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Singularities and exotic spheres

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BRIESKORN has proved [4] that the n-dimensional affine algebraic variety
\[ z_0^2 + z_1^2 + \ldots + z_n^2 = 0 \quad (n \text{ odd}, \ n \geq 1) \]
is a topological manifold though the variety has an isolated singular point (which is normal for \( n \leq 2 \)). Such a phenomenon cannot occur for normal singularities of 2-dimensional varieties, as was shown by NUMFORD ([12], [6]). BRIESKORN's result stimulated further research on the topology of isolated singularities (BRIESKORN [5], MILNOR [11] and the speaker [5], [7]). BRIESKORN [5] uses the paper of F. PHAM [14], whereas the speaker studied certain singularities from the point of view of transformation groups using results of BREDON ([2], [3]), W.C. HSIANG and W.Y. HSIANG [8] and JANICH [9].

§ 1. The integral homology of some affine hypersurfaces.

PHAM [14] studies the non-singular subvariety \( V_a = V(a_0, a_1, \ldots, a_n) \) of \( \mathbb{C}^{n+1} \) given by
\[ a_0 z_0^2 + a_1 z_1^2 + \ldots + a_n z_n^2 = 1 \quad (n \geq 0), \]
where \( a = (a_0, \ldots, a_n) \) consists of integers \( a_j \geq 2 \).

Let \( G_{a_j} \) be the cyclic group of order \( a_j \) multiplicatively written and generated by \( w_j \). Define the group \( G_a = G_{a_0} \times G_{a_1} \times \ldots \times G_{a_n} \) and put \( \varepsilon_j = \exp(2\pi i/a_j) \).
Then $w_0 w_1 \cdots w_n$ is an element of $G_a$ whereas $\varepsilon_0 \varepsilon_1 \cdots \varepsilon_n$ is a complex number. $G_a$ operates on $V_a$ by

$$k_0 k_1 \cdots k_n w_0 \cdots w_n(z_0, \ldots, z_n) = (\varepsilon_0 z_0, \ldots, \varepsilon_n z_n).$$

Let $C_{a_j}$ be the group of $a_j$-th roots of unity and $x \mapsto \hat{x}$ the isomorphism $G_{a_j} \to C_{a_j}$ given by $w_j \mapsto \varepsilon_j = \hat{w}_j$.

PHAM considers the following subspace $U_a$ of $V_a$

$$U_a = \{z \in V_a \text{ and } z_j \text{ real } \geq 0 \text{ for } j = 0, \ldots, n\}$$

**Lemma.** The subspace $U_a$ is a deformation retract of $V_a$ by a deformation compatible with the operations of $G_a$.

For the proof see PHAM [14], p. 338.

$U_a$ can also be described by the conditions

$$z_j = u_j |z_j| \text{ with } u_j \in C_{a_j} \quad (j = 0, \ldots, n).$$

Put $|z_j|^{a_j} = t_j$. Then $U_a$ becomes the space of $(n+1)$-tpls of complex numbers

$$t_o u_o \oplus t_1 u_1 \oplus \cdots \oplus t_n u_n$$

with

$$u_j \in C_{a_j}, \quad t_j \geq 0, \quad \sum_{j=0}^n t_j = 1$$

Thus $U_a$ can be identified with the join $G_{a_0} * G_{a_1} * \cdots * G_{a_n}$ of the finite sets $G_{a_j}$ (see MILNOR [10]).

**Lemma 2.1** in [10] states in particular that the reduced integral homology groups of the join $A * B$ of two spaces $A, B$ without torsion are given by a canonical isomorphism

$$\tilde{H}_{r+1}(A * B) \cong \sum_{i+j=r} \tilde{H}_i(A) \otimes \tilde{H}_j(B),$$

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whereas Lemma 2.2 in [10] shows that \( A \ast B \) is simply connected provided \( B \) is
arcwise connected and \( A \) is any non-vacuous space. These properties of the join
together with its associativity imply

**Theorem.** The subvariety \( V_a \) of \( \mathbb{C}^{n+1} \) is \((n-1)\)-connected. Moreover

\[
\tilde{H}_n(V_a) \cong \tilde{H}_0(C_{a_0}) \otimes \tilde{H}_0(C_{a_1}) \otimes \ldots \otimes \tilde{H}_0(C_{a_n}).
\]

This is a free abelian group of rank \( r = \prod (a_j - 1) \).

The isomorphism (1) is compatible with the operations of \( G_a \).

All other reduced integral homology groups of \( V_a \) vanish.

It can be shown that \( V_a \) has the homotopy type of a connected union
\( S^n \vee \ldots \vee S^n \) of \( r \) spheres of dimension \( n \).

The identification of \( U_a \) with a join was explained to the speaker by

MILNOR.

\( U_a = G_{a_0} \ast G_{a_1} \ast \ldots \ast G_{a_n} \) is an \( n \)-dimensional simplicial complex which has an
\( n \)-simplex for each element of \( G_a \). The \( n \)-simplex belonging to the unit of \( G_a \)
is denoted by \( e \). All other \( n \)-simplices are obtained from \( e \) by operations of
\( G_a \). Thus we have for the \( n \)-dimensional simplicial chain group

\[
C_n(U_a) = J_a e
\]

where \( J_a \) is the group ring of \( G_a \). The homology group \( \tilde{H}_n(U_a) = \tilde{H}_n(V_a) \) is an
additive subgroup of \( J_a e = C_n(U_a) \cong J_a \).

The face operator \( \partial_j \) commutes with all operations of \( G_a \) on \( C_n(U_a) \)
and furthermore satisfies \( \partial_j = \omega_j \partial_j \). Therefore

\[
\text{h} = (1 - \omega_0)(1 - \omega_1) \ldots (1 - \omega_n) e
\]
is a cycle. Thus \( \text{h} \tilde{H}_n(U_a) \). It follows easily that \( \tilde{H}_n(V_a) = J_a \text{h} \). This yields the
THEOREM. The map $w \rightarrow wh (w \in G_a)$ induces an isomorphism

$$J_a/I_a \cong \tilde{H}_n(V_a) = J_a h$$

where $I_a \subset J_a$ is the annihilator ideal of $h$, which is generated by the elements

$$1 + w_j + w_j^2 + \ldots + w_j^{a_j-1}, \quad (j = 0, \ldots, n).$$

Therefore $w_0 w_1 \ldots w_n h$ (where $0 \leq k_j \leq a_j - 2$, $j = 0, \ldots, n$) is a basis of $\tilde{H}_n(V_a)$.

We recall that $\tilde{H}_n(V_a)$ is the integral singular homology group (of course with compact support). $V_a$ is a $2n$-dimensional oriented manifold without boundary (non-compact for $n \geq 1$). Therefore the bilinear intersection form $S$ is well defined over $\tilde{H}_n(V_a)$. It is symmetric for $n$ even, skew-symmetric for $n$ odd. It is compatible with the operations of $G_a$.

PHAM ([14], p.358) constructs an $n$-dimensional cycle $\tilde{h}$ in $V_a$ which is homologous to $h$ and intersects $U_a$ exactly in two interior points of the simplices $e$ and $w_0 w_1 \ldots w_n e$ (sign questions have to be observed). In this way he obtains (using the $G_a$-invariance of $S$) the following result, reformulated somewhat for our purposes.

THEOREM. Put $\eta = (1 - w_0) \ldots (1 - w_n)$. The bilinear form $S$ over $J_a$ $\eta \cong \tilde{H}_n(V_a)$ is given by

$$S(x\eta, y\eta) = f(\overline{y} x\eta), \quad (x, y \in J_a),$$

where $f : J_a \rightarrow Z$ is the additive homomorphism with

$$f(1) = -f(w_0 \ldots w_n) = (-1)^{n(n-1)/2},$$

$$f(w) = 0 \quad \text{for} \quad w \in G_a, \quad w \neq 1, \quad w \neq w_0 \ldots w_n,$$

and where $\gamma \mapsto \overline{y}$ is the ring automorphism of the group ring $J_a$ induced by $w \mapsto w^{-1} (w \in G_a)$.
§ 2. The quadratic form of $V_a$.

Let $G$ be a finite abelian group, $J(G)$ its group ring. The ring
automorphism of $J(G)$ induced by $g \mapsto g^{-1}$ ($g \in G$) is denoted by
$x \mapsto \bar{x}$ ($x \in J(G)$). Give an element $\eta \in J(G)$ and a function $f : G \to \mathbb{Z}$.
The additive homomorphism $J(G) \to \mathbb{Z}$ induced by $f$ is also called $f$.

Put $\hat{f} = \sum_{w \in G} f(w)w$. We assume

a) $f(\bar{x}\eta) = f(x\eta)$ for all $x \in J(G)$, [equivalently $\bar{\eta} = \hat{\eta}$]

or

b) $f(\bar{x}\eta) = -f(x\eta)$ for all $x \in J(G)$, [equivalently $\bar{\eta} = -\hat{\eta}$].

The bilinear form $S$ over the lattice $J(G)\eta$ defined by

$$S(x\eta, y\eta) = f(\bar{y}x\eta), (x, y \in J(G)),$$

is symmetric in case a), skew symmetric in case b). Since $S$ is a form with
integral coefficients, its determinant is well-defined. The signature

$$\tau(S) = \tau^+(S) - \tau^-(S), \text{ case a)},$$

is the number $\tau^+(S)$ of positive minus the number $\tau^-(S)$ of negative diagonal
entries in a diagonalisation of $S$ over $\mathbb{R}$. Let $\chi$ run through the characters
of $G$.

**LEMMA.** With the preceding assumptions

$$\pm \det S = \prod_{\chi(\eta) \neq 0} \chi(\hat{\eta}) \cdot \text{ order of the torsion subgroup of } J(G)/J(G)\eta$$

and in case a)

$$\tau^+(S) = \text{ number of characters } \chi \text{ with } \chi(\hat{\eta}) > 0$$

$$\tau^-(S) = \text{ number of characters } \chi \text{ with } \chi(\hat{\eta}) < 0.$$ 

The proof is an exercise as in [1], p. 444.

The lemma and the last theorem of § 1 imply for the affine hypersurface

$V_a = V(a_0, \ldots, a_n)$ the
THEOREM. Let $S$ be the intersection form of $V_a$. Then

$$
\pm \det S = \prod_{1 \leq k \leq n, \epsilon_j = 1} (1 - \epsilon_j^{k_0^{x_j^1}}) \prod_{1 \leq k \leq n, \epsilon_j = -1} (1 + \epsilon_j^{k_0^{x_j^1}})
$$

where $\epsilon_j = \exp(2\pi i / a_j)$. For $n$ even, we have

$$
\tau^+(S) = \text{number of } (n+1)-\text{tuples of integers } (x_0, \ldots, x_n), 0 < x_j < a_j,
$$

with $0 < \sum_{j=0}^{n} \frac{x_j}{a_j} < 1 \mod 2\mathbb{Z}$

$$
(2)
\tau^-(S) = \text{number of } (n+1)-\text{tuples of integers } (x_0, \ldots, x_n), 0 < x_j < a_j,
$$

with $-1 < \sum_{j=0}^{n} \frac{x_j}{a_j} < 0 \mod 2\mathbb{Z}$.


REMARK. The intersection form $S$ of $V(a_0, \ldots, a_n)$ with $n \equiv 0 \mod 2$ is even, i.e. $S(x,x) \equiv 0 \mod 2$ for $x \in H_n(V)$. Therefore, by a well-known theorem, $\det S = \pm 1$ implies $\tau^+(S) - \tau^-(S) = \tau(S) \equiv 0 \mod 8$.

Following MILNOR we introduce for $a = (a_0, \ldots, a_n)$ the graph $\Gamma(a)$: $\Gamma(a)$ has the $(n+1)$ vertices $a_0, \ldots, a_n$. Two of them (say $a_i, a_j$) are joined by an edge if and only if the greatest common divisor $(a_i, a_j)$ is greater than 1. Then we have [5]

**Lemma.** $\det S$ as given in the preceding theorem equals $\pm 1$ if and only if $\Gamma(a)$ satisfies

a) $\Gamma(a)$ has at least two isolated points, or,

b) it has one isolated point and at least one connectedness component $K$ with an odd number of vertices such that $(a_i, a_j) = 2$ for $a_i, a_j \in K (i \neq j)$.

Now suppose $n$ even and $a = (a_0, \ldots, a_n) = (p, q, 2, \ldots, 2)$ with $p, q$ odd and $(p, q) = 1$. Then $\det S = \pm 1$ and
where \( N_{p,q} \) is the number of \( q \cdot x (1 \leq x \leq \frac{p-1}{2}) \) whose remainder \( \mod p \) of smallest absolute value is negative. This follows from the preceding theorem. Observe that by the above remark \( \tau(S) \) is divisible by 4 (even by 8) and that this is related to one of the proofs of the quadratic reciprocity law ([1], p. 450).

In particular, for \( n \) even and \((a_0, \ldots, a_n) = (3, 6k-1, 2, \ldots, 2)\) the signature \( \tau(S) \) equals \((-1)^{n/2} \cdot 8k\).

§ 3. Exotic spheres.

A \( k \)-dimensional compact oriented differentiable manifold is called a \( k \)-sphere if it is homeomorphic to the \( k \)-dimensional standard sphere. A \( k \)-sphere not diffeomorphic to the standard \( k \)-sphere is said to be exotic. The first exotic sphere was discovered by MILNOR in 1956. Two \( k \)-spheres are called equivalent if there exists an orientation preserving diffeomorphism between them. The equivalence classes of \( k \)-spheres constitute for \( k \geq 5 \) a finite abelian group \( \Theta_k \) under the connected sum operation. \( \Theta_k \) contains the subgroup \( bP_{k+1} \) of those \( k \)-spheres which bound a parallelizable manifold. \( bP_{4m} (m \geq 2) \) is cyclic of order

\[
2^{2m-2}(2^{2m-1} - 1) \text{ numerator } \frac{4B_m}{m},
\]

where \( B_m \) is the \( m \)-th Bernoulli number. Let \( g_m \) be a generator of \( bP_{4m} \). If a \((4m-1)\)-sphere \( \Sigma \) bounds a parallelizable manifold \( B \) of dimension \( 4m \), then the signature \( \tau(B) \) of the intersection form of \( B \) is divisible by 8 and

\[
(1) \quad \Sigma = + \frac{\tau(B)}{8} g_m
\]
(\mathcal{E}_m \text{ should be chosen in such a way that we have always the plus-sign in (1)).}

For \( m = 2 \) and \( 4 \) we have

\[
\begin{align*}
\mathbb{bP}_8 &= \Theta_7 = \mathbb{Z}_{28}, \\
\mathbb{bP}_{12} &= \Theta_{11} = \mathbb{Z}_{992}.
\end{align*}
\]

All these results are due to MILNOR-KERVAIRE. The group \( \mathbb{bP}_{2n} \) (\( n \text{ odd, } n \geq 3 \)) is either 0 or \( \mathbb{Z}_2 \). It contains only the standard sphere and the KERVAIRE sphere (obtained by plumbing two copies of the tangent bundle of \( S^n \)). It is known that \( \mathbb{bP}_{2n} \) is \( \mathbb{Z}_2 \) (equivalently that the KERVAIRE sphere is exotic) if \( n \equiv 1 \) mod 4 and \( n \equiv 5 \) (E. BROWN-F. PETERSON).

Let \( V^0 = V^0(a_0, a_1, \ldots, a_n) \subset \mathbb{C}^{n+1} \) (where \( a_j \geq 2 \)) be defined by

\[
z_0 + z_1 + \ldots + z_n = 0.
\]

This affine variety has exactly one singular point, namely the origin of \( \mathbb{C}^{n+1} \).

Let

\[
S^{2n+1} = \{ z \in \mathbb{C}^{n+1}, \sum_{j=0}^{n} z_j \overline{z}_j = 1 \}.
\]

Then \( \Sigma_a = \Sigma(a_0, \ldots, a_n) = V^0 \cap S^{2n+1} \) is a compact oriented differentiable manifold (without boundary) of dimension \( 2n-1 \).

**THEOREM.** Let \( n \equiv 3 \). Then \( \Sigma_a \) is \( (n-2) \)-connected. It is a \( (2n-1) \)-sphere if and only if the graph \( \Gamma(a) \) defined in \( \S \ 2 \) satisfies the condition a) or b). If \( \Sigma_a \) is a \( (2n-1) \)-sphere, then it belongs to \( \mathbb{bP}_{2n} \). If, moreover, \( n = 2m \), then

\[
\Sigma_a = \tau \mathcal{E}_m,
\]

where \( \tau = \tau^+ - \tau^- \) and \( \tau^+, \tau^- \) are as in \( \S \ 2 \ (2) \). In particular

\[
\begin{align*}
\sum_{i=0}^{2m} z_i \overline{z}_i &= 1, \\
z^3_0 + z^3_1 + z^2_2 + \ldots + z^2_{2m} &= 0.
\end{align*}
\]
is a \((4m-1)\)-sphere embedded in \(S^{4m+1} \subset \mathbb{CP}^{2m+1}\) which represents the element 
\((-1)^ {2m} k \cdot e \cdot \mathbb{bP}^{4m}\). Example: For \(m = 2\) and \(k = 1, \ldots, 28\) we get the 28 classes of 7-spheres, for \(m = 3\) and \(k = 1, \ldots, 992\) the 992 classes of 11-spheres.

**COROLLARY.** The affine variety \(V^0(a_0, \ldots, a_n), n \geq 3\), is a topological manifold if and only if the graph \(\Gamma(a)\) satisfies a) or b) of § 2.

For this theorem and for the case \(n\) odd see BRIESKORN [5].

**Proof.** If we remove from \(V^0_a\) the points with \(z_n = 0\) we get a space \(\tilde{V}_a\) whose fundamental group has \(\pi_1(V_a - \{0\}) \cong \pi_1(\Sigma_a)\) as homomorphic image. \(\tilde{V}_a\) is fibred over \(\mathbb{C}^*\) with \(V(a_0, \ldots, a_{n-1})\) as fibre which is simply-connected. Thus \(\pi_1(\tilde{V}_a) \cong \mathbb{Z}\) and \(\pi_1(\Sigma_a)\) is commutative. Because of this and by SMAL POINCARE we have to study only the homology of \(\Sigma_a\).

Let \(V_a \subset \mathbb{C}^{n+1}\) be the affine variety

\[
\begin{align*}
z_0 + z_1 + \ldots + z_n = \varepsilon
\end{align*}
\]

\((V_a = V^1_a)\). Let \(D^{2n+2}\) be the full ball in \(\mathbb{C}^{n+1}\) with center 0 and radius 1 and \(S^{2n+1}\), as before, its boundary. \(\Sigma_a\) is diffeomorphic to \(\Sigma^\varepsilon = S^{2n+1} \cap V^\varepsilon_a\) for \(\varepsilon > 0\) and small. It is the boundary of \(B^\varepsilon = D^{2n+2} \cap V^\varepsilon_a\) whose interior (for \(\varepsilon\) small) is diffeomorphic to \(V^\varepsilon_a\) and \(V_a\). The exact homology sequence of the pair \((B^\varepsilon_a, V^\varepsilon_a)\) shows that \(\Sigma_a\) is \((n-2)\)-connected.

Using POINCARE duality we get the exact sequence

\[
0 \rightarrow H_n(\Sigma_a) \rightarrow H_n(V_a) \xrightarrow{\sigma} \text{Hom}(H_n(V_a), \mathbb{Z}) \rightarrow H_{n-1}(\Sigma_a) \rightarrow 0
\]

where the homomorphism \(\sigma\) is given by the bilinear intersection form \(S\) of \(V_a\) (see § 2). This determines \(H^*(\Sigma_a)\) completely: \(H_n(\Sigma_a) = 0\) if and only
if \( \det S \neq 0 \). If \( \det S \neq 0 \), then \( |\det S| \) equals the order of \( H_{n-1}(\Sigma_a) \).

The manifold \( E^a \) is parallelizable since its normal bundle is trivial.

This finishes the proof in view of § 2.

§ 4. Manifolds with actions of the orthogonal group.

\( O(n) \) denotes the real orthogonal group with \( O(m) \subset O(n), m < n \), by

\[ A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}, (A \in O(m), 1 = \text{unit of } O(n-m)). \]

Let \( X \) be a compact differentiable manifold of dimension \( 2n-1 \) on which \( O(n) \) acts differentiably \( (n \geq 2) \). Suppose each isotropy group is conjugate to \( O(n-2) \) or \( O(n-1) \). Then the orbits are either Stiefel manifolds \( O(n)/O(n-2) \) (of dimension \( 2n-3 \)) or spheres \( O(n)/O(n-1) \) (of dimension \( n-1 \)). Suppose that the 2-dimensional representation of an isotropy group of type \( O(n-2) \) normal to the orbit is trivial whereas the \( n \)-dimensional representation of an isotropy group of type \( O(n-1) \) normal to the orbit is the 1-dimensional trivial representation plus the standard representation of \( O(n-1) \). Under these assumptions the orbit space is a compact 2-dimensional manifold \( X' \) with boundary, the interior points of \( X' \) corresponding to orbits of type \( O(n)/O(n-2) \), the boundary points of \( X' \) to the orbits of type \( O(n)/O(n-1) \). Suppose finally that \( X' \) is the 2-dimensional disk.

It follows from the classification theorems of [8] and [9] that the classes of manifolds \( X \) with the above properties under equivariant diffeomorphisms are in one-to-one correspondence with the non-negative integers. We let \( W^{2n-1}(d) \) be the \((2n-1)\)-dimensional \( O(n) \)-manifold corresponding to the integer \( d \geq 0 \). The fixed point set of \( O(n-2) \) in \( W^{2n-1}(d) \) is a 3-dimensional \( O(2) \)-manifold, namely \( W^3(d) \), which by ([9], § 5, Korollar 6) is the lens
Thus in order to determine the \(d\) associated to a given \(O(n)\)-manifold of our type we just have to look at the integral homology group \(H_1\) of the fixed point set of \(O(n-2)\). \(W^{2n-1}(0)\) is \(S^n \times S^{n-1}\), the manifold \(W^{2n-1}(1)\) is \(S^{2n-1}\), the actions of \(O(n)\) are easily constructed. Consider for \(d \geq 2\) the manifold \(\Sigma(d,2,\ldots,2)\) in \(\mathbb{C}^{n+1}\) given by

\[
\sum_{i=0}^{n} z_i \overline{z}_i = 1
\]

(see § 3). It is easy to check that this is an \(O(n)\)-manifold satisfying all our assumptions. The operation of \(A \in O(n)\) on \((z_0, z_1, \ldots, z_n)\) is, of course, given by applying the real orthogonal matrix \(A \in O(n)\) on the complex vector \((z_1, \ldots, z_n)\) leaving \(z_0\) untouched. The fixed point set of \(O(n-2)\) is \(\Sigma(d,2,2)\) which is \(L(d,1)\), see [6].

**THEOREM.** The \(O(n)\)-manifold \(\Sigma(d,2,\ldots,2)\) given by (1) is equivariantly diffeomorphic with \(W^{2n-1}(d)\), \(n \geq 2\). It can also be obtained by equivariant plumbing of \(d-1\) copies of the tangent bundle of \(S^n\) along the graph \(A_{d-1}\)

\[\text{---} \quad \ldots \quad \text{---} \quad d-1 \text{ vertices}\]

For the proof it suffices to establish the \(O(n)\)-action on the manifold obtained by plumbing and check all properties:

\(O(n)\) acts on \(S^n\) and on the unit tangent bundle of \(S^n\). Since the action of \(O(n)\) on \(S^n\) has two fixed points the plumbing can be done equivariantly. The fixed point set of \(O(n-2)\) is the manifold obtained by plumbing \(d-1\) tangent bundles of \(S^2\) which is well-known to be \(L(d,1)\), (see [6], resolution of the singularity of \(z_0^2 + z_1^2 + z_2^2 = 0\)).
The above theorem gives another method to calculate the homology of $\Sigma(d,2,\ldots,2)$ and to prove that $\Sigma(d,2,\ldots,2)$ for $d$ odd and an odd number of $2$'s is a sphere. In particular, $\Sigma(3,2,2,2,2,2)$ is the exotic 9-dimensional Kervaire sphere (see § 3). The calculation of the ARF invariant of the $A_{d-1}$-plumbing shows more generally that

$$\Sigma(d,2,\ldots,2), \quad (d \text{ odd, an odd number of } 2\text{'s})$$

is the standard sphere for $d \equiv +1 \mod 8$ and the Kervaire sphere for $d \equiv +3 \mod 8$, in agreement with a more general result in [5].

**Remarks.** The $O(n)$-manifold $W^{2n-1}(d)$ coincides with Bredon's manifolds $M_k^{2n-1}$ for $d = 2k+1$, see Bredon [3]. $\Sigma(3,2,2,2)$ is the standard 5-sphere (since $\Theta_5 = 0$). Therefore $S^5$ admits a differentiable involution $\alpha$ with the lens space $L(3,1)$ as fixed point set and a diffeomorphism $\beta$ of period 3 with the real projective 3-space as fixed point set. Compare [3].

$\alpha$ and $\beta$ are defined on $\Sigma(3,2,2,2)$ given by (1) as follows

$$\alpha(z_0, z_1, z_2, z_3) = (z_0, z_1, z_2, -z_3)$$

$$\beta(z_0, z_1, z_2, z_3) = (\varepsilon z_0, z_1, z_2, z_3), \quad \text{where } \varepsilon = \exp(2\pi i/3).$$

Many more such examples of "exotic" involutions etc. which are not differentiably equivalent to orthogonal involutions etc. can be constructed.

§ 5. Manifolds associated to knots.

Let $X$ be a compact differentiable manifold of dimension $2n-1$ on which $O(n-1)$ acts differentiably ($n \geq 3$). Suppose each isotropy group is conjugate to $O(n-3)$ or $O(n-2)$ or is $O(n-1)$. Then the orbits are either Stiefel manifolds $O(n-1)/O(n-3)$ (of dimension $2n-5$) or spheres $O(n-1)/O(n-2)$ (of dimension $n-2$) or points (fixed points of the whole action). The
representations of the isotropy groups $O(n-3), O(n-2)$ and $O(n-1)$ respectively normal to the orbit are supposed to be the 4-dimensional trivial representation, the 3-dimensional trivial plus the standard representation of $O(n-2)$, the 1-dimensional trivial plus the sum of two copies of the standard representation of $O(n-1)$. The orbit space $X'$ is then a 4-dimensional manifold with boundary. We suppose that $X'$ is the 4-dimensional disk $D^4$.

Then the points of the interior of $D^4$ correspond to Stiefel-manifold-orbits, the points of $\partial D^4 = S^3$ to the other orbits. The set $F$ of fixed points corresponds to a 1-dimensional submanifold of $S^3$, also called $F$.

We suppose $F$ non-empty and connected, it is then a knot in $S^3$. We shall call an $O(n-1)$-manifold of dimension $2n-1$ a "knot manifold" if all the above conditions are satisfied.

Let $K$ be the set of isomorphism classes of differentiable knots (i.e. isomorphism classes of pairs $(S^3, F) - F$ a compact connected 1-dimensional submanifold - under diffeomorphisms of $S^3$). For the following theorem see JÄNICH ([9], § 6), compare also W.C. HSIANG and W.Y. HSIANG [8].

**THEOREM.** For any $n \geq 3$ there is a one-to-one correspondence

$$\kappa_n : K \rightarrow \&_{2n-1}$$

where $\&_{2n-1}$ is the set of isomorphism classes of $(2n-1)$-dimensional knot manifolds under equivariant diffeomorphisms. $\kappa_n^{-1}$ associates to a knot manifold the knot $F$ considered above.

**REMARK.** The 2-fold branched covering of $S^3$ along a knot $F$ is an $O(1)$-manifold which will be denoted by $\kappa_2(F)$. 


If we plumb 8 copies of the tangent bundles of $S^n$ ($n \geq 1$) according to the tree $E_8$, we get a $(2n-1)$-dimensional manifold $M^{2n-1}(E_8)$. For $n=2$ this is $S^3/G$, where $G$ is the binary pentagonal dodecahedral group [6]. For $n$ odd, $M^{2n-1}(E_8)$ is the standard sphere, as the ARF invariant shows. For $n = 2m+4$, the manifold $M^{4m-1}(E_8)$ is an exotic sphere, it is the famous Milnor sphere which represents the generator $\pm e_m$ of $bP_{4m}$ (see §3).

$M^{2n-1}(E_8)$ admits an action of $O(n-1)$ as follows: $O(n-1)$ operates as subgroup of $O(n+1)$ on $S^n$ and thus on the unit tangent bundle of $S^n$. The action on $S^n$ leaves a great circle fixed.

When plumbing the eight copies of the tangent bundle we put the center of the plumbing operation always on this great circle; (for one copy, corresponding to the central vertex of the $E_8$-tree, we need three such centers, therefore, we cannot have an action of $O(n)$, which has only 2 fixed points on $S^n$.) Then the action of $O(n-1)$ on each copy of the tangent bundle is compatible with the plumbing and extends to an action of $O(n-1)$ on $M^{2n-1}(E_8)$ which, for $n \geq 3$, becomes a knot manifold as can be checked. The resulting knot can be seen on a picture attached at the end of this lecture. The speaker had convinced himself that this is the torus knot $t(3,5)$, but Zieschang and Vogt showed him a better proof. This implies the

**Theorem.** Suppose $n \geq 3$. Then $\kappa_n(t(3,5))$ is equivariantly diffeomorphic to $M^{2n-1}(E_8)$ with the $O(n-1)$-action defined by equivariant plumbing. (This is still true for $n=2$, see Remark above).
We now consider the manifold $\Sigma(p,q,2,2,\ldots,2) \subset \mathbb{C}^{n+1}$ given by the equations (see § 3)

$$z_0^p + z_1^q + z_2^2 + \ldots + z_n^2 = 0$$

$$\sum_{i=0}^n z_i \overline{z}_i = 1 \quad (n \geq 3).$$

This is an $O(n-1)$-manifold, the action being defined similarly as in § 4.

Suppose $(p,q) = 1$. Then it can be shown that $\Sigma(p,q,2,2,\ldots,2)$ is a knot manifold: It is $\kappa_n(t(p,q))$ where $t(p,q)$ is the torus knot. Therefore, by the preceding theorem we have an equivariant diffeomorphism

$$\mathbb{R}^{2n-1}(E) \cong \Sigma(3,5,2,\ldots,2).$$

This gives a different proof (based on the classification of knot manifolds) that $\Sigma(3,5,2,\ldots,2)$ represents for $m \equiv 2 \mod 4$ a generator of $bP_{4m}$.

(see § 3).

§ 6. A theorem on knot manifolds.

Let $F$ be a knot in $S^3$. Then the signature $\tau(F)$ can be defined in the following way which MILNOR explained to the speaker in a letter. MILNOR also considers higher dimensional cases. We cite from his letter, but restrict to classical knots:

Let $X$ be the complement of an open tubular neighbourhood of $F$ in $S^3$.

Then the cohomology

$$H^* = H^*(\hat{X}, \partial \hat{X}; \mathbb{R})$$
where $\hat{X}$ is the infinite cyclic covering of $X$, satisfies Poincaré duality just as if $\hat{X}$ were a 2-dimensional manifold bounded by $F$.

In particular the pairing

$$U : H^1 \otimes H^1 \to H^2 \cong \mathbb{R}$$

is non-degenerate. Let $t$ denote a generator for the group of covering transformations of $\hat{X}$. Then for $a, b \in H^1$ the pairing

$$<a, b> = a \cup t^* b + b \cup t^* a$$

is symmetric and non-degenerate. Hence, the signature

$$\tau^+(F) - \tau^-(F) = \tau(F)$$

is defined.

There exist earlier definitions of the signature by MURASUGI [13] and TROTTER [17]. The signature is a cobordism invariant of the knot. A cobordism invariant mod 2 was introduced by ROBERTELLO [15] inspired by an earlier paper of KERVAIRE-MILNOR. Let $F$ be a knot and $\Delta$ its Alexander polynomial, then the ROBERTELLO invariant $c(F)$ is an integer mod 2, namely

$$c(F) = 0, \text{ if } \Delta(-1) \equiv \pm 1 \text{ mod } 8$$
$$c(F) = 1, \text{ if } \Delta(-1) \equiv \pm 3 \text{ mod } 8$$

We recall that the first integral homology group of $\kappa_2(F)$, the 2-fold branched covering of the knot $F$ (see a remark in § 5), is always finite, its order is odd, and equals up to sign the determinant of $F$. We have $+ \det F = \Delta(-1)$.

**THEOREM.** Let $F$ be a knot, then $\kappa_n(F)$, $n \geq 2$, is the boundary of a parallelizable manifold. For $n$ odd, $\kappa_n(F)$ is homeomorphic to $S^{2n-1}$ and thus represents an element of $bP_{2n}$, it is the standard sphere if
c(F) = 0, the KERVAIRE sphere if c(F) = 1. If n = 2m, then κ_{2m}(F) is (2m-2)-connected and \( H_{2m-1}(\kappa_{2m}(F),\mathbb{Z}) \sim H_1(\kappa_2(F),\mathbb{Z}) \). For m ≥ 2 it is homeomorphic to \( S^{4m-1} \) if and only if \( \det F = \pm 1 \). Then \( \kappa_{2m}(F) \) represents (up to sign) an element of \( bP_{4m} \) which is \( \pm \frac{r(F)}{8} \cdot g_m \) (see § 3).

The proof uses an equivariant handlebody construction starting out from a Seifert surface [16] spanned in the knot F. For simplicity, not out of necessity, we have disregarded orientation questions in § 5 and § 6.

REMARK. § 2(3) gives up to sign a formula for the signature of the torus knot \( t(p,q) \), \( (p,q \) odd with \( (p,q) = 1) \).


ERRATUM

Page 314-07. Ligne 4 du bas, au lieu de "Let $g_m$ be a generator of $bP_{4m}$." lire: "Let $g_m$ be the Milnor generator of $bP_{4m}$, see p. 314-14."