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THE JONES-WITTEN INVARIANTS OF KNOTS

par **Michael ATIYAH**

1. INTRODUCTION

One of the most remarkable developments of recent years has been the work initiated by Vaughan Jones [2] [3] on knot invariants. This has all the hallmarks of great mathematics. It produces simple new invariants which solve classical problems and it involves a very wide range of ideas and techniques from practically all branches of mathematics and physics. Here is a list of the areas which have been significantly involved in the theory up to the present : combinatorics, group representations, algebraic geometry, differential geometry, differential equations, topology, Von Neumann algebras, statistical mechanics, quantum field theory. Moreover the subject continues to develop rapidly and a final picture has not yet emerged.

Given this very wide field I have to be very selective for a one-hour presentation. I will concentrate on some aspects and I shall have to omit all the technicalities. Moreover, to shorten the exposition, I will discuss only the simplest case of the theory. Fuller accounts can be found in the papers of Vaughan Jones [3] and Witten [9].

In 1984 Vaughan Jones surprised the experts in knot theory by producing a *polynomial invariant*, now known as the Jones polynomial $V(q)$, which was superficially similar to the classical Alexander polynomial but was, in essential features, rather different. In particular $V(q)$ could distinguish (some) knots from their mirror images. For this and other reasons $V(q)$ turned out to be a very effective tool in knot theory and, as a result, old conjectures of P.G. Tait from the 19th century have now been established.

The Jones polynomial can be profitably studied from many angles and it has been generalized in several ways to produce further knot invariants. Much of this work has involved important ideas from theoretical physics, essentially physics of 2 dimensions. However a major break-through came in 1988 when Witten [10] gave a simple interpretation of the Jones polynomial in terms of 3-dimensional physics. These ideas of Witten are based on a heuristic use of the Feynman integral, but they lead to very explicit results and calculations which can be verified by alternative rigorous methods. A full mathematical treatment of Witten's theory has yet to appear, so my account has to be somewhat sketchy and incomplete.

Not only does Witten's theory provide a physical "meaning" for the Jones invariants but it also extends them to knots in an arbitrary compact oriented 3-manifold. This is a major generalization which had been attempted unsuccessfully via other methods. Finally, and most significantly, Witten's generalization allows us to define "relative invariants", for 3-manifolds with boundary. In this case the invariants are not numbers but take their values in a vector space associated with the boundary. This facility, of allowing manifolds with boundary, makes the theory much more flexible and greatly facilitates computation, even for the "absolute" case of closed 3-manifolds. The situation may roughly be compared with the story of Lefschetz numbers in classical topology. The number of fixed points of a self-map (analogous to the Jones invariant of a knot) re-interpreted as the Lefschetz number, through the induced map on homology, becomes part of a larger theory (analogous to Witten's theory) and hence more computable.

In the next section I will summarize the key features of the Jones polynomial, before going on in section 3 to describe Witten's theory. In section 4 I will outline the way in which Witten's theory may be developed mathematically. I will make no attempt in this presentation to give the physical interpretation via Feynman integrals. For this I refer to Witten's papers [9] [10]. For a general survey of "topological quantum field theories" see also [1] [8].

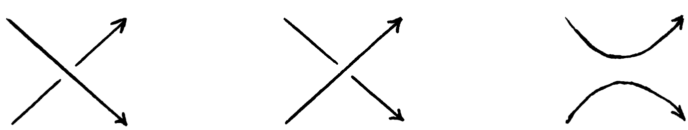
2. THE JONES POLYNOMIAL

We shall deal with oriented knots and links. These are just oriented 1-dimensional submanifolds of the 3-space S^3 : a knot being the case of one component. For an oriented link L the Jones polynomial $V_L(q)$ is a finite Laurent series in the variable $q^{\frac{1}{2}}$ with integer coefficients. Its first basic properties are :

(2.1) $V_L(q) = 1$ when L in the standard unknotted circle,

(2.2) $V_{L^\star}(q) = V_L(q^{-1})$ where $L^\star$ is the mirror image of L .

$V_L(q)$ can be characterized by a *skein relation.* For this we consider a generic plane projection of L , so that all crossing points have just 2 branches, one "over" and one "under". Focussing attention on one crossing point we can then consider the 3 versions of L obtained by allowing the 3 different possibilities as shown below :



L_+ $\quad$ L_- $\quad$ L_0

The skein relation for $V_L(q)$ is the linear relation :

$$q^{-1}V_{L_+}(q) - qV_{L_-}(q) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\ V_0(q) \tag{2.3}$$

It is not hard to show that (2.1) and (2.3) uniquely determine $V_L(q)$. The difficulty is to prove consistency, *i.e.* that $V_L(q)$ depends on the link L (up to isotopy) and not on any particular plane projection.

Note.- In fact $V_L(q)$ does not depend on the orientation of L . However this is not true for the generalizations of $V_L(q)$, except that reversing the orientation of *all* components of L will always preserve the generalized Jones polynomials.

Example.- For a (right-handed) trefoil knot $V(q) = -q^4 + q^3 + q$. By (2.2) this distinguishes it from its mirror image, the left-handed trefoil.

Although it is possible to verify the consistency of (2.3) by direct combinatorial methods this is not very enlightening. A better approach, explained in [3], is based on the use of ***braids***.

The Artin braid group on n strands B_n can be defined as the fundamental group of the configuration space C_n of n unordered distinct points in the plane. There is an elementary geometric construction which assigns to any braid β an oriented link $\hat{\beta}$ in S^3 . All links arise in this way and the equivalence relation on the union of all B_n given by

$$\beta_1 \sim \beta_2 \Longleftrightarrow \widehat{\beta_1} \text{ isotopic to } \widehat{\beta_2}$$

is explicitly known. Thus one may construct link invariants from suitable braid invariants.

To get the right braid invariants to produce $V_L(q)$, Jones introduces certain representations $\rho_\lambda(q)$ of B_n .

These are representations depending on a parameter q and a partition λ of n . For $q = 1$ they reduce to the irreducible representations ρ_λ of the symmetric group S_n pulled back to B_n via the natural homomorphism $B_n \to S_n$. The representations $\rho_\lambda(q)$ come from representations of the Hecke algebra.

The Jones polynomial $V_L(q)$ for $L = \hat{\beta}$ is now defined as a certain linear combination of the characters of $\rho_\lambda(q)$ evaluated at β . The only partitions λ which are needed here are the partitions of n into (at most) 2 parts.

Note.- For generalizations of $V_L(q)$ one needs all partitions of n . These generalizations lead to polynomials satisfying suitable generalizations of (2.3).

In this braid group approach to the Jones polynomial it is still a mystery why suitable linear combinations of the characters $\rho_\lambda(q)$ should give link invariants. The underlying reason becomes clear in Witten's theory as we shall see.

3. THE WITTEN THEORY

Witten considers oriented links L in an arbitrary compact oriented 3-manifold Y. Moreover both Y and L are assumed to be *framed.* For Y this means we fix a trivialization of the tangent bundle while for L it means that we fix a trivialization of its normal bundle in Y. In both cases only the homotopy class of the pairing will be significant.

Witten's invariant of the pair (Y, L) is a complex valued function of a positive integer k : we denote it by $W_{Y,L}(k)$. There are two extreme cases of special interest:

i) $Y = S^3$.

ii) $L = \phi$.

In the first case when $Y = S^3$ there are standard choices (up to homotopy) for the framing of Y and of any link L. The standard framing on Y is the one which when stabilized (by adding trivial bundles) extends to the interior 4-ball. The standard framing of L is characterized by the property that each component L_j of L has zero linking number with the "parallel" copy L'_j we get by using the framing. With these standard choices of framing the Witten invariant now depends only on the oriented link L. It is related to the Jones polynomial $V_L(q)$ by the formula :

$$W_{S^3,L}(k) = V_L(exp\ \frac{2\pi i}{k+2}) \ . \tag{3.1}$$

This shows that the Witten invariant in this case determines, and is essentially equivalent to the Jones polynomial. Thus Witten's invariant for general Y is indeed a generalization of the Jones invariant.

Case ii) when there is no link leads to an invariant $W_Y(k)$ for an oriented 3-manifold Y. Witten's invariant is naturally normalized so that (for a suitable framing) $W_Y(k) = 1$ for $Y = S^1 \times S^2$. On the other hand the formula for $Y = S^3$ is less trivial, namely

$$W_{S^3}(k) = \sqrt{| \frac{2}{k+2}\ sin\ \frac{\pi}{k+2}} \ . \tag{3.2}$$

As mentioned in section 1, Witten's invariant also extends to a "relative" invariant when Y is a 3-manifold with boundary Σ . We assume in this situation that the 1-manifold L meets the boundary transversally in a finite set of points $\partial L = \mathcal{P} = (P_1, \ldots, P_n)$. The framings on Y, L induce a *stable framing* on Σ and a *normal framing* of each point P_i on Σ (*warning* : we do not at present allow homotopies of these framings).

Witten's relative theory assigns a finite-dimensional complex vector space $H_k(\Sigma, \mathcal{P})$ to each such framed pair $(\Sigma, \mathcal{P})$ and each positive integer k . Reversing the orientation of Σ (and using the natural corresponding framings) converts H into its dual *i.e.*

$$H_k(\Sigma^\star, \mathcal{P}) = H_k^\star(\Sigma, \mathcal{P}) \ . \tag{3.3}$$

Moreover these vectors spaces are *multiplicative* in the sense that for the disjoint sums :

$$H_k(\Sigma_1 \cup \Sigma_2 \ , \ \mathcal{P}_1 \cup \mathcal{P}_2) = H_k(\Sigma_1, \mathcal{P}_1) \otimes H_k(\Sigma_2, \mathcal{P}_2) \ . \tag{3.4}$$

If now

$$\partial(Y, L) = (\Sigma, \mathcal{P}) \ ,$$

then Witten's relative invariant is a vector

$$W_k(Y, L) \in H_k(\Sigma, \mathcal{P}) \ . \tag{3.5}$$

These vector spaces and invariants have various naturality properties which we shall not describe, but the key property is the relation between relative and absolute invariants. Of course if Y is closed so that Σ is empty then formally we want the relative and absolute invariants to coincide and this requires

$$H_k(\phi, \phi) = C \ . \tag{3.6}$$

The significant case however is when we cut a closed 3-manifold Y (and link L) into two parts along a common surface Σ . Thus

$$\begin{aligned} &Y = Y_1 \cup Y_2 \quad \text{along} \quad \Sigma : \quad \partial Y_1 = \Sigma \ , \ \partial Y_2 = \Sigma^\star, \\ &L = L_1 \cup L_2 \quad \text{along} \quad \mathcal{P}. \end{aligned}$$

The vectors

$$W_k(Y_1, L_1) \quad , \quad W_k(Y_2, L_2)$$

then lie in dual spaces (by (3.3)), so that this scalar product is well-defined. The key fact is then

$$W_k(Y, L) = \langle W_k(Y_1, L_1), W_k(Y_2, L_2) \rangle \ . \tag{3.7}$$

By decomposing Y in various ways one can use the relative invariants to help calculate the absolute invariants. In particular (3.7) implies the existence of a skein relation for the absolute invariant when $Y = S^3$. For this one has to know that, for S^2 with 4 points $P_1, \dots P_4$,

$$\dim \, H_k(S^2, P_1, \dots P_4) = 2 \ . \tag{3.8}$$

Now apply (3.7) with Y_1 a small ball in the neighbourhood of a crossing point of the link L (viewed as nearly planar). Keeping L_2 fixed but taking the 3 possible choices of L_1 (joining the P_i in pairs) we get 3 different vectors in the 2-dimensional space of (3.8). These must satisfy a linear relation (with coefficients independent of L). Taking the scalar product with the vector (in the dual space) coming from (Y_2, L_2) and applying (3.7) we deduce the existence of a skein relation. In this way, once the coefficients have been determined, Witten can identify his absolute invariant with that of Jones as asserted in (3.1).

If we take $Y = \Sigma \times I$, $L = \mathcal{P} \times I$ (with I the unit integral) and use (3.3) and (3.4) we see that $W_k(Y, L)$ can be viewed as an *automorphism* of $H_k(\Sigma, \mathcal{P})$. This depends on framings of (Y, L) *i.e.* on *homotopies* of framings for $(\Sigma, \mathcal{P})$. In fact all such automorphisms are scalar multiplications and depend only on the initial and final framings. The computation of these scalars is an important but delicate part of the theory.

If we ignore scalar factors and replace the vector space $H_k(\Sigma, \mathcal{P})$ by the associated projective space its functorial properties provide an action on it of the group of components of $Diff^+(\Sigma, \mathcal{P})$, the orientation preserving diffeomorphisms of Σ preserving $\mathcal{P}$.

In particular when $\Sigma = S^2$ these are essentially the representations of the braid group arising from the Hecke algebra which Jones employs to get his invariants.

4. MODULI SPACES

Witten defines his theory heuristically by a Feynman integral. This is not rigorous but it does lead to a rigorous theory of the vector spaces $H_k(\Sigma, \mathcal{P})$ which I will now describe.

For simplicity consider first the case when $\mathcal{P} = \phi$, so that we simply have a surface Σ with no marked points. To construct $H_k(\Sigma)$ we first pick a *complex structure* τ on Σ : denote the resulting Riemann surface by Σ_τ . We can then define the moduli space $M(\Sigma_\tau)$ of holomorphic $SL(2, C)$-bundles over Σ_τ . This is a projective variety with a Zariski open set representing *stable* vector bundles, and has been much studied by algebraic geometers [4]. It has an ample generating line-bundle L and we define

$$H_k(\Sigma_\tau) = H^0(M(\Sigma_\tau), L^k)$$

to be the space of holomorphic sections of L^k . This is of course a standard construction in algebraic geometry. What is not so obvious from this point of view, is that *the projective space* of $H_k(\Sigma_\tau)$ *is essentially independent of the complex structure* τ.

More precisely by allowing τ to vary over Teichmüller space we get a holomorphic vector bundle and this bundle has a natural connection whose curvature is a scalar. This shows in particular that the group of components of $Diff^+(\Sigma)$ acts on the associated projective space of covariant constant sections.

The second essential ingredient in the Witten theory is the construction of the vector $W_k(Y) \in H_k(\Sigma)$ when $\partial Y = \Sigma$. There is no way known at present which is both simple and rigorous. A rigorous procedure is to use a sequence of elementary surgeries (or a Morse function) and to build $W_k(Y)$ from these elementary steps. One then has to verify that the result is independent of the surgeries (or Morse functions) chosen.

If we use a Morse function then we are reduced to studying the behaviour of the vector spaces $H_k(\Sigma_\tau)$ as $\tau \to \tau_0$, the complex structure of a singular Riemann surface with ordinary double points. It can be shown that the vector bundle formed by the $H_k(\Sigma_\tau)$ extends over τ_0 and the subspace preserved (up to scalars) by the local monodromy of the connection can be identified with $H_k(\Sigma_0)$ where Σ_0 is the desingularization of Σ_{τ_0} .

The homomorphism $H_k(\Sigma_0) \to H_k(\Sigma_\tau)$ is then the vector associated by (3.5) to a 3-manifold Y with $\partial Y = \Sigma_\tau - \Sigma_0$.

The framings on Y, Σ which I have been ignoring at this stage are needed to replace the projective spaces by vector spaces.

For the general case of surfaces Σ with marked points all this theory extends in a natural way. The moduli space $M(\Sigma_\tau)$ has a natural generalization to a moduli space $M(\Sigma_\tau, \mathcal{P})$, using bundles with parabolic structures at the marked points in the sense of Seshadri [6].

Much of the literature on this topic is to be found in papers on *rational conformal field theory*, and the spaces $H_k(\Sigma, \mathcal{P})$ above are referred to as the spaces of conformal blocks. A rigorous mathematical treatment in this framework is given in [7].

A different and more differential-geometric approach is being developed by Axelrod, Witten and Hitchin, but these versions have not yet appeared.

A verification that Witten's invariants of 3-manifolds are well-defined has been given by Reshetikin and Turaev [5], using surgery techniques (the Kirby calculus).

Finally I should recall that I have implicitly just been describing the simplest case of the Witten-Jones theory. In general one picks a compact Lie group G and an irreducible representation V . I discussed only the case $G = SU(2)$, $V = C^2$ (note that $V = V^\star$ which need not hold in general). The theory extends to the more general case without serious modification. In particular the relevant moduli spaces are still well-defined.

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