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REPORT ON IGUSA'S LOCAL ZETA FUNCTION

BY JAN DENEF

Igusa's local zeta functions are related to the number of solutions of congruences mod p^m and to exponential sums mod p^m. This report is a survey on what is known about these zeta functions. There are several conjectures and intriguing connections with topology and singularity theory. They will be emphasized throughout the paper (especially sections 2 and 4). The case of curves is well-understood and is explained in section 5 without any calculations. Much less is known in higher dimensions, although there is by now a lot of experimental evidence (section 6) for the monodromy conjecture which relates the poles of Igusa's zeta functions to local monodromy. Relative invariants of prehomogeneous vector spaces are discussed in section 7. They provide very interesting examples which motivated the conjectures. The adelic situation is only mentioned briefly in 7.7. We will not treat the connection with Siegel-Weil formulas, but refer for this to Igusa's book [30, Chap. 4] and his survey paper [34]. At the end we briefly discuss the theory of p-adic subanalytic sets which yields very general rationality results.

1. FIRST PROPERTIES OF LOCAL ZETA FUNCTIONS

1.1 Local zeta functions

(1.1.1) Let K be a p-adic field, i.e. $[K\colon Q_p] < \infty$. Let R be the valuation ring of K, P the maximal ideal of R, and $\overline{K} = R/P$ the residue field of K. The cardinality of $\overline{K}$ is denoted by q, thus $\overline{K} = F_q$. For $z \in K$, ord $z \in \mathbb{Z} \cup \{+\infty\}$ denotes the valuation of z, $|z| = q^{-\operatorname{ord} z}$, and $\operatorname{ac}(z) = z\pi^{-\operatorname{ord} z}$ where π is a fixed uniformizing parameter for R.

(1.1.2) Let $f(x) \in K[x], x = (x_1, \ldots, x_n), f \notin K$. Let $\Phi : K^n \to \mathbb{C}$ be a Schwartz-Bruhat function, i.e. a locally constant function with compact support. Finally,

let χ be a character of $R^\times$, i.e. a homomorphism $\chi : R^\times \to \mathbb{C}^\times$ with finite image, where $R^\times$ denotes the group of units of R. We formally put $\chi(0) = 0$.
(1.1.3) To the above data one associates Igusa's local zeta function

$$Z_\Phi(s,\chi) = Z_\Phi(s,\chi,K,f) := \int_{K^n} \Phi(x)\chi(\mathrm{ac}f(x))|f(x)|^s|dx| ,$$

for $s \in \mathbb{C}, \mathrm{Re}(s) > 0$, where $|dx|$ denotes the Haar measure on K^n so normalized that R^n has measure 1. These zeta functions were introduced by Weil [83] and their basic properties for general f were first studied by Igusa [28], [30]. We will see below that $Z_\Phi(s,\chi)$ is rational in q^{-s}, so that it extends to a meromorphic function on $\mathbb{C}$.

We will write Z, resp. Z_0, instead of Z_Φ, when Φ is the characteristic function of R^n, resp. PR^n. Throughout this paper, we put $t = q^{-s}$. Note that $Z_\Phi(s,\chi)$ is a power series in t. The coefficient of t^m in a power series $P(t)$ is denoted by $\mathrm{Coeff}_{t^m}P(t)$. We denote the trivial character by χ_{triv} and the support of Φ by Supp Φ.

(1.1.4) *Remark.* Note that $\mathrm{ac}(f(x))$ and hence also $Z_\Phi(s,\chi,K,f)$ depend on the choice of the uniformizing parameter π. More canonically one introduces

$$Z_\Phi(w,K,f) = \int_{K^n} \Phi(x)w(f(x))|dx|$$

as a function of a quasicharacter w of $K^\times$ (i.e. a continuous homomorphism $w\colon K^\times \to \mathbb{C}^\times$). Every quasicharacter w of $K^\times$ is of the form $w(y) = \chi(\mathrm{ac}(y))|y|^s$. Thus studying $Z_\Phi(w,K,f)$ is equivalent with studying $Z_\Phi(s,\chi,K,f)$.

Sometimes it is also helpfull to think of Z_Φ as a distribution $\Phi \mapsto Z_\Phi$.

1.2 Number of solutions of congruences

(1.2.1) Suppose $f(x)$ has coefficients in R. Let N_m be the number of solutions of $f(x) \equiv 0 \bmod P^m$ in R/P^m and put $P(t) := \sum_{m=0}^\infty q^{-nm}N_m t^m$. The Poincaré series $P(t)$ is directly related to $Z(s,\chi_{triv})$ by the formula

$$P(t) = \frac{1 - tZ(s,\chi_{triv})}{1-t}.$$

Indeed, since $q^{-nm}N_m$ equals the measure of $\{x \in R^n | \mathrm{ord}\ f(x) \geqslant m\}$, this follows directly from

$$\int_{\mathrm{ord}\ f(x)\geqslant m} \Phi(x)|dx| = Z_\Phi(0,\chi_{triv}) - \mathrm{Coeff}_{t^m}\frac{tZ_\Phi(s,\chi_{triv})}{1-t} . \tag{1.2.2}$$

To verify this last equality note that the left-hand side equals

$$\int \Phi(x)|dx| - \sum_{k \leqslant m-1} \int_{\text{ord } f(x)=k} \Phi(x)|dx| = Z_\Phi(0, \chi_{triv}) - \sum_{k \leqslant m-1} \text{Coeff}_{t^k} Z_\Phi(s, \chi_{triv}).$$

1.3 Rationality of local zeta functions

(1.3.1)*Resolutions.* Put $X = \text{Spec } K[x]$ and $D = \text{Spec } K[x]/(f(x))$. Choose an (embedded) resolution (Y, h) for $f^{-1}(0)$ over K, meaning that Y is an integral smooth closed subscheme of projective space over $X, h\colon Y \to X$ is the natural map, the restriction $h\colon Y \setminus h^{-1}(D) \to X \setminus D$ is an isomorphism, and the reduced scheme $(h^{-1}(D))_{red}$ associated to $h^{-1}(D)$ has only normal crossings (i.e. its irreducible components are smooth and intersect transversally, cf. [25]). Let $E_i, i \in T$, be the irreducible components of $(h^{-1}(D))_{red}$. These consist of the components $E_i, i \in T_s$, of the strict transform of D, and the exceptional divisors $E_i, i \in T \setminus T_s$. For each $i \in T$ let N_i be the multiplicity of E_i in the divisor of $f \circ h$ on Y and let $\nu_i - 1$ be the multiplicity of E_i in the divisor of $h^*(dx_1 \wedge \cdots \wedge dx_n)$. The (N_i, ν_i) are called the *numerical data* of the resolution. For $i \in T$ and $I \subset T$ we consider the schemes

$$\overset{\circ}{E}_i := E_i \setminus \bigcup_{j \neq i} E_j \ , \ E_I := \bigcap_{i \in I} E_i \ , \ \overset{\circ}{E}_I := E_I \setminus \bigcup_{j \in T \setminus I} E_j.$$

When $I = \emptyset$, we put $E_\emptyset = Y$. Finally let $C_f \subset X$ be the singular locus of $f\colon X \to \mathbb{A}^1_K$.

(1.3.2) Theorem (Igusa [28],[30]). *Assume the notation of 1.1 and 1.3.1, then*
(i) $Z_\Phi(s, \chi)$ *is a rational function of* q^{-s}. *Its poles are among the values* $s = -\nu_i/N_i + 2\pi\sqrt{-1}k/N_i \log_e q$ *with* $k \in \mathbb{Z}$ *and* $i \in T$ *such that the order of* χ *divides* N_i.
(ii) If $C_f \cap \text{Supp } \Phi \subset f^{-1}(0)$, *then* $Z_\Phi(s, \chi) = 0$ *for almost all* χ.

Proof of (i). Consider the set $Y(K)$ of K-rational points of Y as a K-analytic manifold. We have

$$Z_\Phi(s, \chi) = \int_{Y(K)} (\Phi \circ h)\chi(\text{ac}(f \circ h))|f \circ h|^s |h^*(dx_1 \wedge \cdots \wedge dx_n)|.$$

Let $b \in Y(K)$ and $\{i \in T | b \in E_i\} = \{i_1, \ldots, i_r\}$. There are local coordinates $y_1, \ldots, y_n$ for $Y(K)$ centered at b such that

$$f \circ h = \epsilon y_1^{N_{i_1}} \ldots y_r^{N_{i_r}}, \quad h^*(dx_1 \wedge \cdots \wedge dx_n) = \eta y_1^{\nu_{i_1}-1} \ldots y_r^{\nu_{i_r}-1} dy_1 \wedge \cdots \wedge dy_n,$$

where ϵ and η are analytic in a neighbourhood U of b and $\epsilon(b) \neq 0$, $\eta(b) \neq 0$. Note that $|\epsilon|, |\eta|$ and $\chi(\mathrm{ac}\ \epsilon)$ are constant on U when U is small enough. Because $h^{-1}(\mathrm{Supp}\ \Phi)$ is compact, we see that $Z_\Phi(s,\chi)$ is a finite $\mathbb{C}$-linear combination of products of factors of the form $q^{ks}, k \in \mathbb{Z}$, or $\int_{\mathrm{ord} z \geqslant e} \chi^{N_i}(\mathrm{ac}\ z)|z|^{N_i s+\nu_i-1}|dz|$. But this last integral is zero unless the order of χ divides N_i, in which case it is a rational function of q^{-s} with denominator $1-q^{-N_i s-\nu_i}$. This proves (i). For (ii), see [30, p. 91-96].

Remark. The rationality of $Z_\Phi(s,\chi)$ can also be proved without the use of resolution of singularities, see [10] and section 8.

1.4. Exponential sums and integration on fibers

(1.4.1) Let Ψ be the standard additive character on K, thus, for $z \in K$, $\Psi(z) = \exp(2\pi i \mathrm{Tr}_{K/Q_p}(z))$, where Tr denotes the trace. Weil [83] introduced the following two functions

$$E_\Phi(z) = E_\Phi(z,K,f) = \int_{K^n} \Phi(x)\Psi(zf(x))|dx|,$$
$$F_\Phi(y) = F_\Phi(y,K,f) = \int_{f^{-1}(y)} \Phi(x)\left|\frac{dx}{df}\right|,$$

for $z \in K$ and $y \in K \setminus V_f$, where $V_f = f(C_f)$. The function $E_\Phi(z)$ is locally constant and bounded on K. One is interested in its behaviour for $|z| \to \infty$. The simplest case is when $C_f \cap \mathrm{Supp}\Phi = \emptyset$, then $E_\Phi(z) = 0$ for $|z|$ large enough. The function $F_\Phi(y)$ is locally constant on $K \setminus V_f$ and has compact support. One is interested in its behaviour when y tends to a point of V_f. For a nice introduction, see Serre [73].

We will write E, F, resp. E_0, F_0, instead of E_Φ, F_Φ when Φ is the characteristic function of R^n, resp. PR^n.

(1.4.2) Suppose $f(x) \in R[x]$, $m \in \mathbb{N}\setminus\{0\}$. If $u \in R^\times$, then obviously $E(u\pi^{-m}) = q^{-nm}\sum_{x\in(R/P^m)^n} \Psi(uf(x)/\pi^m)$, which is a classical exponential sum mod P^m.

Let $a \in R \setminus V_f$ and denote by $N_m(a)$ the number of solutions in R/P^m of the congruence $f(x) \equiv a \mod P^m$. Then one verifies that $F(a) = N_m(a)/q^{(n-1)m}$, for m big enough (depending on a). This stable quotient $F(a)$ is classically known as the local singular series associated to f and a, and plays an important role in the circle method.

(1.4.3) Note that $E_\Phi(z) = \int_K F_\Phi(y)\Psi(zy)|dy|$ is the Fourier transform of $F_\Phi(y)$ on K and that $Z_\Phi(\omega, K, f) = \int_K F_\Phi(y)\omega(y)|dy|$ is the Mellin transform of $(1 - q^{-1})yF_\Phi(y)$ on $K^\times$. This gives the relation between exponential sums and local zeta functions. By decomposing Φ and translation one reduces to the case where $C_f \cap \operatorname{Supp} \Phi \subset f^{-1}(0)$. Then, due to 1.3.2 (i) and (ii), the following are related by formulas (see [28], [30]):

(i) Principal parts of the Laurent expansions of the $Z_\Phi(s, \chi)$ around their poles,
(ii) Terms of an asymptotic expansion of $E_\Phi(z)$ as $|z| \to \infty$,
(iii) Terms of an asymptotic expansion of $F_\Phi(y)$ as $y \to 0$.

Still more information is provided by the following.

(1.4.4) Proposition. *Let $u \in R^\times$ and $m \in \mathbb{Z}$. Then $E_\Phi(u\pi^{-m})$ equals*

$$Z_\Phi(0, \chi_{triv}) + \operatorname{Coeff}_{t^{m-1}} \frac{(t-q)Z_\Phi(s, \chi_{triv})}{(q-1)(1-t)} + \sum_{\chi \neq \chi_{triv}} g_{\chi^{-1}} \chi(u) \operatorname{Coeff}_{t^{m-c(\chi)}} Z_\Phi(s, \chi)$$

where $c(\chi)$ denotes the conductor of χ, i.e. the smallest $c \geqslant 1$ such that χ is trivial on $1 + P^c$, and g_χ denotes the Gaussian sum

$$g_\chi = (q-1)^{-1} q^{1-c(\chi)} \sum_{v \in (R/P^{c(\chi)})^\times} \chi(v)\Psi(v/\pi^{c(\chi)}).$$

Proof. Replacing f by uf we see that it suffices to prove the theorem for $u = 1$. We introduce for any $e \in \mathbb{N} \setminus \{0\}$ the integral

$$I_{\Phi,e}(s) = \int_{K^n} \Phi(x)\Psi(\pi^{-e} \operatorname{ac} f(x))|f(x)|^s|dx|.$$

Direct verification shows

$$E_\Phi(\pi^{-m}) = \int_{\operatorname{ord} f(x) \geqslant m} \Phi(x)|dx| + \sum_{k \leqslant m-1} \operatorname{Coeff}_{t^k} I_{\Phi,m-k}(s).$$

The proposition follows now from 1.2.2, since Fourier transformation on $(R/P^e)^\times$ yields $I_{\Phi,e}(s) = \sum_{c(\chi)=e} g_{\chi^{-1}} Z_\Phi(s, \chi)$.

(1.4.5) Corollary (Igusa [28], [30]). *Suppose that* $C_f \cap \operatorname{Supp} \Phi \subset f^{-1}(0)$. *Then for* $|z|$ *big enough* $E_\Phi(z)$ *is a finite* $\mathbb{C}$*-linear combination of functions of the form* $\chi(ac(z))|z|^\lambda(\log_q |z|)^\beta$ *with coefficients independent of* z, *and with* $\lambda \in \mathbb{C}$ *a pole of* $(s+1)Z_\Phi(s,\chi_{triv})$ *or of* $Z_\Phi(s,\chi), \chi \neq \chi_{triv}$, *and with* $\beta \in \mathbb{N}, \beta \leqslant$ *(multiplicity of pole* $\lambda) - 1$. *Moreover all poles* λ *appear effectively in this linear combination.*

Proof. This follows from 1.3.2 (i), (ii) and 1.4.4, by writing $Z_\Phi(s,\chi)$ in partial fractions.

1.5 Igusa's conjecture on exponential sums

Let F be a number field, $f(x) \in F[x]\setminus\{0\}$ a homogeneous polynomial and $\sigma \in \mathbb{R}$. Suppose $1 < \sigma < \min \nu_i/N_i$, where the minimum is taken over all i except those with $N_i = \nu_i = 1$ and the (N_i, ν_i) are the numerical data of a fixed resolution of $f^{-1}(0)$ over F. By 1.4.5, for each p-adic completion K of F there exists $C(K) \in \mathbb{R}$ satisfying $|E(z,K,f)| \leqslant c(K)|z|^{-\sigma}$ for all $z \in K$.

Conjecture (Igusa [30]). *In the above inequality one can take* $c(K)$ *independent of* K.

This is related to the validity of a certain Poisson formula, see [30, p. 122, 170]. Igusa [27] proved the conjecture when $C_f = \{0\}$, by using Deligne's bound [8] for exponential sums over F_q, which in turn depends on the Riemann hypothesis for varieties over F_q. He also verified it for certain relative invariants of prehomogeneous vector spaces [26], [29], [30, p. 123-127]. Recently Sperber and Denef proved the conjecture for polynomials $f(x)$ which are non-degenerate with respect to their Newton polyhedron $\Delta(f)$ (see 5.3) assuming that $\Delta(f)$ has no vertex in $\{0,1\}^n$ (and only considering toric resolutions).

1.6 The Archimedean case

Replacing K by $\mathbb{R}$ or $\mathbb{C}$ and Φ by a C^∞ function with compact support, one defines

$$Z_\Phi(s,K,f) = \int_{K^n} \Phi(x)|f(x)|^{\delta s} dx,$$

for $s \in \mathbb{C}, \operatorname{Re}(s) > 0$, where $\delta = 1$ if $K = \mathbb{R}$ and $\delta = 2$ if $K = \mathbb{C}$. One proves that $Z_\Phi(s,K,f)$ extends to a meromorphic function on $\mathbb{C}$ whose poles are rational, either by resolution of singularities [2],[5] or by the theory of Bernstein polynomials [4].

2. MONODROMY AND BERNSTEIN POLYNOMIALS

2.1 Monodromy

(2.1.1) Let $f : \mathbb{C}^n \to \mathbb{C}$ be a non constant polynomial map and fix $b \in \mathbb{C}^n$ with $f(b) = a$. Let $B \subset \mathbb{C}^n$ be a small enough ball with center b. Milnor [63] proved that the restriction $f|_B$ is a locally trivial C^∞ fibration over a small enough *pointed* disc $A \subset \mathbb{C} \setminus \{a\}$ with center a. Hence the diffeomorphism type of the so called *Milnor fiber* $F_b := f^{-1}(t) \cap B$ of f *around* b does not depend on $t \in A$ and the counter clockwise generator of the fundamental group of A induces an automorphism T of $H^{\cdot}(F_b, \mathbb{C})$ which is called the *local monodromy* of f at b. It is well known that the eigenvalues of T are roots of unity (see [22, Exposé I]). When b is an isolated critical point of f, a result of Milnor [63] states that $H^i(F_b, \mathbb{C}) = 0$ for $i \neq 0$, $n-1$, and $H^0(F_b, \mathbb{C}) = \mathbb{C}$ with trivial monodromy action.

(2.1.2) Theorem (A'Campo's formula [1]). *Suppose b is an isolated critical point of f, with $f(b) = 0$, $n \geqslant 2$. We adopt the notation of 1.3.1 with $K = \mathbb{C}$. Then the characteristic polynomial of the monodromy action on $H^{n-1}(F_b, \mathbb{C})$ equals*

$$\left((z-1)^{-1} \prod_{i \in T} (z^{N_i} - 1)^{\chi(\overset{\circ}{E}_i \cap h^{-1}(b))}\right)^{(-1)^{n-1}},$$

where χ denotes the Euler characteristic with respect to singular cohomology. In particular $\exp(2\pi\sqrt{-1}/k)$, for $k \in \mathbb{N}, k > 1$, is an eigenvalue of the local monodromy at b if and only if $\sum_{k|N_i} \chi(\overset{\circ}{E}_i \cap h^{-1}(b)) \neq 0$.

2.2 The Bernstein polynomial

I.N. Bernstein [4] has proved for any polynomial $f(x), x = (x_1, \ldots, x_n)$, over a field K of characteristic zero, that there exist $P \in K[x, \frac{\partial}{\partial x}, s]$, and $b(s) \in K[s] \setminus \{0\}$ which satisfy the functional equation $Pf^{s+1} = b(s)f^s$. The monic polynomial $b(s)$ of smallest degree which satisfies this functional equation is called the *Bernstein polynomial* of f and will be denoted by $b_f(s)$. If f is not constant, then $s+1$ divides $b_f(s)$. If further $f = 0$ has no singular points over the algebraic closure of K, then $b_f(s) = s + 1$. A basic theorem of Kashiwara states that all roots of $b_f(s)$ are negative rational numbers. When $K = \mathbb{R}$ or $\mathbb{C}$ one easily verifies, using the functional equation and integration by parts, that the poles of $Z_\Phi(s, K, f)$ are among the values $s = \alpha - j$ with α a root of $b_f(s)$ and $j \in \mathbb{N}$. Note however that this integration by parts does not make sense in the p-adic case.

The roots of $b_f(s)$ are related to the geometry of f. Indeed by Malgrange [58], if α is a root of $b_f(s)$ then $\exp(2\pi\sqrt{-1}\alpha)$ is an eigenvalue of the local monodromy of f at some point of $f^{-1}(0)$, and all eigenvalues are obtained in this way. (Note that $b_f(s)$ is the least common multiple of all local Bernstein polynomials, see e.g. [23, Lemma 2.5.2].)

Thus, in the *Archimedean* case if s is a pole of $Z_\Phi(s, K, f)$ then $\exp(2\pi\sqrt{-1}s)$ is an eigenvalue of monodromy. (For a direct proof in the isolated singularity case, see Malgrange [57].) Moreover, Barlet [3] has proved that each eigenvalue is obtained in this way, when $K = \mathbb{C}$. We refer to Loeser [50] for information on the exact location of the poles and to Loeser [51] for an estimate on the largest pole.

2.3 The monodromy conjecture

Motivated by the situation in the Archimedean case 2.2 and the study of concrete examples, it is natural to propose the following conjectures for any polynomial $f(x)$ over a numberfield $F \subset \mathbb{C}$.

(2.3.1) Conjecture (Igusa [36]). *For almost all p-adic completions K of F , if s is a pole of $Z(s, \chi, K, f)$, then* $\mathrm{Re}(s)$ *is a root of $b_f(s)$.*

This conjecture has been verified in special cases, see 5.2.5, 5.3, 7.3 and 7.4 below.

(2.3.2) Monodromy conjecture (Igusa). *For almost all p-adic completions K of F, if s is a pole of $Z(s, \chi, K, f)$, then* $\exp(2\pi\sqrt{-1}\ \mathrm{Re}\ (s))$ *is an eigenvalue of the local monodromy of f at some complex point of $f^{-1}(0)$.*

Note that the first conjecture implies the second by what we said above. But for the second one there is by now a massive amount of experimental evidence, see section 6. Both conjectures might be true for all p-adic completions and for Z replaced by Z_Φ. In most numerical examples, theorem 1.3.2 yields a very big list of candidate poles. However, due to miraculous cancellations, usually many of these candidates are no pole. This strange phenomenon would be explained by the monodromy conjecture.

Loeser [53] studied zeta functions $\int_{K^n} \Phi(x)|f_1(x)|^{s_1} \dots |f_k(x)|^{s_k} |dx|$ of several variables and formulated a conjecture on the existence of certain asymptotic expansions (generalizing 1.4.3 (iii)). Surprisingly this conjecture implies a relation between the polar locus of these zeta functions and the geometry of the discriminant of $f_1, \dots, f_k$. Moreover there are connections with the monodromy conjecture.

3. EXPLICIT FORMULAS

We continue to use the notation of 1.1 and 1.3.1. Reduction mod P is denoted by $^-$.

(3.1) We call a Schwartz-Bruhat function Φ *residual* if Supp $\Phi \subset R$ and $\Phi(x)$ only depends on $x \mod P$. Such Φ induces $\overline{\Phi}\colon \overline{K}^n \to \mathbb{C}$.

(3.2) We say that the resolution (Y, h) for $f^{-1}(0)$ has *good reduction* mod$P$ if $\overline{Y}$ and all $\overline{E}_i$ are smooth, $\cup_{i \in T} \overline{E}_i$ has only normal crossings, and the schemes $\overline{E}_i$ and $\overline{E}_j$ have no common components whenever $i \neq j$ (cf. [12]). Here the reduction mod P of any closed subscheme Z of Y is denoted by $\overline{Z}$ and defined as the reduction mod P of the closure of Z in projective space over Spec $R[x]$. If in addition $N_i \notin P$ for all $i \in T$ then we say that the resolution has *tame* good reduction. When f and (Y, h) are defined over a number field F, we have good reduction for almost all completions K of F. When the resolution (Y, h) has good reduction we have $\overline{E}_I = \cap_{i \in I} \overline{E}_i$ and we put $\overset{\circ}{\overline{E}}_I = \overline{E}_I \setminus \cup_{j \in T \setminus I} \overline{E}_j$ and $\overset{\circ}{\overline{E}}_i = \overline{E}_i \setminus \cup_{j \neq i} \overline{E}_j$. Finally let $C_{\overline{f}}$ be the singular locus of $\overline{f}\colon \overline{K}^n \to \overline{K}$.

3.3 Theorem [14]. *Let $f \in R[x], \overline{f} \neq 0$. Suppose that $f^{-1}(0)$ has a resolution with tame good reduction mod P and that Φ is residual. If χ is not trivial on $1+P$ then $Z_\Phi(s, \chi)$ is constant as function of s. If moreover $C_{\overline{f}} \cap$ Supp $\overline{\Phi} \subset \overline{f}^{-1}(0)$, then $Z_\Phi(s, \chi) = 0$.*

3.4 Theorem [12], [14]. *Let $f \in R[x], \overline{f} \neq 0$. Suppose that (Y, h) is a resolution for $f^{-1}(0)$ with good reduction mod P, and that Φ is residual. Let χ be a character of $R^\times$ of order d which is trivial on $1 + P$. Then*

$$Z_\Phi(s, \chi) = q^{-n} \sum_{\substack{I \subset T \\ \forall i \in I:\ d | N_i}} c_{I,\chi,\Phi} \prod_{i \in I} \frac{q-1}{q^{N_i s + \nu_i} - 1},$$

where $c_{I,\chi,\Phi} = \sum_{a \in \overset{\circ}{\overline{E}}_I(\overline{K})} \overline{\Phi}(a) \Omega_\chi(a)$.

Here $\Omega_\chi(a)$ is defined as follows: If $a \in \overset{\circ}{\overline{E}}_I(\overline{K})$ and $d|N_i$ for all $i \in I$ then we can write $\overline{f} \circ \overline{h} = uw^d$ with $u, u^{-1}, w \in \mathcal{O}_{\overline{Y},a}$ and we put $\Omega_\chi(a) := \chi(u(a))$. We will write $c_{I,\chi}$, resp. $c_{I,\chi,0}$, instead of $c_{I,\chi,\Phi}$ when Φ is the characteristic function of R^n, resp. PR^n. To denote the dependence on K, we will sometimes write $c_{I,\chi,\Phi}(K)$.

3.5 Cohomological interpretation. Assume the hypothesis of 3.4 and choose a prime ℓ with $\ell \nmid q$. Note that χ induces a character of $F_q^\times$ which we denote again by χ. Let $\mathcal{L}_\chi$ be the Kummer $\overline{Q}_\ell$-sheaf on $\mathbb{A}^1_{F_q} \setminus \{0\}$ associated to this last character (see [9, Sommes Trig.]). Put $U = \overline{Y} \setminus (\overline{f} \circ \overline{h})^{-1}(0)$. Let ν be the open immersion $\nu \colon U \hookrightarrow \overline{Y}$ and $\alpha \colon U \to \mathbb{A}^1_{F_q} \setminus \{0\}$ the map induced by $\overline{f} \circ \overline{h}$. We define $\mathcal{F}_\chi := \nu_* \alpha^* \mathcal{L}_\chi$. It is easy to verify that $\mathcal{F}_\chi$ is lisse of rank one on $U_d := \overline{Y} \setminus \cup_{d \nmid N_i} \overline{E}_i$. Moreover, if $a \in U_d(F_q)$ then the action of the geometric Frobenius on the stalk of $\mathcal{F}_\chi$ at a is multiplication by $\Omega_\chi(a)$. Hence Grothendieck's trace formula yields

$$c_{I,\chi} = \sum_i (-1)^i \mathrm{Tr}(F, H^i_c(\overset{\circ}{\overline{E}}_I \otimes F^a_q, \mathcal{F}_\chi)), \tag{3.5.1}$$

$$c_{I,\chi,0} = \sum_i (-1)^i \mathrm{Tr}(F, H^i_c((\overset{\circ}{\overline{E}}_I \cap \overline{h}^{-1}(0)) \otimes F^a_q, \mathcal{F}_\chi)), \tag{3.5.2}$$

where F denotes the Frobenius and F^a_q the algebraic closure of F_q. For further use we still mention

3.6 Lemma. *The higher direct images $R^j \nu_*(\mathcal{F}_\chi|_U)$ are zero outside U_d for all $j \geqslant 0$. The same holds also for the open immersion $\nu_I \colon \overset{\circ}{\overline{E}}_I \hookrightarrow \overline{E}_I$.*

(3.7) We should also mention that Langlands [47] has given a formula, in terms of principal-value integrals, for the principal parts of the Laurent expansions of $Z_\Phi(s, \chi)$ around its poles.

4. CONSEQUENCES OF THE EXPLICIT FORMULAS

Unless stated otherwise, we keep the notation of 1.1 and 1.3.1. When χ is a character of $\overline{K}^\times$ we denote the induced character of $R^\times$ also by χ. We say a property holds for almost all P if it holds for almost all completions of a number field F (all data being defined over F). For any scheme V of finite type over a field $L \subset \mathbb{C}$, we denote by $\chi(V)$ the Euler characteristic of $V(\mathbb{C})$ with respect to singular cohomology.

4.1 Degree of local zeta functions

(4.1.1) Because $Z_\Phi(s,\chi)$ is a rational function of q^{-s} we can consider its degree $\deg Z_\Phi(s,\chi)$ which is defined as the degree of the numerator minus the degree of the denominator (as polynomials in q^{-s}). If the hypothesis of theorem 3.4 holds, then it is clear from the explicit formula that $\deg Z_\Phi(s,\chi) \leqslant 0$. Clearly the degree is < 0 if and only if $\lim_{s\to -\infty} Z_\Phi(s,\chi) = 0$.

(4.1.2) Proposition [12]. *For almost all P, $\deg Z_0(s,\chi_{triv}) = 0$. If moreover f is homogeneous, then $\deg Z(s,\chi_{triv}) = -\deg f$.*

Proof. From theorem 3.4 it follows that

$$q^n \lim_{s\to -\infty} Z_0(s,\chi_{triv}) \equiv \sum_{I\subset T} c_{I,\chi_{triv},0} \equiv q^n Z_0(0,\chi_{triv}) = 1 \mod q.$$

Hence $\lim_{s\to -\infty} Z_0(s,\chi_{triv}) \neq 0$. This proves the first assertion. The second assertion follows from the first by the formula $Z(s,\chi) = q^{n+s\deg f} Z_0(s,\chi)$.

(4.1.3) When $f(0) = 0$ we proved [15], using the method of vanishing cycles [22], that for almost all P and any character χ of $\overline{K}^\times$ of order d we have

$$\lim_{s\to -\infty} Z_0(s,\chi) = (1-q)q^{-n} \sum_i (-1)^i \mathrm{Tr}(\mathrm{Frob}, H^i(F_0, Q_\ell)^\chi),$$

where F_0 is the Milnor fiber of $\overline{f}$ at 0, $H^i(F_0,Q_\ell)^\chi$ denotes the component of the cohomology on which the semi-simplification of the local monodromy acts like χ, and Frob is a suitable lifting of the Frobenius. In particular this implies that $\deg Z_0(s,\chi) < 0$ when there is no eigenvalue of the local monodromy of f at 0 with order d.

4.2 The functional equation

(4.2.1) We denote by $K^{(e)}$ the unramified extension field of K of degree e, and put

$$Z(s,e,\chi) := Z(s, \chi \circ N_{K^{(e)}/K}, K^{(e)}, f),$$

where N denotes the norm. D. Meuser [61] has shown that $Z(s,e,\chi_{triv})$, as function of s and e, is a rational function of $q^{-es}, \alpha_1^e, \ldots, \alpha_r^e$ for some $\alpha_1,\ldots,\alpha_r \in$