higgs energy $\kappa^2 \int_{B_{\kappa^{-1}}(x_0)} |1 - |u|^2|^2$ (i.e larger than a value $\delta > 0$ independent of κ).

The joint use of the upper bounding of the energy and the conservation law of the energy-momentum tensor (5), from which we deduce the Pohozaev identity on any geodesic ball, gives in particular (cf [BR1])

$$\forall 0 < \alpha < 1 \quad \forall x \in M \qquad \kappa^2 \int_{B_{\kappa^{-\alpha}}(x_0)} |1 - |u|^2|^2 \le C_\alpha, \tag{20}$$

where C_α is independent of κ. The use of the intermediate scales $\kappa^{-\alpha}$ in [BR1] between the natural scales of the problem (1 and κ^{-1}) enables us to get rid of the magnetic field in the Pohozaev identities and to obtain (20). These scales (which are not really necessary in the case $A = 0$ in dimension 2 [BBH]) are extensively used in the gauge invariant model ([BR1], [Ri1], [Ri2]...). Combining (19) and (20), we get the following covering of $\mathcal{M}$ on a whole ball of radius $\kappa^{-\alpha}$.

$$\forall x \in M \qquad \mathcal{M} \cap B_{\kappa^{-\alpha}}(x) \subset \cup_{j=1}^{N_\alpha} B_{\kappa^{-1}}(x_j) \quad \text{and } N_\alpha = O(1) \tag{21}$$

where $(x_j)_{j\in\{1\dots N_\alpha\}}$ is a family of points in $B_{\kappa^{-\alpha}}(x)$ et N_α and N_α is bounded independently of κ.

Once again, the use of the stress-energy tensor (5) enables us to prove the quantization result below:

LEMMA 2.5 (eta-compactness). — *There exists $\eta > 0$ independent of κ such that for any radius $1 \ge \rho \ge \kappa^{-1}$ and any x in M, one has*

$$\begin{aligned} &\int_{B_\rho(x)} f_\kappa(u, A) \le \eta \ \log(\rho\kappa) \\ &\Longrightarrow B_{\rho/2}(x) \cap \mathcal{M} = \varnothing \end{aligned} \tag{22}$$

where $f_\kappa(u, A)$ is the free energy density $f(u, A) = |d_A u|^2 + \frac{\kappa^2}{2}|1 - |u|^2|^2 + |dA|^2$.

This lemma tells us that the contribution of the bad set to the total energy on a ball of radius ρ is at least greater than $\eta \ \log(\rho\kappa)$. The proof in 2 dimensions is quite straightforward (see [Ri1]), while in larger dimensions, it becomes much more technical (see [Ri2] [LR2]).

The combination of the bounding of the energy (13), (21) and the lemma of eta-compactness enables us to conclude easily that the set $\mathcal{M}$ is contained in a uniform number of bounded balls of radius κ^{-1}:

$$\mathcal{M} \subset \cup_{j=1}^{n} B_{\kappa^{-1}}(x_j) \qquad n = O(1). \tag{23}$$

We can then extract a sub-sequence of the original sequence $\kappa \to +\infty$ such that the family $(x_j)_{j=1..n}$ converges in M.

Outside $\mathcal{M}$, $h = *dA$ verifies the following elliptic equation deduced from the second Ginzburg-Landau equation (1)

$$d^* \left[\frac{1}{|u|^2} dh \right] + h = 0 \qquad \text{in } M \setminus \cup_{j=1}^{n} B_{\kappa^{-1}}(x_j). \tag{24}$$

What prevents this equation from being verified on all M is the index of u around the zero set. Then if $d_j = \deg(\frac{u}{|u|}; \partial B_{\kappa^{-1}}(x_j))$, one verifies that

$$\int_{\partial B_{\kappa^{-1}}(x_j)} \frac{dh}{|u|^2} + \int_{B_{\kappa^{-1}}(x_j)} h = 2\pi d_j. \tag{25}$$

From (24) and (25), we then have that h is a κ^{-1}-approximation of the following linear problem

$$d^* dk + k = 2\pi \sum_{j=1}^{n} \delta_{x_j}. \tag{26}$$

From (17), we can deduce that the d_j are uniformly bounded and thus that the $W^{1,p}$ norms ($p < 2$) of k are bounded independently of κ. These boundings can be transmitted easily to $h = *dA$ and by bootstrapping then in the Ginzburg-Landau equations (1), we prove theorem 2.4 (moreover to show that $n = N$, we have used the minimality of the solution).

As opposed to the non-interacting case $\kappa = 1/\sqrt{2}$, the vortices x_j and their limits $(p_1...p_N)$ cannot be anywhere in the domain M, the configuration $(p_1...p_N)$ minimizes a certain energy W of $M^N \setminus \Delta$ in $\mathbb{R}$, known as the renormalized energy (Δ denotes the diagonal of M^N).

THEOREM 2.6 ([BBH]). — *The configuration of limiting vortices $(p_1...p_N)$ given by theorem 2.4 is a fundamental state of the following function defined on $M^N \setminus \Delta$*

$$\begin{aligned} W(z_1...z_N) = \;& 2\sum_{i \neq j} \int_M dG_i \cdot dG_j + 2\sum_{j=1}^{N} \int_M dR \cdot dG_j \\ & + \int_M |dR|^2 + \int_M |k|^2. \end{aligned} \tag{27}$$

The fact that the vortices strongly repel one another can be clearly seen in W. Indeed, we can verify that if two vortices z_i and z_j come close to each other, the others remaining fix, then we have $W \simeq 2\pi \log |z_i - z_j|$.

The proof of theorem 2.6 relies on the following decomposition. We take some $\delta > 0$ independent of κ. We divide M into two disjoint parts $M = \cup_{k=1}^{N} B_\delta(p_k) \cup M_\delta$. We decompose the total energy as a sum of the energies on each ball $B_\delta(p_k)$ and on M_δ. According to theorem 2.4, $|u|$ converges uniformly towards 1 while $*dA$ converges uniformly towards $h_\star$ which is solution of (16) on M_δ. Using then the second Ginzburg-Landau equation, we get

$$\lim_{\kappa \to +\infty} \int_{M_\delta} f_\kappa(u, A) = \int_{M_\delta} |dh_\star|^2 + |h_\star|^2. \tag{28}$$

Moreover, an explicit calculation enables us to verify that

$$\lim_{\delta \to 0} \int_{M_\delta} |dh_\star|^2 + |h_\star|^2 - 2\pi N \log \frac{1}{\delta} = W(p_1 ... p_N) + C, \tag{29}$$

where C is independent of the positions of the vortices. Using a covering of $\mathcal{M}$ by a finite number of balls of radii κ^{-1}, we can show by means of the convergence given by theorem 2.4 that the principal part of the energy around a limiting vortex is independent of its position and of the existence of the other vortices

$$\int_{B_\delta(p_k)} f_\kappa(u, A) = 2\pi \log(\delta\kappa) + C_0 + o_\delta(1), \tag{30}$$

where C_0 is a universal constant. By combining (28), (29) and (30) we can easily verify that to optimize $\mathcal{F}_\kappa$, we have to optimize the configuration of vortices $(p_1 ... p_N)$ with respect to W. This proves theorem 4.

Moreover, we have shown that $\mathcal{F}_\kappa(u_\kappa, A_\kappa)$ has the following asymptotic expansion.

$$\mathcal{F}_\kappa(u_\kappa, A_\kappa) = 2\pi N \log \kappa + W(p_1 ... p_N) + C_0 N + o(1). \tag{31}$$

This asymptotic expansion can be interpreted as follows: each vortex interacts with itself with an energy with principal part $2\pi \log \kappa$, W is the interaction energy amongst vortices, and finally C_0 is the renormalized energy of an isolated particle.

If we now follow a sequence (u_κ, A_κ) of Ginzburg-Landau solutions (1) in the "London limit" ($\kappa \to +\infty$), which is now not necessarily a sequence of minima, we prove results corresponding to theorems 2.4 and 2.6, the condition being to remain in "reasonable" energy levels $\mathcal{F}_\kappa(u_\kappa, A_\kappa) = O(\log \kappa)$. The number of limiting vortices $p_1 ... p_Q$ is now not necessarily N. To each vortex is associated an integer multiplicity d_j which is the index of the limiting singular section $u_\star$ around p_j. The equation verified by the limiting field $h_\star$ becomes

$$d^* dh_\star + h_\star = 2\pi \sum_{j=1}^{Q} d_j \delta_{p_j} \qquad \text{in } \mathcal{D}'(M) \tag{32}$$

while the positions of the p_j is a critical point of the function W given by (27) where G_j is replaced by $d_j G_j$.

2.3. The space of solutions in the "London limit"

Study of the zero set in the London limit. — We saw that in the London limit ($\kappa \to +\infty$), the zero set of the minimizing section u_k is forced to converge towards a very precise locus of points of the domain: $p_1...p_N$ which is a critical point of the function W. We now want to have a more complete description of this zero set, and eventually, to see in which limit the property obtained in the non-interactive case, saying that two solutions having the same zeros are gauge equivalent, remains valid in the strongly repulsive case. This would then bring the study of the space of solutions to that of the eventual zero set and would justify the tendency in physics to call Ginzburg-Landau vortices the solutions to the Ginzburg-Landau equations themselves.

Consider a sequence of minima (u_κ, A_κ) of $\mathcal{F}_k$ converging as in theorem 2.2 towards a couple section-singular connections $(u_\star, A_\star)$. We first look for the zero set of $|u_\kappa|$— we shall omit the index κ except where necessary. The limiting section being of index 1 to p_1, the sum of the indices of u on the bad balls $B_{\kappa^{-1}}(x_i)$ converging towards p_1 is also $+1$. We can always suppose that $|u|(x_j) < 1/2$.

If ever the centers x_l and x_k of two bad balls separate faster that $O(\kappa^{-1})$, we obtain a contradiction for the following reason. Suppose that this actually occurs, then up to the extraction of a sub-sequence, we can suppose that

$$\kappa|x_k - x_l| \longrightarrow +\infty. \tag{33}$$

Consider then the dilation of the section around the point x_k given by (in a local Coulomb gauge): $\hat{u}_\kappa(z) = u(\kappa^{-1}z + x_k)$. The convergence of the field dA established in the proof of theorem 2.4 as well as in the L^∞ bound of the covariant derivative (17) enable to conclude that, up to an extraction of a sub-sequence, $\hat{u}_\kappa$ converges in $C^1_{loc}(\mathbb{R}^2)$ towards a solution of

$$\begin{cases} \Delta\hat{u} + \hat{u}(1 - |\hat{u}|^2) = 0 \qquad \text{on } \mathbb{R}^2 \\[2ex] |\hat{u}|(0) < 1 \\[2ex] \displaystyle\int_{\mathbb{R}^2} |1 - |\hat{u}|^2|^2 < +\infty. \end{cases} \tag{34}$$

It is shown in [BMR] that the Higgs energy of the solutions of (34) is quantized:

$$\int_{\mathbb{R}^2} |1 - |\hat{u}|^2|^2 \;\in\; 2\pi\mathbb{N}^*. \tag{35}$$

Thus, if we go back to the original scale, for a sufficiently large radius R, independent of κ, with κ sufficiently large, then at the limit the contribution of the ball $B_{\kappa^{-1}R}(x_k)$ to the Higgs energy is at least 2π. This also holds for x_l and thus, since both points separate faster than $O(\kappa)$, from (33) we can deduce that for all $\delta > 0$ independent of

κ

$$\liminf_{\kappa\to+\infty} \kappa^2 \int_{B_\delta(p_1)} |1-|u_\kappa|^2|^2 \geq 4\pi. \tag{36}$$

Moreover, the Pohozaev identity on the ball $B_\delta(p_1)$ deduced from the energy-momentum conservation law, combined with the convergences of theorem 2.4, easily gives

$$\lim_{\kappa\to+\infty} \kappa^2 \int_{B_\delta(p_1)} |1-|u_\kappa|^2|^2 = 2\pi + o_\delta(1). \tag{37}$$

(36) and (37) are then in contradiction, and so (33) cannot occur even for a sub-sequence. Thus, there exists $\lambda > 0$ independent of κ such that the part of the bad set $\mathcal{M}$ converging towards p_1 can be covered by exactly one ball of radius $\lambda\kappa^{-1}$. So let x_1 be the center of this ball and more generally x_i be the center of the ball converging towards p_i and containing $\mathcal{M}$ in the vicinity of p_i. We can always suppose that $u(x_i) = 0$ since the limiting index of $u_\star$ is non-zero at p_i, but equal to 1, and that u must become 0 somewhere in the part of $\mathcal{M}$ which converges towards p_i which is contained in $B_{\lambda\kappa^{-1}}(x_i)$. The dilation argument previously applied from x_1 then says that, in a local Coulomb gauge, $\hat{u}_\kappa(z) = u_\kappa(\kappa z + x_1)$ converges in $C^1_{loc}(\mathbb{R}^2)$ up to the extraction of a sub-sequence towards a solution $\hat{u}$ of the following problem.

$$\begin{cases} \Delta\hat{u} + \hat{u}(1-|\hat{u}|^2) = 0 \qquad \text{in } \mathbb{R}^2 \\[2ex] |\hat{u}(0)| = 0 \\[2ex] \displaystyle\int_{\mathbb{R}^2} |1-|\hat{u}|^2|^2 < +\infty \\[2ex] \operatorname{ind}(\hat{u}, +\infty) = +1. \end{cases} \tag{38}$$

(It is shown in [BMR] that from the first three lines of (38), one can show that $|\hat{u}|$ converges uniformly towards 1 at infinity and thus that the index $\hat{u}$, $\operatorname{ind}(\hat{u}, +\infty)$ is well-defined.) Equation (38) is known as the "profile" equation of the Ginzburg-Landau vortices. The problem of the multiplicity of the solutions of (38) is not a standard problem in non-linear elliptic equations; indeed $\hat{u}$ does take real but complex values and the classical approaches using the maximum principle to show an eventual symmetry of the solution cannot apply here.

P. Mironescu in [Mi] gave the following proof of the uniqueness (up to rotations) of the "profile".

It is quite standard to verify that there exists a unique solution of (38) of the form $\rho(r)e^{i\theta}$ ((r,θ) being the polar co-ordinates on $\mathbb{R}^2$). Let us divide $\hat{u}$, which is any solution of (38), by this axially symmetric solution. The ratio $w = \hat{u}/\rho(r)e^{i\theta}$ is a critical point on $\mathbb{R}^2 \setminus \{0\}$ of the functional

$$\mathcal{E}(w) = \int \rho^2 |\nabla w|^2 + 2\frac{\rho^2}{r}\left(iw, \frac{\partial w}{\partial \theta}\right) + \rho^4 |1 - |w|^2|^2.$$

w is the critical point for the infinitesimal action of dilations $r\frac{\partial}{\partial r}$ on $\mathbb{R}^2 \setminus \{0\}$. This then gives the Pohozaev identity

$$0 = \int_{\mathbb{R}^2} r\frac{\rho'}{\rho}\left|\frac{\partial w}{\partial r}\right|^2 + \frac{r\rho\rho' + \rho^2}{2}|1 - |w|^2|^2 \tag{39}$$

(the index 1 being used at infinity to get rid of the boundary terms). A simple study of the modulus of the radial solution ρ enables to conclude that $\rho' > 0$. The identity (39) then tells us that $|w| = 1$, $\frac{\partial w}{\partial r} = 0$ and thus $\hat{u} = \rho(r)e^{i(\theta+\alpha)}$ where α is a constant. This proves the uniqueness of the profile up to rotations.

Since $\hat{u}_\kappa$ converges (up to the extraction of a sub-sequence) in $C^1_{loc}(\mathbb{R}^2)$ towards this profile (unique up to rotations), also since $\hat{u}$ becomes zero exactly at 0 and since $\nabla^2\hat{u}(0)$ has rank 2, one can then verify that on $B_\lambda(0)$ $\hat{u}_k^{-1}(\{0\}) = \{0\}$. Thus on the ball $B_\delta(p_1)$, u is equal to zero at a unique point.

We have up to now used the fact that the couple (u_κ, A_κ) is minimal in order to get the convergence of theorem 2.4 and the index 1 of $u_\star$ at each p_j. From the extension of the results 2.4 and 2.6 to any of the critical configurations $(u_\kappa, A_{,\kappa})$ under the energy levels $O(\log(\kappa))$ described at the end of section 2.2, we have shown the following proposition.

Proposition 2.7. — *Let (u_κ, A_k) be a sequence of configurations ($\kappa \to +\infty$), solutions of the Ginzburg-Landau equation* (1), *verifying the energy bounding $\mathcal{F}_\kappa(u_\kappa, A_\kappa) = O(\log\kappa)$ and converging towards a couple section-connection $(u_\star, A_\star)$ singular at $p_1 ... p_Q$. If the index of $u_\star$ at p_j is ± 1 for all j, then there exists κ_0 such that for $\kappa > \kappa_0$, we have*

$$|u_\kappa|^{-1}(\{0\}) = \{x_1 ... x_Q\} \qquad \textit{and} \qquad x_j - p_j \to 0 \ \textit{ for all } j.$$

The role of the renormalized energy W. — Let us again place ourselves in the hypotheses of proposition 2.7. The section u_κ for κ sufficiently large thus intersects exactly Q times the zero section with each time an intersection index ± 1. Each of those zero points of $|u_k|$ converges towards a limiting vortex $p_1 ... p_Q$. Considering how important the part played by the position of the zero is in P. Mironescu's uniqueness argument, a natural question is whether, more generally, two solutions of (1) having the same zero sets in the London limit, are equal (up to the gauge invariance). This proposition would generalize to the strongly repulsive case the same proposition proved above by Jaffe and Taubes for the non-interacting case. The answer to this question is yes under the same hypotheses as in proposition 2.7.

PROPOSITION 2.8 ([PR]). — *Let* (u_κ, A_κ) *and* (v_κ, B_κ) *be two sequences of solutions of the Ginzburg-Landau equations* (1) *in the London limit* ($\kappa \to +\infty$) *verifying the common energy bounding* $\mathcal{F}_\kappa(u_\kappa, A_k) = O(\log \kappa)$, $\mathcal{F}_\kappa(v_k, B_k) = O(\log \kappa)$, *and both converging towards the same singular configuration* $(u_\star, A_\star)$. *We suppose that the index of* $u_\star$ *at each singularity is* ± 1. *Then there exists* κ_0 *such that for* $\kappa > \kappa_0$

$$|u_\kappa|^{-1}(\{0\}) = |v_\kappa|^{-1}(\{0\}) \qquad \Longrightarrow \qquad (u_\kappa, A_\kappa) \simeq (v_\kappa, B_\kappa),$$

where $\simeq$ *denotes the gauge equivalence.*

The proof of the preceding proposition relies amongst other things on the following generalization of P. Mironesu's argument. In a Coulomb gauge in the vicinity of any limiting vortex p_k, we reconsider under the hypotheses of the proposition, the ratio of the two solutions $w = \frac{u}{v}$. Instead of making dilations with respect to the conformal field $X = r\frac{\partial}{\partial r}$ centered at the common zero of u and v, which do not give any results in that case, we dilate with respect to the following field, constructed from the second solution of v,

$$Y = |v|^2 \frac{(iv, *dv)}{|(iv, *dv)|^2}$$

where we identify the field with the dual 1-form given by the scalar product. We can observe that Y coincides with the usual conformal dilation field $X = r\frac{\partial}{\partial r}$ in the case where v has a radial symmetry. The action of this field on the functional for which (w, A) is a critical point gives a Pohozaev identity in the vicinity of p_k, which all put together enable us to conclude that for sufficiently large κ $(u_\kappa, A_\kappa) \simeq (v_\kappa, B_\kappa)$.

The use of Y rather that the usual field $X = r\frac{\partial}{\partial r}$ is *a posteriori* natural in view of the functional for which (w, A) is a critical point and of the property of $|v|$-conformality of Y (see [PR]).

The proposition 2.8 is the first step to describe the space of solutions below the $O(\log \kappa)$ energy levels in the London limit. The second step consists in constructing a sequence (v_κ, B_κ) converging towards a $(u_\star, A_\star)$ whose configuration of associated vortices $p_1 ... p_Q$ is any critical point of W (non-degenerate up to the actions of isometries of M) and whose multiplicities d_j are ± 1. Such a construction has been done in [LL] for $d_j = +1$ by means of variational arguments, by establishing a link between the level sets of the function W and the functional $\mathcal{F}_\kappa$. In [PR] this construction is done for $d_j = \pm 1$ by means of the local inversion theorem which also has the advantage to bring the local uniqueness. The argument of linearization around a solution is rendered complex by the existence of a "kernel at infinity": the inverse of the linearized operator of (1) around an accumulation of vortices which converges towards $(u_\star, A_\star)$, blows-up in the standard norms $W^{2,2} - L^2$ when κ tends to infinity. This is due to the action of the invariance group of the limiting profile equation (38) (here the isometries of the plane). We are thus led to develop the Lyapunov-Schmidt reduction argument which first consists in working in a direction perpendicular to this kernel at infinity, which we reintroduce in the final non-linear argument. This technique applied to the elliptic P.D.E. has often been used in a number of constructive problems of differential geometry like the existence of minimal surfaces, the existence of surfaces of constant mean curvature, the Yamabe problem, the Yang-Mills monopoles... by N. Kapouleas, R. Mazzeo, F. Pacard, R. Schoen, K. Uhlenbeck, S.T. Yau... with each time new difficulties. (v_κ, B_κ) being thus constructed benefits from a local uniqueness in an adapted functional space. If (u_κ, A_κ) is another solution of (1), converging towards the same limiting configuration whose singularities are $p_1...p_Q$, we know (proposition 2.7) that the zeros of $|u_\kappa|$ converge towards those singularities and are thus close to those of $|v_\kappa|$. The argument of the proof of proposition 2.8 is then converted in order to obtain, no longer directly that $(u_\kappa, A_\kappa) \simeq (v_\kappa, B_\kappa)$, a sufficiently small bound of the "separation" of (u_κ, A_κ) and (v_κ, B_κ) which, by means of the local uniqueness established previously, enables to conclude that $(u_\kappa, A_\kappa) \simeq (v_\kappa, B_\kappa)$. We thus have the following theorem.

THEOREM 2.9 ([PR]). — *Let $(p_1...p_Q)$ be a critical point of the renormalized energy W for the multiplicities $(d_1...d_Q)$ in $\{-1,+1\}$. Suppose that this critical point is non-degenerate up to the action of isometries of M. Let $(u_\star, A_\star)$ be the singular couple section-connection associated to (p_j, d_j). Then let (u_κ, A_κ) and (v_κ, B_κ) be two sequences of Ginzburg-Landau equations* (1) *verifying the common energy bounding $\mathcal{F}_\kappa(u_\kappa, A_\kappa) = O(\log\kappa)$ and $\mathcal{F}_\kappa(v_\kappa, B_\kappa) = O(\log\kappa)$, if their singular closure set is the configuration $(u_\star, A_\star)$, then for sufficiently large κ*

$$(u_\kappa, A_\kappa) \simeq (v_\kappa, B_\kappa)$$

up to the action of isometries of M ($\simeq$ is the gauge equivalence).

Comments. — Theorem 2.9 tells us that under certain hypotheses, the study of the critical points of the functional $\mathcal{F}_\kappa$ in the London limit—*infinite dimensional* problem—can be brought to that of the critical points of the function W—*finite dimensional* problem. The limits of our description of solutions to the Ginzburg-Landau equations in the strongly repulsive case are the following. First, in order to be in the context of the BBH asymptotic, we have restricted the critical points to energies below $C\log\kappa$. How about other eventual solutions to (1)? In the case $A = 0$ on a star-shaped domain, it has been shown that *all* the Ginzburg-Landau solutions are under a certain energy level $C\log\kappa$. The second restriction in the hypotheses of theorem 2.9 is the constraint on the limiting multiplicities d_j who have to belong to $\{+1,-1\}$. How about the branches of solutions linking a critical point of W where some of $|d_j|$'s are greater than 1? This question is completely open. Neither the variational methods, in the spirit of [LL], or [AB1], [AB2], nor critical points methods, seen above, have enabled up to now the construction of solutions (other than the axially symmetric solutions) converging towards a critical point of W with multiplicities $d_j \neq \pm 1$. We do not even have a precise idea of the aspect of such solutions near the vortices. The understanding of the possible limiting "profiles" is also lacking in that case (see conjectures on that theme in [OS]). Nevertheless, it seems that the case $d_j = \pm 1$ is "generic" in the following sense: for a "generic" perturbation of the metric, a critical point of W having multiplicity indices greater than 1 should be transformed into a critical point whose indices are ± 1.

To end up the study of the free energy, we should mention that the techniques presented above help to bring a partial answer to the Jaffe and Taubes conjectures on all $\mathbb{R}^2$ in the strongly repulsive case.

THEOREM 2.10 ([Ri3]). — *The conjecture* 2.3 *is true in the London limit (sufficiently large κ) if we replace "stable critical point" by "minimum in the homotopy class".*

3. INTERACTION WITH AN EXTERNAL MAGNETIC FIELD; TOWARDS THE ABRIKOSOV LATTICES

3.1. Introduction

Up to now, we have studied the Ginzburg-Landau free energy $\mathcal{F}_\kappa$ without the interaction term with the magnetic field $2h_e \int dA$. Moreover, we have placed ourselves on a plane torus in order to ignore in a first approach, the border effects which are however very important in phenomenological studies, and about which it would be very interesting to recover the conclusions by the mathematical study itself. Finally, in the absence of external magnetic field, in order not to work with a trivial fundamental state, we have imposed a global vorticity through the choice of a certain bundle on our torus of non zero Euler class. The analysis of our model problem has enabled us to isolate and understand the mechanism

$$\text{vorticity} \qquad \Longrightarrow \qquad \text{vortex formation}$$

and to bring the study of the space of solutions to that of the positions of the vortices, governed by the renormalized energy.

In her PhD thesis [Se1], then in her works in collaboration with E. Sandier [SS1] [SS2], S. Serfaty considers the complete functional $\mathcal{G}_\kappa$ including the interaction term with the external magnetic field, on Ω a simply connected bounded domain of $\mathbb{R}^2$. The vorticity is thus free and should spontaneously appear by increasing the external magnetic field and thus the total free energy of the system as it has been observed for type II superconductors ($\kappa > 1/\sqrt{2}$).

We still place ourselves in the London limit ($\kappa \longrightarrow +\infty$) and the magnetic field h_e is assumed to be uniform (we will confuse the 2-form and the corresponding constant).

3.2. The Meissner solution

When the applied field $h_e = 0$, the fundamental state of $\mathcal{G}_\kappa$ is clearly reached by the *pure state* solution $h = *dA = 0$. When we increase the field h_e, the superconducting character of the sample changes a little without however forming any vortex ($|u_\kappa| \geq 1/2$ for sufficiently large κ). We expect this configuration which is vortex-free, stable, and absolute minimum for not too strong external fields, to be unique. This is what S. Serfaty shows by combining energy estimates and convexity arguments.

THEOREM 3.1 ([Se3]). — *In the London limit (κ sufficiently large), there exists a unique stable, vortex free ($|u_\kappa| \geq 1/2$) critical point of $\mathcal{G}_\kappa$ which minimizes $\mathcal{G}_\kappa$ amongst the vortex-free critical points, with the condition $h_e = O(\kappa^\alpha)$ where α is a positive constant. This solution known as the "Meissner solution" verifies in particular*

$$\begin{cases} -div\left(\dfrac{\nabla h}{|u|^2}\right) + h = 0 & in\ \Omega \\ \\ h = h_e & on\ \partial\Omega. \end{cases} \tag{40}$$

The constant $1/2$ is of course a constant chosen at random between 0 and 1. In fact the Meissner solution remains stable up to a critical value of the applied field $h_e = H_{sh} \simeq C\kappa$ known as the "super-heating field", and above which it is no longer stable. This separation has been studied in detail by H. Berestycki, A. Bonnet and J. Chapman in [BBC]. Moreover, the uniqueness of the Meissner solution in the vicinity of the field H_{sh} is proven by A. Bonnet, J. Chapman and R. Monneau in [BCM].

What is then natural to ask oneself is for which value H_{c_1} of the external field h_e, the Meissner solution ceases to be a minimum of $\mathcal{G}_\kappa$. The calculations of Abrikosov on all $\mathbb{R}^2$ predict $H_{c_1} \simeq \frac{\log\kappa}{2}$. As we shall see, the work of S. Serfaty has enabled to show, that following a conjecture in [BR2], H_{c_1} is in fact larger on a bounded domain. This is a consequence of border effects and is more precisely as follows: in the principal term of H_{c_1}, the factor $1/2$ in front of $\log\kappa$ has to be replaced by $k_1 = 1/2\max|\xi_0|$ where ξ_0 is the solution of

$$\begin{cases} \Delta\xi_0 = \xi_0 + 1 & \text{in } \Omega \\ \\ \xi_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{41}$$

ξ_0 is also solution of $-\Delta^2\xi_0 + \Delta\xi_0 = 0$. We can easily verify using the maximum principle that $\Delta\xi_0 > 1$ and $\xi_0 < 0$ inside Ω and thus that $|\xi_0| < 1$. E. Sandier and S. Serfaty then show the following result.

THEOREM 3.2 ([SS1]). — *There exists a constant $C_1 > 0$ such that if*

$$h_e \leq k_1 \log\kappa - C_1 \log\log\kappa$$

the Meissner solution is an absolute solution of $\mathcal{G}_\kappa$ and there exists C_2 such that if

$$h_e \geq k_1 \log \kappa + C_2$$

the Meissner solution is no longer an absolute minimum of $\mathcal{G}_\kappa$.

It is then tempting to place oneself in the vicinity of $k_1 \log \kappa$ and to try to observe the vortices appearing one after the other as the external field is increased. Technically, this implies that we should work with the absolute minimum of $\mathcal{G}_\kappa$ for which we would try to cover precisely the "bad set" $\mathcal{M}$, in the spirit of the BBH asymptotic, by means of uniformly bounded number of balls of radii κ^{-1} of which we would then study the minimal value (eventually 0) and the energetically most advantageous respective positions.

With the above scheme, we unfortunately come across a major difficulty which has not yet been solved and which is exposed in the next section. The BBH analysis was built on the initial estimate of the free energy $\mathcal{F}_\kappa(u_\kappa, A_k) = O(\log \kappa)$ which can be expressed by a bound of the number of vortices *a priori* independent of κ (knowing that the cost for one "isolated" vortex is about $2\pi \log \kappa$). In the case we are actually considering, where the total vorticity is moreover not under control and has become a variable of the problem, the bound of the energy is *a priori*

$$|\mathcal{G}_k(u_\kappa, A_\kappa)| = O(\log^2 \kappa). \tag{42}$$

From that we cannot deduce a better estimate than $\mathcal{F}_\kappa(u_\kappa, A_\kappa) = O(\log^2 \kappa)$ for the free energy part. This thus tells us that the number of vortices that have to be considered *a priori* is $O(\log \kappa)$ (which must be true for a field $h_e = 2k_1 \log \kappa$, but not for $h_e = k_1 \log \kappa + O(1)$ where we expect rather a uniformly bounded number of vortices). The difficulty of working with an *a priori* so large number of vortices is responsible for the fact that we have to get out the BBH asymptotic scheme and that the lack of precision is larger as can be seen between $k_1 \log \kappa - C \log \log \kappa$ and $k_1 \log \kappa + C_2$ in theorem 3.2 which is yet a very nice achievement.

The proof of theorem 3.2 has its roots in the asymptotic arguments of the previous section. The point is to realize the best possible lower bound of the total energy of a critical configuration with vortices and to show that for a sufficiently low external field, the Meissner solution is energetically preferable. In order to obtain such a lower bound, we begin by renormalizing the induced field by "extracting" the external field described following [BR2]. Let ξ be the solution $\Delta \xi = h$ on Ω, which is equal to 0 on $\partial\Omega$ ($A = *d\xi$ is the Coulomb gauge of h on Ω), we note ζ the difference $\zeta = \xi - h_e \xi_0$ (where ξ_0 is given by (41)). $*d\zeta$ is the Coulomb connection on all Ω, from which we have removed the influence of the external field. We show without much difficulty that the energy of the minimal configuration (u_κ, A_κ) can be decomposed as follows:

$$
\begin{aligned}
\mathcal{G}_\kappa(u,A) &= \mathcal{G}_\kappa(1, h_e * d\xi_0) + \mathcal{F}_\kappa(u, *d\zeta) + \int_\Omega |\Delta\zeta|^2 \\
&\quad -2h_e \int_\Omega (iu, du) \wedge d\xi_0 + o(1).
\end{aligned}
\tag{43}
$$

We suppose then that $h_e = O(\log\kappa)$ and we proceed to a decomposition of the domain, in the same spirit as that used to prove theorem 2.6, but more complex in the present case where the "bad set" is *a priori* far much bigger. We shall have $\Omega = \Omega_s \cup \Omega_v$ where Ω_s and Ω_v are disjoint and Ω_v will be a union of disjoint balls $\{B_{r_i}(a_i)\}_{i\in I}$ covering $\mathcal{M}$, the "bad set", and containing as much self energy as possible, without taking into account the interaction energy of the vortices amongst themselves as in the proof of theorem 2.6. Precisely, by means of a technique introduced by R. Jerrard [J], we show (see [SS1]) the existence of a family of balls $\mathcal{B} = \{B_{r_i}(a_i)\}_{i\in I}$ such that

- $\mathcal{M} \subset \cup_{i\in I} B_{r_i}(a_i)$
- $\displaystyle\int_{B_{r_i}(a_i)} f_\kappa(u, *d\zeta) \geq 2\pi|d_i|\,(\log\kappa - O(\log\log\kappa))$ where $d_i = \deg\left(\frac{u}{|u|}; \partial B_{r_i}(a_i)\right)$
- $r_i = O(\log^{-6}\kappa)$
- $\mathrm{Card}\, I = O(\log^2\kappa)$.

We then choose $\Omega_v = \cup_{i\in I} B_{r_i}(a_i)$. The $O(\log^{-6}\kappa)$ radii replace the radii $\delta = O(1)$ of the proof of theorem 2.6 since the number of vortices tends towards infinity *a priori* as $O(\log\kappa)$. The role of the a_i is a bit like intermediate vortices of degree d_i and this choice of covering gives in particular

$$
\int_\Omega (iu, du) \wedge d\xi_0 = 2\pi \sum_{i\in I} d_i \xi_0(a_i) + o(1).
\tag{44}
$$

Combining (43), the choice of Ω_v, (44) and the minimizing character of (u, A) for $\mathcal{G}_\kappa$, we have

$$
0 \geq 2\pi \sum_{i\in I} |d_i|(\log\kappa + O(\log\log\kappa)) - 4\pi\, h_e \,\max|\xi_0| \left(\sum_{i\in I} |d_i|\right)
\tag{45}
$$

and so if $\sum_{i\in I} |d_i| \neq 0$, we must get $h_e \geq k_1 \log\kappa + O(\log\log\kappa)$, which establishes the first part of theorem 3.2. The second part will be a consequence of theorem 3.3.

This technique of extracting the external magnetic field and then decomposing the domain in order to optimize the lower bound of the energy is a recurrent theme in the work of E. Sandier and S. Serfaty and appears through different forms to prove the theorems below.

3.3. Family of stable solutions having $1, 2, 3\ldots$vortices; the blooming of the lattice

For a while, let us give up the idea of considering the absolute minimum of $\mathcal{G}_\kappa$, but yet keeping an external field in the vicinity of $k_1 \log\kappa$.

In her first work [Se1], S. Serfaty constructs a family of stable solutions which have a finite number of vortices for the fields $1 \ll h_e \ll \kappa^\alpha$ (α being positive non explicit constant). To counteract the difficulty of showing evidence of a finite number of vortices for the fundamental states themselves, S. Serfaty makes an ansatz which consists in considering, instead of the absolute minimum (u_κ, A_κ) of $\mathcal{G}_\kappa$ in all $W^{1,2}(\Omega, \mathbb{C}) \times W^{1,2}(\Omega, \mathbb{R}^2)$, the (u_κ, A_κ) contained in the following set.

$$\mathcal{D}_n = \left\{u \in W^{1,2}(\Omega, \mathbb{C}) \quad \text{such that} \quad 2\pi n \log\kappa + B \le F_\kappa(u_T) \le (2\pi n + 1)\log\kappa\right\}$$

where

$$F_\kappa(u_T) = \int_\Omega |du_T|^2 + \frac{\kappa^2}{2}|1 - |u_T|^2|^2.$$

u_T is the projection of the function u on the unitary disc ($u_T = u$ if $|u| \le 1$ and $u_T = u/|u|$ otherwise) and B is a negative constant which remains fixed throughout the proof. The idea is that, in the Coulomb gauge on Ω, the cost for a vortex is $(F_\kappa(u_T) \simeq 2\pi \log\kappa)$. Thus, restraining to $\mathcal{D}_n$, we *a priori* constrain the configuration (u, A) not to have more than n vortices so that we can get back to the BBH asymptotic in order to establish a complete description of (u, A). The main difficulty of this approach comes from the fact that it is not at all evident that a minimum of $\mathcal{G}_\kappa$ in $\mathcal{D}_n$, once we have established its existence, should be in the interior of $\mathcal{D}_n$ and thus be a solution of the Ginzburg-Landau equation (1). Suppose that we have proven the existence of a minimum (u_κ, A_κ) of $\mathcal{G}_\kappa$ in $\mathcal{D}_n$. We are tempted to use once more the arguments of section 2 to cover the "bad set" $\mathcal{M}$... in order to establish an asymptotic expansion of $\mathcal{F}_\kappa(u_\kappa, A_\kappa)$ as in (31) and verify that (u_κ, A_k) is well in the interior of $\mathcal{D}_n$ (i.e. $2\pi n \log\kappa + B < F(u_T) < (2\pi n + 1)\log\kappa$) and thus solution of (1). The difficulty of this approach resides in the fact that as long as we do not know that (u, A) is solution of (1), we do not have an estimate of the form $\|d|u|\|_{L^\infty} = O(\kappa)$, no longer dispose of the quantization result (19) and the set $\mathcal{M}$ can be very diffuse, etc. We also no longer have any control of its size, which was given in section 2 by Pohozaev identities. The idea is then, not to work any longer with (u, A) itself, but with its parabolic regularization v which minimizes

$$\min_{v \in H^1(\Omega, \mathbb{C})} \int_\Omega |\nabla v|^2 + \frac{\kappa^2}{2}|1 - |v|^2|^2 + \kappa^{2\gamma} \int_\Omega |u_\kappa - v|^2$$

for some $\gamma > 0$. The advantage of replacing u by its parabolic regularisation v is that (v, A) will both be energetically very close to (u, A) and benefit from the properties of the minimum of $\mathcal{F}_\kappa$ in section 2 (such as $|v| \le 1$, $|d|v|| = O(\kappa)$...) which come from the elliptic equation verified by v. This idea of the parabolic regularization has been introduced in the context of Ginzburg-Landau equations by L. Almeida and F. Bethuel in [AB2], in order to define an approximated configuration of vortices for any function u in the energy zone $F(u) = O(\log\kappa)$. We then establish an asymptotic expansion for v in the spirit of section 2.2, which enables to deduce that (v, A) (and thus (u, A) too) which is energetically close to it, is in the interior of $\mathcal{D}_n$. The existence of a minimum $\mathcal{G}_k$ in $\mathcal{D}_n$ also rests on the use of the parabolic regularization. We then prove the following theorem which we present in the case where Ω is a disc centered at 0: $\Omega = B_R(0)$.

THEOREM 3.3 ([Se1]). — *Let D be an arbitrary positive constant, $h_e(\kappa)$ a function of κ tending towards $+\infty$ at infinity and verifying $h_e(\kappa) \leq \kappa^\alpha$, where α is a positive constant. Then there exists a positive constant $\kappa_0(D)$ such that for $\kappa \geq \kappa_0(D)$ and for all $n \in \mathbb{N}^*$, $n < \frac{D}{\pi}$, there exists a family of stable critical points (u_κ, A_k) of $\mathcal{G}_\kappa$, solution of the Ginzburg-Landau equations* (1), *verifying*

– $|u_\kappa|^{-1}(\{0\}) = \{a_1^\kappa ... a_n^\kappa\}$ *where $a_1^\kappa ... a_n^\kappa$ are isolated points of $B_R(0)$.*

– *For all $j = 1...n$, $ind(u_\kappa, a_j^\kappa) = +1$.*

– *The configuration $\tilde{a}_j = a_j\sqrt{h_e(\kappa)}$ converges, up to the extraction of a sub-sequence, towards a configuration of n points of $\mathbb{R}^2$ minimizing*

$$W(x_1...x_n) = -2\pi \sum_{i\neq j} \log|x_i - x_j| + 2\pi\ddot{\xi}_0(0) \sum_{i=1}^{n} |x_i|^2. \tag{46}$$

– *The asymptotic expansion of $\mathcal{G}_k(u_\kappa, A_\kappa)$ is*

$$\begin{aligned}\mathcal{G}_\kappa(u_\kappa, A_\kappa) = \mathcal{G}_\kappa(1, h_e * d\xi_0) \quad &+2\pi n \left(|\log\kappa - \frac{h_e}{k_1}| \right) + \pi(n^2 - n)\log h_e \\ &+W(\tilde{a}_1...\tilde{a}_n) + Q_n + o(1)\end{aligned} \tag{47}$$

where Q_n only depends on n.

This theorem is very rich in information which have to be extracted one by one, according to the questions posed in the introduction of this talk.

The information on the location of the zeros is not as precise in Serfaty's original work but can be deduced from the arguments in part 2.3 of the present survey.

As opposed to the case without border of section 2, the vortices here tend towards a same point which coincides with the point where $|\xi_0|$ is maximum. Had there been several points where $|\xi_0|$ is maximal, the vortices would have separated to form groups of n_j vortices, in an optimal way amongst those points (we optimize $\sum_j(n_j^2 - n_j)$).

We can calculate, as a function of h_e, the number of vortices n which optimize the energy and find through the Serfaty ansatz $F(u_T) \leq 2\pi D$ a very precise estimate of H_{c_1}. We can clearly see that the optimal number of vortices is an increasing function of h_e for fixed κ (large κ). We can easily verify the existence of $k_2 > 0$ such that, for $h_e > k_1 \log\kappa + k_2$, the Meissner solution ceases to be minimizing amongst those solutions with a finite number of vortices, which proves the second part of theorem 2.2. More generally, denoting by H_n the value of the field where the solution with n vortices is energetically preferable, we calculate ($n > 1$)

$$H_n - k_1 \log\kappa \simeq (n-1)k_1 \log\log\kappa. \tag{48}$$

Figure 2 graphically represents those results.

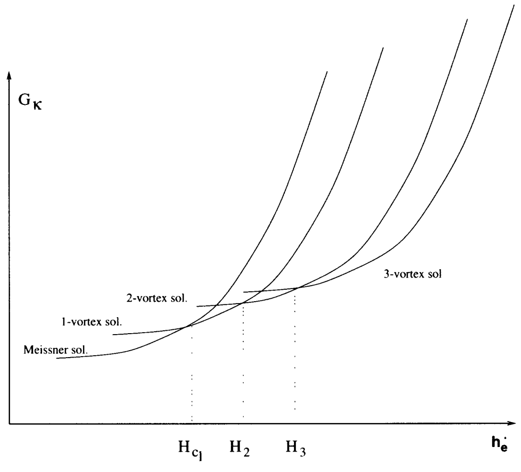

FIGURE 2. Solutions with 1,2,3... vortex

To know if the solution with n vortices for $H_n \leq h_e \leq H_{n+1}$ is a fundamental state on the whole space is an open seemingly difficult problem discussed in the next subsection. An important point is also the stable character of these solutions proven for $1 \ll h_e \ll \kappa^\alpha$ and which we have seen above for the Meissner solution and the "super-heating" field. This stability is responsible for hysteric phenomena observed in experiments. The stable character of the solutions having a finite number of vortices for fields smaller than H_{c_1} has been observed by Q. Du and F.H. Lin in [DL].

Although the expression for the renormalized energy W is relatively simple, the eventual symmetries of its fundamental states, which constitute the bloomings of the Abrikosov lattices, are still to be completely explored. In [GS], S. Gueron et I. Shafrir have done stability analyses of the symmetric configurations as well as numerical studies. They have made the following observations on the probable fundamental states (see figure 3)

– $n \leq 3$: these are regular polygons centered in 0

– $7 \leq n \leq 10$: these are regular, star-shaped (regular polygon + origin)

– $4 \leq n \leq 6$: the two previous types of configurations are locally minimizing and are observed

– $n \geq 11$: these are clusters of regular concentric polygons which "converge" towards a triangular lattice centered in 0 when n becomes large.

3.4. An interesting open problem: remove S. Serfaty's Ansatz

It would be nice to really characterize the phase change at H_{c_1} through a rigorous proof of the spontaneous creation of vorticity for absolute minimizers. In other words an interesting question would be to remove S. Serfaty ansatz and to replace the

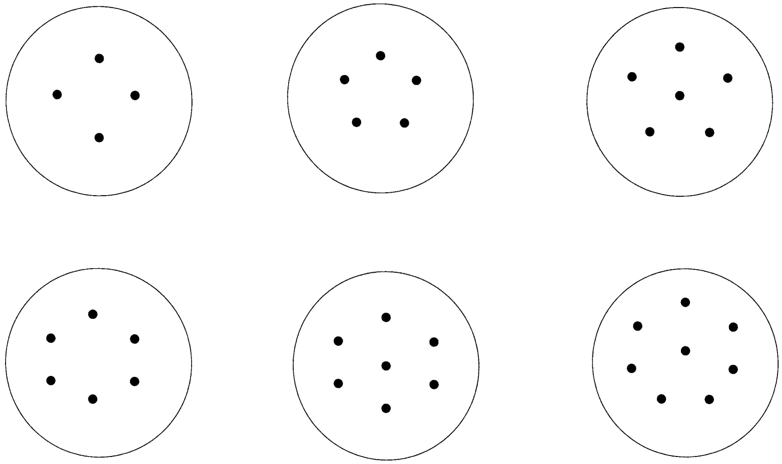

FIGURE 3. Blooming of Abrikosov Lattices.

constrained minimization of $\mathcal{G}_\kappa$ inside $\mathcal{D}_n$ by the minimization of $\mathcal{G}_\kappa$ inside the whole space $W^{1,2}(\Omega,\mathbb{C}) \times W^{1,2}(\Omega,\mathbb{R}^2)$. The hope is to prove that the solutions obtained in theorem 3.3 are the absolute minimizers of $\mathcal{G}_\kappa$ for the various ranges of external fields given by (48).

There is an interesting discrete model derived from the original one in the large $\kappa-$limit. Assume each vortex has zero size and $|u| \equiv 1$ out of the vortices located at $x_1 ... x_N$ with vorticity $d_1 ... d_N$. Following [BR2] one can extract the main terms in the energy (43):

$$\mathcal{H}(x_i, d_i) = -\sum_{i \neq j} d_i d_j \ \log|x_i - x_j| + \sum_i d_i^2 \ \log\kappa - 2h_e \sum_i d_i \ \xi_0(x_i). \tag{49}$$

One of the major step in understanding fully the creation of vortices and the phase transition at H_{c_1} would be to show that a minimizing configuration $(x_i, d_i)_{i=1...N}$ of $\mathcal{H}$ for an external field of the order $h_e = H_{c_1} + O(\log\log\kappa) = k_1 \log\kappa + O(\log\log\kappa)$ verifies

$$N = O_\kappa(1) \qquad \text{as} \qquad \kappa \to +\infty \tag{50}$$

or at least

$$\sum_{i=1}^{N} d_i = O(1). \tag{51}$$

This question is the main issue in describing the formation of vortices. If one assumes that all the d_i's have the same sign, answering to this question seems reasonably easy. The difficulty here comes from possible mixing of $+$ and $-$ creating negative clusters which are not *a-priori* energetically less favorable.

3.5. The second renormalization: the free boundary problem

In this last section, come back on the study of the fundamental states of $\mathcal{G}_\kappa$ but for much more intense fields that H_{c_1}, inside the mixed phase (see figure 1) and still in the London limit ($\kappa \to +\infty$). In that case, the number of vortices will *a priori* tend towards infinity with κ. One must then develop adapted methods to account for the limiting set occupied by the vortices in Ω. The first renormalization method consists in subtracting the interaction of each vortex on itself ($2\pi log\kappa$), that is $2\pi N \log\kappa$, to the minimal energy. For reasons seen above, because the *a priori* estimate (42) is insufficient, the precision on N to order $O(1)$ is very difficult to reach for the fundamental state for fields larger than $k_1 \log\kappa + O(\log\log\kappa)$. The second renormalization proposed by E. Sandier and S. Serfaty in [SS2] consists in dividing $\mathcal{G}_\kappa$ by h_e^2 and in studying the Γ-limit of the ratio, for at least the fundamental states. The result is then as follows.

THEOREM 3.4 ([SS2]). — *Let $h_e(\kappa)$ be a positive function so that $\lambda = \lim_{\kappa\to+\infty} \frac{\log\kappa}{h_e}$ exists. If $\lambda = 0$, we suppose $h_e(\kappa) = o(\kappa^2)$. Let $k_\star$ be a solution of the following problem*

$$\min \left\{ \begin{array}{l} E(k) = \lambda|\mu|(\Omega) + \displaystyle\int_\Omega |\nabla k|^2 + |k-1|^2 \\ \\ k \in W^{1,2}(\Omega) \quad k = 1 \quad \text{on } \partial\Omega \\ \\ \mu = -\Delta k + k \qquad \text{is a Radon measure} \end{array} \right\}$$

Then $k_\star$ is unique, $\mu = -\Delta h_\star + h_\star$ is positive and coincides with the characteristic function $\mathbf{1}_\omega$ of the locus of points ω where $k_\star$ is minimal and equal to $1 - \frac{\lambda}{2}$

$$\mu = \left(1 - \frac{\lambda}{2}\right) \mathbf{1}_\omega \tag{52}$$

and

$$\frac{h}{h_e} \rightharpoonup k_\star \qquad \text{weakly in } W^{1,2}(\Omega). \tag{53}$$

Moreover, the lack of strong convergence is exactly given by the measure of the following defect

$$\lambda\mu = \lim_{\kappa\to+\infty} |\nabla(h/h_e)|^2 - |\nabla k|^2. \tag{54}$$

Finally we have the following expansion of the renormalized fundamental energy

$$\lim_{\kappa\to+\infty} \frac{\mathcal{G}_\kappa(u_\kappa, A_\kappa)}{h_e^2} = E(k_\star) \tag{55}$$

where (u_κ, A_κ) is a minimal configuration.

Remark 2. — When λ is equal to 0, (i.e. $h_e \gg \log\kappa$) μ is the characteristic function of all Ω and h/h_e converges strongly towards 1.

The absence of strong convergence for the case $\lambda > 0$ can be understood in the following sense. Apart from the bad set which is covered by balls $\mathcal{M} \subset \Omega_v = \cup_{i\in I} B_{\kappa^{-1}}(x_i)$ (supposedly disjoint), $h = *dA$ verifies

$$-\mathrm{div}\left(\frac{\nabla h}{|u|^2}\right) + h = 0 \qquad \text{in } \Omega \setminus \Omega_v$$

h/h_e is then very close to the solution of

$$-\Delta k + k = 2\pi \sum_{i\in I} d_i \frac{\delta_{x_i}}{h_e}$$

where d_i is the degree of $u/|u|$ on the boundary $B_{\kappa^{-1}}(x_i)$. We recall that the second Ginzburg-Landau equation (1) gives in particular

$$-\int_{\partial B_{\kappa^{-1}}(x_i)} \frac{1}{|u|^2}\frac{\partial h}{\partial \nu} + \int_{B_{\kappa^{-1}}(x_i)} h = 2\pi d_i$$

and that, moreover, again by means of this equation,

$$\frac{\mathcal{G}_\kappa(u,A)}{h_e^2} \simeq \int_{\Omega\setminus\Omega_v} \frac{1}{|u|^2}\left|d\frac{h}{h_e}\right|^2 + \left|\frac{h}{h_e} - 1\right|^2 \simeq \int_{\Omega\setminus\Omega_v} \left|d\frac{h}{h_e}\right|^2 + \left|\frac{h}{h_e} - 1\right|^2.$$

Let μ be the limit of the vortex density per unit external field.

$$\mu = \lim_{\kappa\to+\infty} 2\pi \sum_{i\in I} d_i \frac{\delta_{x_i}}{h_e}.$$

If we consider the energy of the limit $k_\star$, solution of $-\Delta k_\star + k_\star = \mu$ instead of the limit of the energy

$$\int_{\Omega\setminus\Omega_v} |dk|^2 + |k-1|^2,$$

we forget that μ has been obtained by means of Dirac sums and we miss the part of the energy coming from the interaction of each Dirac on itself divided by h_e^2 which is

$$\frac{2\pi N \log\kappa}{h_e^2} \longrightarrow \mu(\Omega) \lim_{\kappa\to+\infty} \frac{\log\kappa}{h_e} = \lambda\mu(\Omega).$$

The difficulty of the analysis of this second renormalization is not only due to the above handwaving understanding of the mechanism of the energy partitioning, but essentially in the application of rigorous arguments which enable to establish theorem 3.4. The proof is somewhat in the same spirit as that of theorem 3.2, where, in order to obtain an optimal lower bound of the energy, we proceed by decomposing the domain which separates the self energy of the vortices from the rest of the energy. This decomposition of $\mathcal{M}$ relies on a more refined covering of the "bad set" than that of theorem 3.2, where the method of enlargement of balls introduced in [Sa] is used.

As described by the above simple reasoning, μ is the limiting density of vortices per unit applied field. It is either maximal and equal to $1 - \lambda/2$ in the sub-domain ω of Ω where $k_\star = 1 - \frac{\lambda}{2}$ also, or equal to zero in its complementary (see figure 4). The problem verified by $k_\star$ is a free boundary problem in the sense that the knowledge of

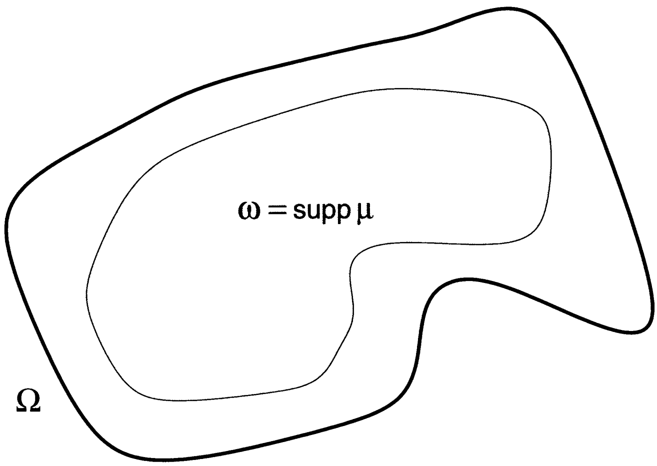

FIGURE 4. Free-boundary problem

ω determines $k_\star$ in a unique way. This is an obstacle-problem, now quite classical, considered in particular in [R]. It has been shown by A. Bonnet and R. Monneau in [BM] that $\partial\omega$ is regular for almost all the values of λ and that, whenever it is the case, ω is determined by the existence of the solution (...) of the following problem.

$$\begin{cases} -\Delta k_\star + k_\star = 0 & \text{in } \Omega \setminus \omega \\ k_\star = 1 - \dfrac{\lambda}{2} & \text{in } \omega \\ \dfrac{\partial k_\star}{\partial \nu} = 0 & \text{on } \partial\omega \\ k_\star = 1 & \text{on } \partial\Omega. \end{cases} \tag{56}$$

3.6. Conclusion

What about the Abrikosov lattices? It would be interesting, in the previous approach, to place oneself at a scale $1/\sqrt{h_e}$ -the minimal mean relative distance between two vortices- and to try then to understand the limit $\kappa \to +\infty$. We can hope in this asymptotic, to show the existence of an infinite renormalized energy on countable configurations of points of $\mathbb{R}^2$, which govern the set up of the vortices amongst themselves and try to understand the fundamental states of this energy by restraining ourselves, in a first approach, to periodic lattices. The first difficulty of this analysis is to understand the terms of lower order in the expansion of the energy (55). The first term only gives the mean density of vortices and does not see their relative positions to $O(1/\sqrt{h_e})$.

In his PhD thesis, M. Dutour [D] adopts a different approach which we shall not develop more extensively here since that would lead us beyond the scope of this talk. One places oneself on any flat torus T, which can be seen as a unit cell of a lattice of $\mathbb{R}^2$, and whose size and geometry are variables of the problem. On that cell T which is considered occupied by only one vortex (one fixes a complex line bundle E on T of Euler class equal to 1), one studies the minima of $\mathcal{G}_\kappa$ the Ginzburg-Landau functional under the action of an external uniform field h_e. When the external field increases, the increase in the vortex density, in that model, gives rise to a decrease in the size of the cell T, etc. M. Dutour thus gives a very complete description of the phase diagram (figure 1) namely in the vicinity of H_{c_2}, where he manages to show the existence of exactly two solutions; the first with vortices (close to one another by $\simeq \kappa^{-1}$) and the second in the normal state $dA = h_e$. He establishes whether each of these solutions is minimizing or not with respect to h_e, and accounts for the large precision of the Abrikosov bifurcation from the critical value 1/2. The optimality of such or such lattice is only partially discussed, but this approach seems to open a new direction towards a rigorous understanding of the relative positions of the vortices in the fundamental state (very far from the boundary which has disappeared from this model) and eventually of the energetically favorable nature of the Abrikosov triangular lattices.

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