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SYLOW 2-SUBGROUPS OF SIMPLE GROUPS

by John G. THOMPSON

I will primarily limit this lecture to a discussion of results obtained by two students, Goldschmidt and MacWilliams.

For each group $X$, let $m(X)$ be the minimal number of generators of $X$ and let $d(X) = \max\{m(A)\}$, where $A$ ranges over all the normal abelian subgroups of $X$.

Suppose $G$ is simple and $T$ is a Sylow 2-subgroup of $G$. In studying the minimal simple groups, it became clear that the case $d(T) \leq 2$ was anomalous. I handled the problem by first determining all the possibilities for a Sylow 2-subgroup and then using techniques available in any minimal simple group.

For further work in simple groups, it is desirable to classify all simple $G$ such that $d(T) \leq 2$. The case $d(T) = 1$ is non trivial, but seems well on the way to a solution, so we assume $d(T) = 2$.

The most naive way to tackle this problem is first to classify all 2-groups $T$ with $d(T) = 2$. This is difficult, but one result about 2-groups is helpful.

**Lemma 1.** If $T$ is a 2-group with $d(T) \leq 2$, then every subgroup of $T$ is generated by 4 elements.

This result then leads fairly rapidly to
THEOREM 1 (MacWilliams).—Suppose $T$ is a Sylow 2-subgroup of the simple group $G$, $d(T) = 2$ and $T.C(T) \subseteq N(T)$. Then $|T| = 4, 64$ or $128$ and $T$ is determined by $|T|$.

If $|T| = 64$, $T$ is isomorphic to a Sylow 2-subgroup of $U_3(4)$ and if $|T| = 128$, $T$ is isomorphic to a Sylow 2-subgroup of the new simple groups of Janko of orders $604,800$ and $50,232,960$.

The structure of $T$ in case $T.C(T) = N(T)$ is not yet determined. Several of the families of known simple groups satisfy these hypotheses.

Goldschmidt's work had a different origin. Initially, he studied simple groups with a Sylow 2-subgroup whose class of nilpotency is 2. One of the results obtained is that a Sylow 2-subgroup has exponent 4. However, this emerges from a more general set up, the starting point being

**LEMMA 2.**—Suppose $p$ is a prime and $P$ is a Sylow $p$-subgroup of a group $G$. Let $n$ be the smallest integer such that $n(p-1) \geq c - 1$, where $c$ is the class of nilpotency of $P$. Let $W = \langle x^P \mid x \in Z(P) \rangle$, where $Z(P)$ is the center of $P$. Then $W$ is weakly closed in $P$ (that is, $g \in G$ and $W^g \subseteq P$ imply $W = W^g$).

This is elementary, but clever. The crucial result is

**THEOREM 2 (Goldschmidt).**—Suppose $T$ is a Sylow 2-subgroup of $G$, $W$ is a weakly closed subgroup of $T$, $1 \subseteq W \subseteq Z(T)$ and $t$ is an involution of $T - W$.

If $W \subseteq Q_{2',2}(C(x))$ for every involution $x$ of $Wt$, then $G$ is not simple.
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Here $O_{2',2}(X)$ is the largest normal subgroup of $X$ with a normal $2$-complement. The proof is character-theoretic.

If one couples this result with work of Gorenstein, we get

**THEOREM 3 (Goldschmidt).** - If $W$ is weakly closed in $T$ and $W \subseteq Z(T)^2$ where $T$ is a Sylow $2$-subgroup of $G$, then $W \subseteq O_{2',2}(G)$.

All these results are fragmentary, but given the state of finite group theory, this is not surprising.