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Singularities and exotic spheres


<http://www.numdam.org/item?id=SB_1966-1968__10__13_0>
BRIESKORN has proved [4] that the n-dimensional affine algebraic variety \( z_0^2 + z_1^2 + \ldots + z_n^2 = 0 \) (n odd, \( n \equiv 1 \)) is a topological manifold though the variety has an isolated singular point (which is normal for \( n \equiv 2 \)). Such a phenomenon cannot occur for normal singularities of 2-dimensional varieties, as was shown by MUMFORD ([12], [6]). BRIESKORN's result stimulated further research on the topology of isolated singularities (BRIESKORN [5], MILNOR [11] and the speaker [5], [7]). BRIESKORN [5] uses the paper of F. PHAM [14], whereas the speaker studied certain singularities from the point of view of transformation groups using results of BREDON ([2], [3]), W.C. HSIANG and W.Y. HSIANG [8] and JÄNICH [9].

§ 1. The integral homology of some affine hypersurfaces.

PHAM [14] studies the non-singular subvariety \( V_a = V(a_0, a_1, \ldots, a_n) \) of \( \mathbb{C}^{n+1} \) given by

\[
\frac{a_0}{z_0} + \frac{a_1}{z_1} + \ldots + \frac{a_n}{z_n} = 1 \quad (n \equiv 0),
\]

where \( a = (a_0, \ldots, a_n) \) consists of integers \( a_j \geq 2 \).

Let \( G_{a_j} \) be the cyclic group of order \( a_j \) multiplicatively written and generated by \( w_j \). Define the group \( G_a = G_{a_0} \times G_{a_1} \times \ldots \times G_{a_n} \) and put \( \epsilon_j = \exp(2\pi i/a_j) \).
Then \( k_0 k_1 k_n \) is an element of \( G_a \) whereas \( \varepsilon_0 \varepsilon_1 \varepsilon_n \) is a complex number. \( G_a \) operates on \( V_a \) by

\[
(\varepsilon_0 z_0, \ldots, \varepsilon_n z_n).
\]

Let \( \hat{G}_{a_j} \) be the group of \( a_j \)-th roots of unity and \( x \mapsto \hat{x} \) the isomorphism \( G_{a_j} \rightarrow \hat{G}_{a_j} \) given by \( v_j \mapsto \varepsilon_j = \hat{w}_j \).

PHAM considers the following subspace \( U_a \) of \( V_a \)

\[
U_a = \{ z \in V_a \text{ and } z_j \text{ real } \geq 0 \text{ for } j = 0, \ldots, n \}
\]

LEMMA. - The subspace \( U_a \) is a deformation retract of \( V_a \) by a deformation compatible with the operations of \( G_a \).

For the proof see PHAM [14], p. 338.

\( U_a \) can also be described by the conditions

\[
z_j = u_j |z_j| \quad \text{with} \quad u_j \in \hat{G}_{a_j} \quad (j = 0, \ldots, n).
\]

Put \( |z_j|^{a_j} = t_j \). Then \( U_a \) becomes the space of \((n+1)\)-tpls of complex numbers

\[
t_0 u_0 \oplus t_1 u_1 \oplus \ldots \oplus t_n u_n
\]

with

\[
u_j \in \hat{G}_{a_j}, \quad t_j \geq 0, \quad \Sigma_{j=0}^n t_j = 1
\]

Thus \( U_a \) can be identified with the join \( G_{a_0} * G_{a_1} * \ldots * G_{a_n} \) of the finite sets \( G_{a_j} \) (see MILNOR [10]).

LEMMA 2.1 in [10] states in particular that the reduced integral homology groups of the join \( A * B \) of two spaces \( A, B \) without torsion are given by a canonical isomorphism

\[
\tilde{H}_{r+1}(A * B) \cong \sum_{i+j=r} \tilde{H}_i(A) \otimes \tilde{H}_j(B),
\]
whereas LEMMA 2.2 in [10] shows that $A * B$ is simply connected provided $B$ is arcwise connected and $A$ is any non-vacuous space. These properties of the join together with its associativity imply

**THEOREM.** The subvariety $V_a$ of $C^{n+1}$ is $(n-1)$-connected. Moreover

\[
\tilde{H}_n(V_a) \cong \tilde{H}_0(G_a_n) \otimes \tilde{H}_0(G_{a_1}) \otimes \cdots \otimes \tilde{H}_0(G_{a_0}).
\]

This is a free abelian group of rank $r = \prod (a_j - 1)$.

The isomorphism (1) is compatible with the operations of $G_a$. All other reduced integral homology groups of $V_a$ vanish.

It can be shown that $V_a$ has the homotopy type of a connected union $S^n \vee \cdots \vee S^n$ of $r$ spheres of dimension $n$.

The identification of $U_a$ with a join was explained to the speaker by MILNOR.

$U_a = G_{a_0} * G_{a_1} * \cdots * G_{a_n}$ is an $n$-dimensional simplicial complex which has an $n$-simplex for each element of $G_a$. The $n$-simplex belonging to the unit of $G_a$ is denoted by $e$. All other $n$-simplices are obtained from $e$ by operations of $G_a$. Thus we have for the $n$-dimensional simplicial chain group

\[
C_n(U_a) = J_a e
\]

where $J_a$ is the group ring of $G_a$. The homology group $\tilde{H}_n(U_a) = \tilde{H}_n(V_a)$ is an additive subgroup of $J_a e = C_n(U_a) \cong J_a$.

The face operator $\partial_j$ commutes with all operations of $G_a$ on $C_n(U_a)$ and furthermore satisfies $\partial_j = w_j \partial_j$. Therefore

\[
h = (1 - w_0)(1 - w_1) \cdots (1 - w_n) e
\]

is a cycle. Thus $h \in \tilde{H}_n(U_a)$. It follows easily that $\tilde{H}_n(V_a) = J_a h$. This yields the
THEOREM. The map \( w \rightarrow \text{wh}(w G_a) \) induces an isomorphism

\[
J_a / I_a \cong \tilde{H}_n(V_a) = J_a h
\]

where \( I_a \subset J_a \) is the annihilator ideal of \( h \) which is generated by the elements

\[
1 + w_j + w_j^2 + \ldots + w_j^{a_j - 1}, \quad (j = 0, \ldots, n).
\]

Therefore \( w_0 w_1 \ldots w_n \in H_n(V_a) \) (where \( 0 \leq k_j \leq a_j - 2, \quad j = 0, \ldots, n \)) is a basis of \( \tilde{H}_n(V_a) \).

We recall that \( \tilde{H}_n(V_a) \) is the integral singular homology group (of course with compact support). \( V_a \) is a 2n-dimensional oriented manifold without boundary (non-compact for \( n \geq 1 \)). Therefore the bilinear intersection form \( S \) is well defined over \( \tilde{H}_n(V_a) \). It is symmetric for \( n \) even, skew-symmetric for \( n \) odd. It is compatible with the operations of \( G_a \).

Pham ([14], P.358) constructs an \( n \)-dimensional cycle \( \tilde{h} \) in \( V_a \) which is homologous to \( h \) and intersects \( U_a \) exactly in two interior points of the simplices \( e \) and \( w_o w_1 \ldots w_n e \) (sign questions have to be observed). In this way he obtains (using the \( G_a \)-invariance of \( S \)) the following result, reformulated somewhat for our purposes.

THEOREM. Put \( \bar{\eta} = (1-w_o) \ldots (1-w_n) \). The bilinear form \( S \) over \( J_a \bar{\eta} \cong \tilde{H}_n(V_a) \) is given by

\[
S(x\bar{\eta}, y\bar{\eta}) = f(\bar{y} x\bar{\eta}), \quad (x, y \in J_a),
\]

where \( f : J_a \rightarrow \mathbb{Z} \) is the additive homomorphism with

\[
f(1) = -f(w_o \ldots w_n) = (-1)^{n(n-1)/2},
\]

\[
f(w) = 0 \quad \text{for} \quad w \in G_a, \quad w \neq 1, \quad w \neq w_o \ldots w_n,
\]

and where \( y \mapsto \bar{y} \) is the ring automorphism of the group ring \( J_a \) induced by \( w \mapsto w^{-1} (w G_a) \).
§ 2. The quadratic form of $V_a$.

Let $G$ be a finite abelian group, $J(G)$ its group ring. The ring automorphism of $J(G)$ induced by $g \mapsto g^{-1}$ $(g \in G)$ is denoted by $x \mapsto x^{\#} (x \in J(G))$. Give an element $\eta \in J(G)$ and a function $f : G \to \mathbb{Z}$.

The additive homomorphism $J(G) \to \mathbb{Z}$ induced by $f$ is also called $f$. Put $\hat{f} = \sum_{w \in G} f(w)w$. We assume

a) $f(x\eta) = f(x\hat{\eta})$ for all $x \in J(G)$, [equivalently $\hat{f}\eta = \hat{f}\eta$]

or

b) $f(x\eta) = -f(x\hat{\eta})$ for all $x \in J(G)$, [equivalently $\hat{f}\eta = -\hat{f}\eta$].

The bilinear form $S$ over the lattice $J(G)\eta$ defined by

$$S(x\eta, y\eta) = f(\overline{yx}\eta), \quad (x, y \in J(G)),$$

is symmetric in case a), skew symmetric in case b). Since $S$ is a form with integral coefficients, its determinant is well-defined. The signature

$$\tau(S) = \tau^+(S) - \tau^-(S), \text{ case a),}$$

is the number $\tau^+(S)$ of positive minus the number $\tau^-(S)$ of negative diagonal entries in a diagonalisation of $S$ over $\mathbb{R}$. Let $\chi$ run through the characters of $G$.

**LEMMA.** With the preceding assumptions

$$\pm \det S = \prod_{\chi(\eta) \neq 0} \chi(\hat{f}) \cdot \text{order of the torsion subgroup of } J(G)/J(G)\eta$$

and in case a)

$$\tau^+(S) = \text{number of characters } \chi \text{ with } \chi(\hat{f}\eta) > 0$$

$$\tau^-(S) = \text{number of characters } \chi \text{ with } \chi(\hat{f}\eta) < 0.$$

The proof is an exercise as in [1], p. 444.

The lemma and the last theorem of § 1 imply for the affine hypersurface

$$V_a = V(a_0, \ldots, a_n)$$

the
THEOREM. Let $S$ be the intersection form of $V_a$. Then

$$
\pm \det S = \prod_{1 \leq k \leq n, \varepsilon_j=1}^{k_0, k_1, \ldots, k_n} (1-\varepsilon_0 \varepsilon_1 \cdots \varepsilon_n)
$$

where $\varepsilon_j = \exp(2\pi i/a_j)$. For $n$ even, we have

$$
\tau^+(S) = \text{number of (n+1)}-\text{tups of integers } (x_0, \ldots, x_n), 0 < x_j < a_j,
$$

with $0 < \sum_{j=0}^{n} \frac{x_j}{a_j} < 1 \mod 2\mathbb{Z}$

$$
(2) \quad \tau^-(S) = \text{number of (n+1)}-\text{tups of integers } (x_0, \ldots, x_n), 0 < x_j < a_j,
$$

with $-1 < \sum_{j=0}^{n} \frac{x_j}{a_j} < 0 \mod 2\mathbb{Z}$.


REMARK. The intersection form $S$ of $V(a_0, \ldots, a_n)$ with $n \equiv 0 \mod 2$ is even, i.e. $S(x,x) \equiv 0 \mod 2$ for $x \in \tilde{\mathbb{H}}_n(V)$. Therefore, by a well-known theorem, $\det S = \pm 1$ implies $\tau^+(S) - \tau^-(S) = \tau(S) \equiv 0 \mod 8$.

Following MILNOR we introduce for $a = (a_0, \ldots, a_n)$ the graph $\Gamma(a)$: $\Gamma(a)$ has the $(n+1)$ vertices $a_0, \ldots, a_n$. Two of them (say $a_i, a_j$) are joined by an edge if and only if the greatest common divisor $(a_i, a_j)$ is greater than 1. Then we have [5]

**LEMMA.** $\det S$ as given in the preceding theorem equals $\pm 1$ if and only if $\Gamma(a)$ satisfies

a) $\Gamma(a)$ has at least two isolated points, or,

b) it has one isolated point and at least one connectedness component $K$ with an odd number of vertices such that $(a_i, a_j) = 2$ for $a_i, a_j \in K (i \neq j)$.

Now suppose $n$ even and $a = (a_0, \ldots, a_n) = (p,q,2,\ldots, 2)$ with $p$, $q$ odd and $(p,q) = 1$. Then $\det S = \pm 1$ and
where \( N_{p,q} \) is the number of \( q \cdot x \) whose remainder mod \( p \) of smallest absolute value is negative. This follows from the preceding theorem. Observe that by the above remark \( \tau(S) \) is divisible by 4 (even by 8) and that this is related to one of the proofs of the quadratic reciprocity law ([1], p. 450).

In particular, for \( n \) even and \( (a_0, ..., a_n) = (3, 6k-1, 2, ..., 2) \) the signature \( \tau(S) \) equals \((-1)^{n/2} \cdot 8k \).

§ 3. Exotic spheres.

A \( k \)-dimensional compact oriented differentiable manifold is called a \( k \)-sphere if it is homeomorphic to the \( k \)-dimensional standard sphere. A \( k \)-sphere not diffeomorphic to the standard \( k \)-sphere is said to be exotic. The first exotic sphere was discovered by MILNOR in 1956. Two \( k \)-spheres are called equivalent if there exists an orientation preserving diffeomorphism between them. The equivalence classes of \( k \)-spheres constitute for \( k \geq 5 \) a finite abelian group \( \Theta_k \) under the connected sum operation. \( \Theta_k \) contains the subgroup \( bP_{k+1} \) of those \( k \)-spheres which bound a parallelizable manifold. \( bP_{4m} \) is cyclic of order

\[
2^{2m-2}(2^{2m-1} - 1) \text{ numerator } \left( \frac{4B_m}{m} \right),
\]

where \( B_m \) is the \( m \)-th Bernoulli number. Let \( g_m \) be a generator of \( bP_{4m} \).

If a \((4m-1)\)-sphere \( \Sigma \) bounds a parallelizable manifold \( B \) of dimension \( 4m \), then the signature \( \tau(B) \) of the intersection form of \( B \) is divisible by 8 and

\[
\Sigma = + \frac{\tau(B)}{8} g_m
\]

(1)
For $m = 2$ and $4$ we have
\[ b^P_8 = \Theta_7 = \mathbb{Z}_{28}, \quad b^P_{12} = \Theta_{11} = \mathbb{Z}_{292}. \]

All these results are due to MILNOR-KERVAIRE. The group $bP_{2n}$ ($n$ odd, $n \geq 3$) is either $0$ or $\mathbb{Z}_2$. It contains only the standard sphere and the KERVAIRE sphere (obtained by plumbing two copies of the tangent bundle of $S^n$). It is known that $bP_{2n}$ is $\mathbb{Z}_2$ (equivalently that the KERVAIRE sphere is exotic) if $n = 1 \mod 4$ and $n \geq 5$ (E. BROWN-F. PETERSON).

Let $V_0^\circ = V^\circ(a_0, a_1, \ldots, a_n) \subset \mathbb{C}^{n+1}$ (where $a_j \geq 2$) be defined by
\[ a_0 + a_1z_1 + \cdots + a_n z_n = 0. \]

This affine variety has exactly one singular point, namely the origin of $\mathbb{C}^{n+1}$.

Let $S^{2n+1} = \{ z | z \in \mathbb{C}^{n+1}, \sum_{j=0}^{n} z_j \overline{z}_j = 1 \}$.

Then $\Sigma_a = \Sigma(a_0, \ldots, a_n) = V_0^\circ \cap S^{2n+1}$ is a compact oriented differentiable manifold (without boundary) of dimension $2n-1$.

**THEOREM.** Let $n \geq 3$. Then $\Sigma_a$ is $(n-2)$-connected. It is a $(2n-1)$-sphere if and only if the graph $\Gamma(a)$ defined in § 2 satisfies the condition a) or b). If $\Sigma_a$ is a $(2n-1)$-sphere, then it belongs to $bP_{2n}$. If, moreover, $n = 2m$, then
\[ \Sigma_a = \tau \otimes \mathcal{G}_m, \]
where $\tau = \tau^+ - \tau^-$ and $\tau^+, \tau^-$ are as in § 2 (2). In particular
\[ \sum_{i=0}^{2m} z_i \overline{z}_i = 1 \]
\[ z_0 + z_1 z_1 + z_2 + \cdots + z_{2m} = 0 \]
is a \((4m-1)\)-sphere embedded in \(S^{4m+1} \subset \mathbb{C}^{2m+1}\) which represents the element
\((-1)^mk, g_m b_{4m}\). Example: For \(m = 2\) and \(k = 1, \ldots, 28\) we get the 28
classes of 7-spheres, for \(m = 3\) and \(k = 1, \ldots, 992\) the 992 classes of
11-spheres.

**COROLLARY.** The affine variety \(V^0(a_0, \ldots, a_n)\), \(n \geq 3\), is a topological
manifold if and only if the graph \(\Gamma(a)\) satisfies a) or b) of § 2.

For this theorem and for the case \(n\) odd see BRIESKORN [5].

**Proof.** If we remove from \(V^0\) the points with \(z_n = 0\) we get a space \(\tilde{V}_a\)
whose fundamental group has \(\pi_1(V_a - \{0\}) \cong \pi_1(\Sigma_a)\) as homomorphic image.
\(\tilde{V}_a\) is fibred over \(\mathbb{C}^*\) with \(V(a_0, \ldots, a_{n-1})\) as fibre which is simply-connected.
Thus \(\pi_1(\Sigma_a) \cong \mathbb{Z}\) and \(\pi_1(\Sigma_a)\) is commutative. Because of this and by SMALE-
POINCARE we have to study only the homology of \(\Sigma_a\).

Let \(V^\varepsilon_a \subset \mathbb{C}^{n+1}\) be the affine variety
\[
\begin{align*}
    z_0 + z_1 + \ldots + z_n &= \varepsilon \\
    (V_a = V^1_a).
\end{align*}
\]
(\(V_a = V^1_a\)). Let \(D^{2n+2}\) be the full ball in \(\mathbb{C}^{n+1}\) with center 0 and
radius 1 and \(S^{2n+1}\), as before, its boundary. \(\Sigma_a\) is diffeomorphic to
\(\Sigma^\varepsilon_a = S^{2n+1} \cap V^\varepsilon_a\) for \(\varepsilon > 0\) and small. It is the boundary of \(B^\varepsilon_a = D^{2n+2} \cap V^\varepsilon_a\)
whose interior (for \(\varepsilon\) small) is diffeomorphic to \(V^\varepsilon_a\) and \(V_a\). The exact
homology sequence of the pair \((B^\varepsilon_a, V^\varepsilon_a)\) shows that \(\Sigma_a\) is \((n-2)\)-connected.

Using POINCARE duality we get the exact sequence
\[
0 \to H_n(\Sigma_a) \to H_n(V_a) \to \text{Hom}(H_n(V_a), \mathbb{Z}) \to H_{n-1}(\Sigma_a) \to 0
\]
where the homomorphism \(\sigma\) is given by the bilinear intersection form \(S\) of
\(V_a\) (see § 2). This determines \(H^*(\Sigma_a)\) completely: \(H_n(\Sigma_a) = 0\) if and only
if \( \det S \neq 0 \). If \( \det S \neq 0 \), then \( |\det S| \) equals the order of \( H_{n-1}(\Sigma_a) \).

The manifold \( E_a \) is parallelizable since its normal bundle is trivial.

This finishes the proof in view of § 2.

§ 4. Manifolds with actions of the orthogonal group.

\( O(n) \) denotes the real orthogonal group with \( O(m) \subset O(n) \), \( m < n \), by

\[
A \mapsto \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}, \quad (A \in O(m), \text{ 1=unit of } O(n-m)).
\]

Let \( X \) be a compact differentiable manifold of dimension \( 2n-1 \) on which \( O(n) \) acts differentiably \((n \geq 2)\). Suppose each isotropy group is conjugate to \( O(n-2) \) or \( O(n-1) \). Then the orbits are either Stiefel manifolds \( O(n)/O(n-2) \) (of dimension \( 2n-3 \)) or spheres \( O(n)/O(n-1) \) (of dimension \( n-1 \)). Suppose that the 2-dimensional representation of an isotropy group of type \( O(n-2) \) normal to the orbit is trivial whereas the \( n \)-dimensional representation of an isotropy group of type \( O(n-1) \) normal to the orbit is the 1-dimensional trivial representation plus the standard representation of \( O(n-1) \). Under these assumptions the orbit space is a compact 2-dimensional manifold \( X' \) with boundary, the interior points of \( X' \) corresponding to orbits of type \( O(n)/O(n-2) \), the boundary points of \( X' \) to the orbits of type \( O(n)/O(n-1) \). Suppose finally that \( X' \) is the 2-dimensional disk.

It follows from the classification theorems of [8] and [9] that the classes of manifolds \( X \) with the above properties under equivariant diffeomorphisms are in one-to-one correspondence with the non-negative integers. We let \( W^{2n-1}(d) \) be the \((2n-1)\)-dimensional \( O(n) \)-manifold corresponding to the integer \( d \geq 0 \). The fixed point set of \( O(n-2) \) in \( W^{2n-1}(d) \) is a 3-dimensional \( O(2) \)-manifold, namely \( W^3(d) \), which by ([9], § 5, Korollar 6) is the lens
Thus in order to determine the $d$ associated to a given $O(n)$-manifold of our type we just have to look at the integral homology group $H_1$ of the fixed point set of $O(n-2)$. $W^{2n-1}(0)$ is $S^n \times S^{n-1}$, the manifold $W^{2n-1}(1)$ is $S^{2n-1}$, the actions of $O(n)$ are easily constructed. Consider for $d \geq 2$ the manifold $\Sigma(d,2,\ldots,2)$ in $\mathbb{C}^{n+1}$ given by

$$z_0^d + z_1^2 + \ldots + z_n^2 = 0$$

(see § 3). It is easy to check that this is an $O(n)$-manifold satisfying all our assumptions. The operation of $A \in O(n)$ on $(z_0, z_1, \ldots, z_n)$ is, of course, given by applying the real orthogonal matrix $A \in O(n)$ on the complex vector $(z_1, \ldots, z_n)$ leaving $z_0$ untouched. The fixed point set of $O(n-2)$ is $\Sigma(d,2,2)$ which is $L(d,1)$, see [6].

**Theorem.** The $O(n)$-manifold $\Sigma(d,2,\ldots,2)$ given by (1) is equivariantly diffeomorphic with $W^{2n-1}(d)$, $n \geq 2$. It can also be obtained by equivariant plumbing of $d-1$ copies of the tangent bundle of $S^n$ along the graph $A_{d-1}$ vertices.

For the proof it suffices to establish the $O(n)$-action on the manifold obtained by plumbing and check all properties:

$O(n)$ acts on $S^n$ and on the unit tangent bundle of $S^n$. Since the action of $O(n)$ on $S^n$ has two fixed points the plumbing can be done equivariantly. The fixed point set of $O(n-2)$ is the manifold obtained by plumbing $d-1$ tangent bundles of $S^2$ which is well-known to be $L(d,1)$, (see [6], resolution of the singularity of $z_0^d + z_1^2 + z_2^2 = 0$).
The above theorem gives another method to calculate the homology of
$\Sigma(d,2,\ldots,2)$ and to prove that $\Sigma(d,2,\ldots,2)$ for $d$ odd and an odd number
of 2's is a sphere. In particular, $\Sigma(3,2,2,2,2,2)$ is the exotic
9-dimensional KERVAIRE sphere (see § 3). The calculation of the ARF invariant
of the $A_{d-1}$-plumbing shows more generally that

$$\Sigma(d,2,\ldots,2), \quad (d \text{ odd, an odd number of 2's})$$

is the standard sphere for $d \equiv \pm 1 \mod 8$ and the KERVAIRE sphere for
$d \equiv \pm 3 \mod 8$, in agreement with a more general result in [5].

REMARKS. The $O(n)$-manifold $W^{2n-1}(d)$ coincides with BREDON's manifolds
$M^{2n-1}_k$ for $d = 2k+1$, see BREDON [3]. $\Sigma(3,2,2,2)$ is the standard 5-sphere
(since $\theta_5 = 0$). Therefore $S^5$ admits a differentiable involution $\alpha$ with
the lens space $L(3,1)$ as fixed point set and a diffeomorphism $\beta$ of period
3 with the real projective 3-space as fixed point set. Compare [3].

$\alpha$ and $\beta$ are defined on $\Sigma(3,2,2,2)$ given by (1) as follows

$$\alpha(z_0,z_1,z_2,z_3) = (z_0,z_1,z_2,-z_3)$$

$$\beta(z_0,z_1,z_2,z_3) = (\varepsilon z_0,z_1,z_2,z_3), \quad \text{where } \varepsilon = \exp(2\pi i/3).$$

Many more such examples of "exotic" involutions etc. which are not
differentially equivalent to orthogonal involutions etc. can be constructed.

§ 5. Manifolds associated to knots.

Let $X$ be a compact differentiable manifold of dimension $2n-1$ on which
$O(n-1)$ acts differentiably ($n \geq 3$). Suppose each isotropy group is conjugate
to $O(n-3)$ or $O(n-2)$ or is $O(n-1)$. Then the orbits are either Stiefel
manifolds $O(n-1)/O(n-3)$ (of dimension $2n-5$) or spheres $O(n-1)/O(n-2)$ (of
dimension $n-2$) or points (fixed points of the whole action). The
representations of the isotropy groups $O(n-3)$, $O(n-2)$ and $O(n-1)$ respectively normal to the orbit are supposed to be the 4-dimensional trivial representation, the 3-dimensional trivial plus the standard representation of $O(n-2)$, the 1-dimensional trivial plus the sum of two copies of the standard representation of $O(n-1)$. The orbit space $X'$ is then a 4-dimensional manifold with boundary. We suppose that $X'$ is the 4-dimensional disk $D^4$.

Then the points of the interior of $D^4$ correspond to Stiefel-manifold-orbits, the points of $\partial D^4 = S^3$ to the other orbits. The set $F$ of fixed points corresponds to a 1-dimensional submanifold of $S^3$, also called $F$.

We suppose $F$ non-empty and connected, it is then a knot in $S^3$. We shall call an $O(n-1)$-manifold of dimension $2n-1$ a "knot manifold" if all the above conditions are satisfied.

Let $K$ be the set of isomorphism classes of differentiable knots (i.e. isomorphism classes of pairs $(S^3, F)$ - $F$ a compact connected 1-dimensional submanifold - under diffeomorphisms of $S^3$). For the following theorem see JÄNICH ([9], § 6), compare also W.C. HSIANG and W.Y. HSIANG [8].

**THEOREM.** For any $n \geq 3$ there is a one-to-one correspondence

$$\kappa_n : K \rightarrow \mathcal{S}_{2n-1},$$

where $\mathcal{S}_{2n-1}$ is the set of isomorphism classes of $(2n-1)$-dimensional knot manifolds under equivariant diffeomorphisms. $\kappa_n^{-1}$ associates to a knot manifold the knot $F$ considered above.

**REMARK.** The 2-fold branched covering of $S^3$ along a knot $F$ is an $O(1)$-manifold which will be denoted by $\kappa_2(F)$. 
If we plumb 8 copies of the tangent bundles of $S^n$ ($n \geq 1$) according to the tree $E_8$

we get a $(2n-1)$-dimensional manifold $M^{2n-1}(E_8)$. For $n=2$ this is $S^3/G$, where $G$ is the binary pentagonal dodecahedral group [6]. For $n$ odd, $M^{2n-1}(E_8)$ is the standard sphere, as the ARF invariant shows. For $n = 2m+4$, the manifold $M^{4m-1}(E_8)$ is an exotic sphere, it is the famous Milnor sphere which represents the generator $\pm e_m$ of $bP_{4m}$ (see § 3).

$M^{2n-1}(E_8)$ admits an action of $O(n-1)$ as follows: $O(n-1)$ operates as subgroup of $O(n+1)$ on $S^n$ and thus on the unit tangent bundle of $S^n$. The action on $S^n$ leaves a great circle fixed.

When plumbing the eight copies of the tangent bundle we put the center of the plumbing operation always on this great circle; (for one copy, corresponding to the central vertex of the $E_8$-tree, we need three such centers, therefore, we cannot have an action of $O(n)$, which has only 2 fixed points on $S^n$.)

Then the action of $O(n-1)$ on each copy of the tangent bundle is compatible with the plumbing and extends to an action of $O(n-1)$ on $M^{2n-1}(E_8)$ which, for $n \geq 3$, becomes a knot manifold as can be checked. The resulting knot can be seen on a picture attached at the end of this lecture. The speaker had convinced himself that this is the torus knot $t(3,5)$, but Zieschang and Vogt showed him a better proof. This implies the

**Theorem.** Suppose $n \geq 3$. Then $\kappa_n(t(3,5))$ is equivariantly diffeomorphic to $M^{2n-1}(E_8)$ with the $O(n-1)$-action defined by equivariant plumbing. (This is still true for $n=2$, see Remark above).
We now consider the manifold $\Sigma(p,q,2,2,...,2) \subset \mathbb{C}^{n+1}$ given by the equations (see § 3)

\[ z_0^p + z_1^q + z_2^2 + ... + z_n^2 = 0 \]

\[ \sum_{i=0}^{n} z_i \overline{z}_i = 1 \quad (n \geq 3). \]

This is an $O(n-1)$-manifold, the action being defined similarly as in § 4. Suppose $(p,q) = 1$. Then it can be shown that $\Sigma(p,q,2,2,...,2)$ is a knot manifold: It is $\kappa_n(t(p,q))$ where $t(p,q)$ is the torus knot. Therefore, by the preceding theorem we have an equivariant diffeomorphism

\[ H^{2n-1}(E_8) \cong \Sigma(3,5,2,...,2). \]

This gives a different proof (based on the classification of knot manifolds) that $\Sigma(3,5,2,...,2)$ represents for $m \equiv 2$ a generator of $b_{4m}^P$. (compare § 3).

§ 6. A theorem on knot manifolds.

Let $F$ be a knot in $S^3$. Then the signature $\tau(F)$ can be defined in the following way which MILNOR explained to the speaker in a letter. MILNOR also considers higher dimensional cases. We cite from his letter, but restrict to classical knots:

Let $X$ be the complement of an open tubular neighbourhood of $F$ in $S^3$. Then the cohomology

\[ H^* = H^*(\hat{X}, \partial\hat{X}; R) \]
where $\hat{X}$ is the infinite cyclic covering of $X$, satisfies Poincaré duality just as if $\hat{X}$ were a 2-dimensional manifold bounded by $F$.

In particular the pairing

$$U : H^1 \otimes H^1 \to H^2 \cong \mathbb{R}$$

is non-degenerate. Let $t$ denote a generator for the group of covering transformations of $\hat{X}$. Then for $a, b \in H^1$ the pairing

$$\langle a, b \rangle = a \cup t^* b + b \cup t^* a$$

is symmetric and non-degenerate. Hence, the signature

$$\tau^+(F) - \tau^-(F) = \tau(F)$$

is defined.

There exist earlier definitions of the signature by MURASUGI [13] and TROTTER [17]. The signature is a cobordism invariant of the knot. A cobordism invariant mod 2 was introduced by ROBERTELLO [15] inspired by an earlier paper of KERVAIRE-MILNOR. Let $F$ be a knot and $\Delta$ its Alexander polynomial, then the ROBERTELLO invariant $c(F)$ is an integer mod 2, namely

$$c(F) = 0, \text{ if } \Delta(-1) \equiv \pm 1 \text{ mod } 8$$

$$c(F) = 1, \text{ if } \Delta(-1) \equiv \pm 3 \text{ mod } 8$$

We recall that the first integral homology group of $\kappa_2(F)$, the 2-fold branched covering of the knot $F$ (see a remark in § 5), is always finite, its order is odd, and equals up to sign the determinant of $F$. We have

$$\pm \det F = \Delta(-1).$$

**THEOREM.** Let $F$ be a knot, then $\kappa_n(F), n \geq 2$, is the boundary of a parallelizable manifold. For $n$ odd, $\kappa_n(F)$ is homeomorphic to $S^{2n-1}$ and thus represents an element of $bP_{2n}$, it is the standard sphere if
c(F) = 0, the KERVAIRE sphere if c(F) = 1. If n = 2m, then \( \kappa_{2m}(F) \) is (2m-2)-connected and \( H_{2m-1}(\kappa_{2m}(F), \mathbb{Z}) \cong H_1(\kappa_2(F), \mathbb{Z}) \). For \( m \geq 2 \) it is homeomorphic to \( S^{4m-1} \) if and only if \( \det F = +1 \). Then \( \kappa_{2m}(F) \) represents (up to sign) an element of \( bP_{4m} \) which is \( \pm \frac{r(F)}{8} \cdot e_m \) (see § 3).

The proof uses an equivariant handlebody construction starting out from a Seifert surface [16] spanned in the knot \( F \). For simplicity, not out of necessity, we have disregarded orientation questions in § 5 and § 6.

REMARK. § 2(3) gives up to sign a formula for the signature of the torus knot \( t(p,q) \), \( (p,q \) odd with \( (p,q) = 1 \)).
BIBLIOGRAPHY


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Page 314-07. Ligne 4 du bas, au lieu de "Let \( g_m \) be a generator of \( bP_{4m} \)." lire: "Let \( g_m \) be the Milnor generator of \( bP_{4m} \), see p. 314-14."