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Sylow 2-subgroups of simple groups

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I will primarily limit this lecture to a discussion of results obtained by two students, Goldschmidt and MacWilliams.

For each group $X$, let $m(X)$ be the minimal number of generators of $X$ and let $d(X) = \max\{m(A)\}$, where $A$ ranges over all the normal abelian subgroups of $X$.

Suppose $G$ is simple and $T$ is a Sylow 2-subgroup of $G$. In studying the minimal simple groups, it became clear that the case $d(T) \leq 2$ was anomalous. I handled the problem by first determining all the possibilities for a Sylow 2-subgroup and then using techniques available in any minimal simple group.

For further work in simple groups, it is desirable to classify all simple $G$ such that $d(T) \leq 2$. The case $d(T) = 1$ is non trivial, but seems well on the way to a solution, so we assume $d(T) = 2$.

The most naive way to tackle this problem is first to classify all 2-groups $T$ with $d(T) = 2$. This is difficult, but one result about 2-groups is helpful.

**LEMMA 1.** If $T$ is a 2-group with $d(T) \leq 2$, then every subgroup of $T$ is generated by 4 elements.

This result then leads fairly rapidly to
THEOREM 1 (MacWilliams).- Suppose \( T \) is a Sylow 2-subgroup of the simple group \( G \), \( d(T) = 2 \) and \( T \cdot C(T) \subseteq N(T) \). Then \( |T| = 4, 64 \) or 128 and \( T \) is determined by \( |T| \).

If \( |T| = 64 \), \( T \) is isomorphic to a Sylow 2-subgroup of \( U_3(4) \) and if \( |T| = 128 \), \( T \) is isomorphic to a Sylow 2-subgroup of the new simple groups of Janko of orders 604,800 and 502,329,600.

The structure of \( T \) in case \( T \cdot C(T) = N(T) \) is not yet determined. Several of the families of known simple groups satisfy these hypotheses.

Goldschmidt's work had a different origin. Initially, he studied simple groups with a Sylow 2-subgroup whose class of nilpotency is 2. One of the results obtained is that a Sylow 2-subgroup has exponent 4. However, this emerges from a more general set up, the starting point being

LEMMA 2.- Suppose \( p \) is a prime and \( P \) is a Sylow \( p \)-subgroup of a group \( G \). Let \( n \) be the smallest integer such that \( n(p-1) \geq c - 1 \), where \( c \) is the class of nilpotency of \( P \). Let \( W = \langle x^n \mid x \in Z(P) \rangle \), where \( Z(P) \) is the center of \( P \). Then \( W \) is weakly closed in \( P \) (that is, \( g \in G \) and \( Wg \subseteq P \) imply \( W = Wg \)).

This is elementary, but clever. The crucial result is

THEOREM 2 (Goldschmidt).- Suppose \( T \) is a Sylow 2-subgroup of \( G \), \( W \) is a weakly closed subgroup of \( T \), \( 1 \subseteq W \subseteq Z(T) \) and \( t \) is an involution of \( T - W \). If \( W \subseteq O_{2',2}(C(T)) \) for every involution \( x \) of \( Wt \), then \( G \) is not simple.
Here $0_{2',2}(X)$ is the largest normal subgroup of $X$ with a normal $2$-complement. The proof is character-theoretic.

If one couples this result with work of Gorenstein, we get

THEOREM 3 (Goldschmidt).—If $W$ is weakly closed in $T$ and $W \subseteq Z(T)^2$ where $T$ is a Sylow $2$-subgroup of $G$, then $W \subseteq 0_{2',2}(G)$.

All these results are fragmentary, but given the state of finite group theory, this is not surprising.