N. GHOUSSOUB
On operators fixing copies of $c_0$ and $\ell_\infty$


SEMINAIRE
D'ANALYSE FONCTIONNELLE
1980-1981

ON OPERATORS FIXING COPIES OF $c_0$ AND $\ell_\infty$

N. GHOUSSOUB
(University of British Columbia, Vancouver)

Exposé No XII
12 Décembre 1980
In this seminar, we report on a part of a joint work with W.B. Johnson and T. Figiel [13] concerning the structure of non-weakly compact operators on Banach lattices. First, we recall the following two fundamental theorems.

Theorem (A) : (A. Pełczynski [4]). A non-weakly compact operator from a $C(K)$-space into any Banach space must preserve a copy of $c_0$; that is, there exists a subspace of $C(K)$, isomorphic to $c_0$, on which $T$ acts as an isomorphism.

Theorem (B) : (H. Rosenthal [5]). If $K$ is a $\sigma$-Stonian compact space, then every non-weakly compact operator from $C(K)$ into any Banach space must preserve a copy of $\ell_\infty$.

Our goal is to see to which extent, one can replace $C(K)$ in theorems (A) and (B) by a larger class of Banach spaces.

§ I. NON WEAKLY COMPACT OPERATORS :

The existence of the James space [2] eliminates the possibility of replacing $C(K)$ in theorem (A) by any Banach space not containing a subspace isomorphic to $\ell_1$, since $c_0$ and $\ell_1$ do not embed in this space and yet it is not reflexive. However, the result does hold for the identity operator acting on a Banach lattice since if the latter is not reflexive, then it must contain a sublattice isomorphic either to $\ell_1$ or $c_0$ [3]. A natural problem is then to check if the result holds for any operator or equivalently if whether in theorem (A), $C(K)$ can be replaced by any Banach lattice not containing $\ell_1$.

Surprisingly, Pełczynski's theorem does not extend even to this case as we show in the following counterexample.

Example (1) : For every $p$, $1 \leq p < \infty$, there exists a Banach lattice
X\textsubscript{p} and a lattice homomorphism T\textsubscript{p} from X\textsubscript{p} onto c\textsubscript{0} so that

(i) T\textsubscript{p} is strictly singular for each p, 1 \leq p < \infty

(ii) X\textsubscript{p} contains no subspace isomorphic to \ell\textsubscript{1} for p, 1 < p < \infty.

We first give the idea. Let c be the space of converging sequences and set X = \ell\textsubscript{1}(c); that is the space of doubly-indexed sequences 
\[ a = (a_i, j), \text{ where } i = 1, 2, \ldots; j = 1, 2, \ldots, \omega \text{ such that} \]

\[ \lim_{j \to \infty} a_{i, j} = a_{i, \omega} \text{ for } i = 1, 2, \ldots \]

and

\[ \|a\|_X = \sum_{i=1}^{\infty} \sup_{j} |a_{i, j}| < \infty \]

Define the norm one operator T : X \to c\textsubscript{0} by

\[ Ta = (a_{i, \omega})_{i=1}^{\omega}. \]

Clearly, T is weakly compact and X contains lots of sublattices isomorphic to \ell\textsubscript{1}. However, we can turn T into a non-weakly compact operator by adding to the unit ball of X vectors \( (f_n) \) for which \( (Tf_n) \) is not weakly compact in c\textsubscript{0} and taking for the new unit ball in X the absolute convex solid hull of the old unit ball and the \( f_n \)'s, in order to get a normed lattice. The completion of the resulting space probably still contains \ell\textsubscript{1} complementably, but we can kill them by taking the p-convexification of the space for some 1 < p < \infty.

Letting X and T be defined as above we define \( f_n \) \in X by

\[ (f_n)_{i, j} = \begin{cases} 1, & \text{if } i \leq n \leq j \\ 0, & \text{otherwise} \end{cases}. \]

Clearly

\[ Tf_n = \sum_{i=1}^{n} e_i \]

where \( (e_i)_{i=1}^{\omega} \) is the unit vector basis for c\textsubscript{0}.

Let \( X_0 \) be the dense sublattice of X consisting of those vectors \( a = (a_{i, j}) \) whose rows are eventually zero; i.e., for some \( n, a_{i, j} = 0 \) for all \( i \geq n \) and all \( j = 1, 2, \ldots, \omega \).
Let $\| \cdot \|_1$ be the greatest lattice norm on $X_0$ such that

$$\| f_n \|_1 \leq 1, \quad \| x \|_1 \leq \| x \|$$

for $n = 1, 2, \ldots$ and all $x \in X_0$. That is, $\| \cdot \|_1$ is the gauge of the closed absolutely convex solid hull of the unit ball of $X_0$ and the sequence $(f_n)$. Thus $\| x \|_1 < 1$ if and only if there are $g \in X_0^+$ and eventually zero sequence $s_1, s_2, \ldots$ in $\mathbb{R}^+$ so that

$$|x| \leq g + \sum_{i=1}^{\infty} s_i f_i \quad \text{and} \quad \| g \|_X + \sum_{i=1}^{\infty} s_i < 1.$$ 

Let $(X_1, \| \cdot \|_1)$ be the completion of $(X_0, \| \cdot \|_1)$ and for $1 < p < \infty$, let $(X_p, \| \cdot \|_p)$ be the completion of the $p$-convexification of $(X_0, \| \cdot \|_1)$; that is, for $x \in X_0$,

$$\| x \|_p = \left( \sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}.$$ 

(See chapter 1.e in [3] for a discussion of $p$-convexity.)

We claim that $\| T \|_p = 1$ for every $1 \leq p < \infty$; i.e., $T$ has norm one as an operator from $(X_0, \| \cdot \|_p)$ into $c_0$. This claim is a consequence of the observation that for each $i$ and $j$, the coordinatewise evaluation functional on $X_0$ defined by $a \mapsto a_{i, j}$ has $\| \cdot \|_p$-norm one.

(For $p=1$ this is clear, because $|f_n| \leq 1$ for each $n$, the general case then follows from the definition of $\| \cdot \|_p$.)

Since $X_0$ is dense in $X_p$, $T$ extends to a norm one operator, $T_p$, from $X_p$ into $c_0$. Note also that $T_p$ is a lattice homomorphism and for every choice of signs $\varepsilon$ and $n = 1, 2, \ldots$, there is $g \in X_0$, $|g| \leq f_n$, so that $T_p g = \sum_{i=1}^{\infty} \varepsilon_i e_i$, which shows that $T_p$ is a quotient map.

In the sequel, we shall say that a sequence $(x_n)_{n=1}^{\infty}$ in $X_p$ is a special $c_0$-sequence if there exist $K < \infty$ and integers $i_1 < i_2 < \ldots$ such that for every $n = 1, 2, \ldots$,

$$x_n \geq 0, \quad \| x_n \|_p = 1$$

$$(x_n)_{i, j} = 0 \text{ if } i \neq j.$$
Note that if \(1 \leq i < \infty\) and \(x \in X_0\) with

\[x_{\ell,j} = 0 \text{ for } \ell \neq i,\]

then

\[\|x\|_X = \sup_j |x_{i,j}|\; ;\]

consequently,

\[\|x\|_p = \sup_j |x_{i,j}|\]

for \(p = 1\) and hence for all \(1 \leq p < \infty\). In particular, all the terms of a special \(c_0\)-sequence lie in \(X_0\).

We now show that \(X_1\) contains no special \(c_0\)-sequence.

If such a sequence \((x_n)_{n=1}^\infty\) exists in \(X_1\), pick for each \(n\) an index \(j_n < \omega\) so that

\[(x_n)_{i_n,j_n} \geq 1/2 \sup_j (x_n)_{i_n,j} = 1/2 \|x_n\|_1 = 1/2.\]

By passing to a subsequence, we may assume that \(i_{n+1} > j_n\) for each \(n\).

Given an integer \(N\), find \(g \in X_0^+\) and \((s_i)_{i=1}^\infty \subseteq \mathbb{R}^+\) so that

\[\sum_{n=1}^N x_n \leq g + \sum_{i=1}^\infty s_i f_i,\]

\[\|g\|_X + \sum_{i=1}^\infty s_i < \|\sum_{n=1}^N x_n\|_1 + 1.\]

Evaluating both sides of the first inequality at \((i_n,j_n)\), we get

\[1/2 \leq (g)_{i_n,j_n} + \sum_{i=i_n}^{j_n} s_i \text{ for } n = 1, 2, \ldots, N.\]

It follows that

\[N/2 \leq \sum_{n=1}^N (g)_{i_n,j_n} + \sum_{n=1}^N \sum_{i=i_n}^{j_n} s_i \leq \sum_{n=1}^N (g)_{i_n,j_n} + \sum_{n=1}^N \sum_{i=i_n}^{j_n} s_i.\]
XII.5

\[ \|g\|_X + \sum_{i=1}^{\infty} s_i < \sum_{n=1}^{N} x_n \|_1 + 1 \]

which for large \( N \) contradicts the inequality

\[ \| \sum_{n=1}^{N} x_n \|_1 < K. \]

To prove (i), suppose that \( T : X_p \to c_0 \) is an isomorphism on an infinite dimensional subspace \( E \) of \( X_p \) which we may assume is isomorphic to \( c_0 \). Let \( (z_n)_{n=1}^{\infty} \) be a normalized basis for \( E \) which is \( K \)-equivalent to the unit vector basis of \( c_0 \); since \( X_0 \) is dense in \( X_p \), we can assume that each \( z_n \) lies in \( X_0 \).

Since

\[ \| T z_n \| = \max_i |(z_n)_i, \omega | \text{ and } \lim_{n \to \infty} |(z_n)_i, \omega | = 0 \text{ for each } i \in \mathbb{N}, \]

we can find a sequence \( i_1 < i_2 < \ldots \) and \( \delta > 0 \) such that for all \( n \),

\[ |(z_n)_{i_n}, \omega | > \delta. \]

Define the band projection \( P_n : X_p \to X_p \) by

\[ (P_n x)_{i,j} = \begin{cases} x_{i,j}, & \text{if } i = n \\ 0, & \text{if } i \neq n. \end{cases} \]

By the diagonal principle (cf. p. 20 in [2]) it follows that the disjoint sequence \( (P_n z_n)_{n=1}^{\infty} \) is \( K/\delta \)-equivalent to the unit vector basis of \( c_0 \). Consequently,

\[ y_n = \| P_n z_n \|_p^{-1} | P_n z_n \| \]

is a special \( c_0 \)-sequence in \( X_p \) and hence the sequence \( x_n = y_n \) is a special \( c_0 \)-sequence in \( X_1 \), which is a contradiction.

To prove (ii), note that if \( E \) is a subspace of \( X_p \) isomor-
XII.6

phic to \( \ell_1 \), and if \( S X = \sum_{i=1}^{m} p_i X_i \) determines the natural Schauder decomposition of \( X \), then \( S_j \) cannot be an isomorphism for any \( m \) because \( S X \) is isomorphic to \( c_0 \). Thus there exists a normalized sequence \( (x_n)_{n=1}^{\infty} \) in \( E \) which is equivalent to the unit vector basis for \( \ell_1 \) and a disjoint sequence in \( X_0 \) so that

\[
\lim_{n \to \infty} \|x_n - y_n\|_p = 0.
\]

It follows that the sublattice of \( X \) generated by \( (y_n)_{n=1}^{\infty} \) is isomorphic to \( \ell_1 \), which is impossible for \( p > 1 \) because \( X_0 \) is \( p \)-convex.

\[\square\]

§ II. OPERATORS WHOSE ADJOINT ARE NOT WEAK*-SEQUENTIALLY COMPACT:

To study the extensions of theorem (B), we note first that if \( K \) is \( \sigma \)-Stomian, then \( C(K) \) is a Grothendieck space, that is, the weak-star sequential convergence in its dual coincide with the weak convergence. The problem then reduces to the study of the structure of operators whose adjoints are not weak-star sequentially compact and whose domain is a Banach lattice which contains no complemented copy of \( \ell_1 \). The first theorem reduces the problem to \( C(K) \)-spaces, where much is known.

Given any \( u \) in the positive cone \( L^+ \) of a Banach lattice \( L \), denote by \( L_u \) the (not necessarily closed) ideal generated by \( u \). The canonical injection from \( L_u \) into \( L \) is denoted by \( j_u \) or just \( j \) if there is no ambiguity. If we put the natural norm on \( L_u \), defined by

\[
\|x\|_u = \inf \{\lambda > 0 : |x| \leq \lambda u\}
\]

then \( (L_u, \|\cdot\|_u) \) is an abstract M-space with unit \( u \) and hence is isometrically isomorphic to a \( C(K) \) space by Kakutani’s Theorem. The operator \( j_u : (L_u, \|\cdot\|_u) \rightarrow L \) obviously has norm \( \|u\| \).

Theorem 2: Let \( L \) be a Banach lattice which does not contain a copy of \( \ell_1 \) as a sublattice and let \( T \) be an operator from \( L \) into a Banach space \( X \) such that \( T^{\star*} \operatorname{Ball}(X^{\star}) \) is not weak* sequentially compact. Then there exists \( u \in L^+ \) so that \( (Tj_u)^{\star*} \operatorname{Ball}(X^{\star}) \) is not weak* sequen-
To prove the theorem we will need a few lemmas. Given an infinite subset of $\mathbb{N}$, denote by $[M]$ the set of all infinite subsets of $M$. Given a Banach space $L$ and a bounded sequence $(f_n)$ in $L^*$, we define for $x \in L$ and $M \in [\mathbb{N}]$

$$\alpha_M(x) = \limsup_{m \in M} f_m(x) - \liminf_{m \in M} f_m(x).$$

Note that

$$\alpha_M(x) \leq 2 \sup_{m \in M} \|f_m\| \|x\|$$

and there exists $P \in [M]$ so that

$$\lim_{p \in P} f_p(x) = \frac{1}{2} \alpha_M(x).$$

Given $A \subseteq L$, define

$$\alpha_M(A) = \sup\{\alpha_M(x) : x \geq 0, \|x\| \leq 1, x \in A\}$$

and

$$\beta_M(A) = \inf\{\alpha_M(A) : P \in [M]\}.$$

**Lemma (3):** Let $L$ be a Banach space and let $(f_n)$ be a bounded sequence in $L^*$. If $A \subseteq \text{Ball}(L)$ and $M \in [\mathbb{N}]$, then either $\beta_M(A) > 0$ for some $P \in [M]$ or there exists $P \in [M]$ such that $(f_p)_{p \in P}$ converges pointwise on $A$.

**Proof:** If $\beta_M(A) = 0$ for all $P \in [M]$, we can recursively define infinite sets $M \supseteq P_1 \supseteq P_2 \supseteq \ldots$ so that $\alpha_{P_n}(A) < \frac{1}{n}$. If $P$ is a diagonal sequence with respect to the $P_n$'s, then $\alpha_P(A) = 0$; i.e., $(f_p)_{p \in P}$ converges on $A$.

From lemma (3) it follows that if $L$ is a Banach lattice and $(f_n) \subseteq \text{Ball}(L^*)$ has no weak$^*$ convergent subsequence, then we may assume, by passing to a subsequence of $(f_n)$ that $\beta_M(L^*) > 0$.

To prove Theorem 2, we fix a sequence $(f_n) \subseteq T^* \text{Ball}(X^*)$ with $\sup_{n} \|f_n\| < 1$ so that $\beta_M(L^*) > 0$. We assume that $\beta_M(L_X) = 0$ for
all \( x \in L^+ \) and \( M \in [\mathbb{N}] \) since this is the case if \((j)f(x)_{m} \in M\) has a subsequence which converges weak* in \(L_x^*\). The conclusion that this set-up implies that \(L\) must contain a disjoint positive sequence equivalent to the unit vector basis of \(\ell_1\) is an immediate consequence of the next two lemmas. Lemma (4), produces an "almost disjoint" sequence in \(\text{Ball}(L^+)\) which, by Lemma (5), has a subsequence which is a small perturbation of a disjoint \(\ell_1\) sequence.

\textbf{Lemma (4)}: Suppose that \(L\) is a Banach lattice, \((f_n) \subseteq \text{Ball}(L^*)\), \(\alpha_n(L^*) > \delta > 0\), \(\beta_{M}(L_x) = 0\) for all \(M \in [\mathbb{N}]\) and \(x \in L^+\), and \(\varepsilon_n \rightarrow 0\). Then there exists \(f \in \text{weak}^*\) closure \((f_n)\) and \((y_n) \subseteq \text{Ball}(L^+)\) so that for each \(n = 1, 2, \ldots\),

(i) \[\left\| \left( \sum_{i=1}^{n-1} y_i \right) \wedge y_n \right\| < \varepsilon_n\]
(ii) \[|f(y_n)| \geq \delta/2\]

\textbf{Proof}: By induction we construct a sequence \((y_n) \subseteq \text{Ball}(L^+)\) and \((M_n) \subseteq [\mathbb{N}]\) to satisfy for each \(n = 1, 2, \ldots\) condition (i) and

(iii) \[M_{n+1} = M_n\]
(iv) \[|f_m(y_n)| > \delta/2\] for all \(m \in M_n\).

Having done this, we simply let \(f\) be any element of \(\text{Ball}(L^*)\) which is a weak* cluster point of \((f_k)_{k \in M_n}\) for each \(n = 1, 2, \ldots\).

Choosing \(y_1 \in \text{Ball}(L^+\) so that \(\alpha_{M}(y_1) > \delta\), we have that

\[\lim \sup_{m \in M} |f_m(y_1)| > \delta/2\]

so that we can choose \(M_1 \in [\mathbb{N}]\) to satisfy (iv) for \(n = 1\).

Having defined \((M_n)_{n=1}^{N}\) and \((y_n)_{n=1}^{N}\) to satisfy (i), (iii), and (iv) for \(n \leq N\), we pick \(M \in [M_N]\) so that

\[\alpha_{M}([0, \Sigma_{i=1}^{N} y_i]) = 0\]

and choose \(z \in \text{Ball}(L^+)\) so that \(\alpha_{M}(z) > \delta\). Define
Since
\[ y_{N+1} = z - z \wedge \left( \varepsilon_{N+1}^{-1} \sum_{i=1}^{N} y_i \right). \]

Thus we can choose \( I \) so that for all \( m \in \mathbb{M}_{N+1} \),
\[ \alpha_{M}(y_{N+1}) = \alpha_{M}(z) > \delta. \]

Thus we can choose \( M_{N+1} \in [M] \) so that for all \( m \in M_{N+1} \),
\[ |f_m(y_{N+1})| > \delta/2. \]

To check (i), just note that if \( z, x \in L^+ \) and \( \lambda \in \mathbb{R}^+ \), then
\[ (z - z \wedge \lambda x) \wedge x = (z - \lambda x)^+ \wedge x \leq \lambda^{-1} z. \]

**Lemma (5)**: Suppose that \( L \) is a Banach lattice, \( f \in \text{Ball}(L^*) \),
\( (y_n) \subseteq \text{Ball}(L^+) \), and \( 0 < \delta < \delta + \varepsilon \). Suppose that for each \( n = 1, 2, \ldots \),
\[ |f(y_n)| \geq \delta + \varepsilon \] and \[ \lim_{n \to \infty} \|\sum_{i=1}^{n} y_i \wedge y_k\| = 0. \] Then there is a subsequence \( (y_{n(i)}) \) of \( (y_n) \) and a disjoint sequence \( (x_i) \) in \( L^+ \) with \( x_i \leq y_{n(i)} \) so that for each \( i = 1, 2, \ldots \),
\[ \|y_{n(i)} - x_i\| < 4^{-i+1} \varepsilon. \]

Consequently, \( |f(x_i)| > \delta \) for each \( i = 1, 2, \ldots \), and hence \( (x_i) \) is
\( 1/5 \)-equivalent to the unit vector basis for \( l_1^* \) and \( \{x_i\} \) is \( 1/5 \)-complemented in \( L \).

**Proof**: Assume, by passing to a subsequence of \( (y_n) \), that for \( n = 1, 2, \ldots \),
\[ \|y_{n+1} \wedge \sum_{i=1}^{n} y_i\| < 4^{-n} \varepsilon. \]

We define by recursion a double sequence \( (y_{n,k})_{n=1}^{\infty} \subseteq \text{Ball}(L^+) \) to satisfy
\( (y_{n,k})_{n=1}^{k} \) is disjoint for \( k = 1, 2, \ldots \)
\[ y_{n,k+1} \leq y_{n,k} \leq y_n \] for \( 1 \leq n \leq k \).
(d) \[ \| y_n - y_n, n \| < 4^{-n} \varepsilon \text{ for } n = 1, 2, \ldots \]

(e) \[ \| y_{n, k} - y_{n, k+1} \| < 4^{-k} \varepsilon \text{ for } 1 \leq n \leq k. \]

Once this is done, we can in view of (e) set

\[ x_n = \lim_{k \to \infty} y_{n, k} ; \]

from (b) and (c) we have that \((x_n)_{n=1}^{\infty}\) is disjoint and \(0 \leq x_n \leq y_n\) for each \(n = 1, 2, \ldots\). From (d) and (e) we infer that

\[ \| y_n - x_n \| < 4^{-n+1} \varepsilon. \]

We turn now to the construction of the \(y_{n, k}\)'s. Set \(y_1, 1 = y_1\).

Suppose that \((y_{n, k})_{n=1}^{N} k=n\) has been defined. Let

\[ y_{N+1, N+1} = y_{N+1} - \sum_{k=1}^{N} y_{N, k} \]

and, for \(1 \leq n \leq N + 1\), set

\[ y_{n, N+1} = y_{n, N} - y_{n, N} \wedge y_{N+1} \cdot \]

We leave the verification of (b) - (e) to the reader. \(\square\)

By applying Theorem (2) we obtain the following two corollaries of Theorem (2).

**Corollary (6)**: Let \(L\) be a \(\sigma\)-complete Banach lattice which does not contain a copy of \(\ell_1\) as a sublattice. If \(T\) is an operator from \(X\) into some Banach space \(Y\) and \(T^* \text{Ball}(Y^*)\) is not weak\(^*\) sequentially compact, then \(T\) preserves a copy of \(\ell_\infty\).

**Proof**: By Theorem (2) there is \(u \in L^+\) so that \((T_j u)^* \text{Ball}(Y^*)\) is not weak\(^*\) sequentially compact and hence not weakly compact. When \(L_u\) is represented as \(C(K)\) space, \(K\) is \(\sigma\)-Stonian because \(L\) is \(\sigma\)-complete. Therefore, by Theorem (B) \(T^* u\), hence also \(T\), preserves a copy of \(\ell_\infty\). \(\square\)

**Corollary (7)**: If \(L\) is a \(\sigma\)-complete Grothendieck Banach lattice, then every non-weakly compact operator from \(L\) into any Banach space preserves a copy of \(\ell_\infty\).
Proof: A Grothendieck space cannot contain $\ell_1$ (or any other non-reflexive separable space) as a complemented subspace, and non-weakly compact operators from a Grothendieck space have adjoints which are not weak* sequentially compact, and hence Corollary (6) can be applied to any non-weakly compact operator from a $\sigma$-complete Grothendieck Banach lattice.

Problem: It is still unknown whether every non-weakly compact operator from a Grothendieck space into any Banach space preserves a copy of $\ell_\infty$.

References:


