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SOME RESULTS ON INJECTIVE BANACH LATTICES

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In analogy with the definition of injective Banach spaces (or $\mathcal{P}_\lambda$-spaces) one introduces the notion of injective Banach lattices in the following way. A Banach lattice X is called injective if it is complemented by a positive projection in every Banach lattice Y containing it as a sublattice. This notion was considered first by H. Lotz [3] who showed that the class of injective Banach lattices includes the spaces $C(K)$ with K extremally disconnected (i.e., the $\mathcal{P}_1$-spaces), the $L_1(\mu)$ spaces, and direct sums of $L_1(\mu)$ spaces in the sense of $\ell_\infty(\Gamma)$.

The class of direct sums of $L_1(\mu)$ spaces in the sense of $\ell_\infty(\Gamma)$ is in a certain sense universal. If X is an arbitrary Banach lattice and $\{x_j^*\}_{j\in\Gamma}$ is a family of norm one positive functionals such that $\|x\|_X = \sup_{j\in\Gamma} x_j^*(|x|)$ then, for each $j \in \Gamma$, the expression $|||x|||_j = x_j^*(|x|)$ defines a semi-norm on X which is additive on its positive cone. Hence, X endowed with $|||\cdot|||_j$ and divided by the elements of semi norm $|||\cdot|||_j$ equal to zero is an abstract L_1-space X_j and, thus, X is order isometric to a sublattice of the injective lattice $Y = (\sum_{j\in\Gamma} \oplus X_j)_\infty$. It follows easily from this fact that, for an injective Banach lattice X, there is a number $\lambda \geq 1$ such that, whenever W is a Banach lattice, V a sublattice of W and T a positive linear operator from V into X, then there is a positive extension $\hat{T}$ of T from W into X such that $\|\hat{T}\| \leq \lambda\|T\|$. We shall call such a lattice X λ-injective.

All the examples of injective Banach lattices mentioned above are in fact 1-injective. A thorough study of 1-injective Banach lattices was undertaken by D. I. Cartwright [1]. He showed e.g., that whenever x_1, x_2 and y are positive elements of X such that $\|x_1 + x_2 + y\| \leq \|x_1\| + \|x_2\|$, there exist positive elements y_1 and y_2 in X such that $\|x_i + y_i\| \leq \|x_i\|$ for $i = 1,2$ and $y = y_1 + y_2$. Conversely, if a Banach lattice X has the afore property then its second dual X^{**} is 1-injective. In the case of 1-injective lattices of finite dimension, D. I. Cartwright proved that they are order isometric to lattices of the form $(\sum_{j=1}^{m} \oplus \ell_1^{n_j})_\infty$. The complete characterization of 1-injective Banach lattices was however achieved by R. Haydon [2]. He showed that such a lattice is order isometric to the lattice of all continuous sections of a bundle of L_1-spaces over an extremally disconnected space with some additional properties.

The methods used by D. I. Cartwright and R. Haydon give no information on the structure of λ-injective Banach lattices when $\lambda > 1$. As in the case of P_λ-spaces with $\lambda > 1$ this case is more complicated. The natural question concerning such lattices is whether every λ-injective Banach lattice is order isomorphic to a 1-injective lattice.

The aim of this lecture is to present some positive results in this direction obtained jointly with J. Lindenstrauss. The central theorem is the positive solution to the finite-dimensional problem.

<u>Theorem 1</u>. *Let X be a λ-injective Banach lattice with $\dim X < +\infty$. Then there exists an order isomorphism U between X and a 1-injective lattice such that* $\|U\| \, \|U^{-1}\| < 2^{18}\lambda^{14}$.

Combining this result with a standard diagonal argument one obtains a characterization of injective discrete lattices.

<u>Theorem 2</u>. *A discrete injective lattice X (i.e. a lattice which coincides with the band generated by its atoms) is order isomorphic to a 1-injective Banach lattice. In particular, if X has only countably many atoms then it is order isomorphic to one of the following 6 lattices:*

$$\ell_1, \ell_\infty, \ell_1 \oplus \ell_\infty, (\sum_{n=1}^{\infty} \oplus \ell_1^n)_\infty, \ell_1 \oplus (\sum_{n=1}^{\infty} \oplus \ell_1^n) \quad \text{and}$$

$$(\sum_{n=1}^{\infty} \oplus \ell_1)_\infty .$$

The general case of λ-injective lattices with $\lambda > 1$ remains however open. We just mention that after the completion of the work reported here, P. Mangheri and R. Haydon solved another special case: the only order continuous injective lattices are those order isomorphic to $L_1(\mu)$ spaces.

Our solution to the local version of the λ-injective problem implies also a result of some interest concerning unconditional bases (answering thus a question posed to us by W. B. Johnson). An unconditional basis $\{e_n\}_{n=1}^{\infty}$ is said to be block injective if, for any unconditional basis $\{x_j\}_{j=1}^{\infty}$ and every block basis $\{u_n\}_{n=1}^{\infty}$ of $\{x_j\}_{j=1}^{\infty}$ which is equivalent to $\{e_n\}_{n=1}^{\infty}$ there is a bounded projection from $[x_j]_{j=1}^{\infty}$ onto $[u_n]_{n=1}^{\infty}$. The structure of the block injective unconditional bases is described in the following result.

Theorem 3. The only normalized block injective unconditional bases are, up to equivalence and permutation, the unit vector bases of one of the following 6 spaces:

$$\ell_1, c_0, \ell_1 \oplus c_0, (\sum_{n=1}^{\infty} \oplus \ell_1^n)_0, \ell_1 \oplus (\sum_{n=1}^{\infty} \oplus \ell_1^n)_0 \quad \text{and}$$

$$(\sum_{n=1}^{\infty} \oplus \ell_1)_0.$$

The proof of Theorem 3 required besides Theorem 1 also a uniformity argument which ensures that if $\{e_n\}_{n=1}^{\infty}$ is a block injective unconditional basis then there exists a $\lambda \geq 1$ such that, for each m, $[e_n]_{n=1}^{m}$ is a λ-injective Banach lattice.

Additional details and the complete proofs will appear in Advances of Mathematics.

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