F. DELBAEN

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WEAKLY COMPACT SETS IN $L^1/H^1_0$

F. DELBAEN
(Université de Bruxelles)
By \( A \) we mean a uniform algebra in the sense of T.W. Camelin [4], i.e. there is a compact Hausdorff space \( X \) such that \( A \subseteq C(X) \), \( 1 \in A \) and \( A \) separates the points of \( X \). If \( \phi : A \rightarrow \mathbb{C} \) is a nonzero, multiplicative, linear functional then \( M_\phi \) denotes the set of positive representing measures on \( X \). More precisely \( M_\phi = \{ \mu \mid \mu \) a positive measure on \( X \) and \( \int f \, d\mu = \phi(f) \) for all \( f \in A \} \).

We will suppose that \( M_\phi \) is a weakly compact set in the space of all measures on \( X \). In this case it is easily seen that there is \( m \in M_\phi \) such that all other measures in \( M_\phi \) are absolutely continuous with respect to \( m \) (i.e. a slight modification of the proof given in Dunford-Schwartz [3] p. 307 already gives this result).

By \( H^\infty \) we mean the Hardy space which is the weak star closure of \( A \) in \( L^\infty(m) \) where \( m \) is the dominant measure mentioned before. The predual of \( H^\infty \) is \( L^1(m)/N \) where \( N \) is the space of functions annihilating \( H^\infty \) for the bilinear form \( \langle f, g \rangle = \int fg \, dm \).

Since \( M_\phi \) is weakly compact in \( L^1(m) \), all the results of [1] and [2] apply. Of course we identify \( M_\phi \) with the set \( \{ \frac{d\mu}{dm} \mid \mu \in M_\phi \} \subseteq L^1(m) \).

Given an element \( \phi \in L^1(m)/N \) then we can restrict \( \phi \) to the space \( A \) and obtain an element \( \phi|_A \in A^\ast \). It follows immediately from the results of Ahern and Sarason that \( ||\phi|_A|| = ||\phi|| \). ([4] Theorem VI.5.2., p. 152-153).

It follows that \( L^1(m)/N \) can be identified with a closed subspace of \( A^\ast \).

**Lemma**: Let \( \phi \in L^1/N \) and let \( \mu \) be a measure on \( X \) such that

1) \( ||\mu|| = ||\phi|| \)

2) \( \mu|_A = \phi \)

then \( \mu \in L^1(m) \).

**Proof**: The existence of \( \mu \) is given by the Hahn-Banach theorem. Let now \( \nu \in L^1(m) \) such that \( \nu|_A = \phi \) then \( (\nu - \mu) \perp A \).

If \( \mu = \mu_s + \mu_s \) is the Lebesgue decomposition of \( \mu \) with respect to \( m \) then by the abstract F. and M. Riesz theorem \( (\nu - \mu_a) \perp A \) and \( \mu_s \perp A \) ([4] p. 44).
Since \(||\phi|| = ||\mu|| = ||\mu_a|| + ||\mu_s||\) and \(\mu_a/\Lambda = \nu/\Lambda = \phi\)
we obtain that \(||\mu_s|| = 0\) and hence \(\mu = \mu_a \subseteq L^1(m)\).

We will need the following results of [1] and [2].

**Lemma (Chaumat [2], Lemma 2):** Let \(f_n\) be a bounded sequence in \(L^1(m)\) and let \(\mu\) be an element of \((L^\infty)^*\) adherent to the sequence \(f_n\) (for the topology \(\sigma((L^\infty)^*, L^\infty)\)). Let \(\mu = \mu_a + \mu_s\) where \(\mu_a\) is the \(\sigma\)-additive part of \(\mu\) and \(\mu_s\) is the purely finitely additive part of \(\mu\) (Hewitt-Yosida [5]). If \(\mu_s\) is not orthogonal to \(H^\infty\) then there is a subsequence \(f_{n_k}\) such that
\[
\begin{align*}
H^\infty & \to l^\infty \\
g & \mapsto \left( \int g f_{n_k} \, dm \right)_k
\end{align*}
\]
is onto, i.e. \(f_{n_k}\) is an interpolating sequence.

**Lemma ([1] and [2]):** If \(K\) is a bounded subset of \(L^1/\mathbb{N}\) then are equivalent

i) \(K\) is weakly relatively compact

ii) \(\forall \varepsilon > 0; \exists \delta > 0\) such that \(f \in H^\infty\), \(||f||_{\infty} \leq \delta\) and \(||f||_1 \leq \delta\)
imply \(\sup_{\phi \in K} |\phi(f)| < \varepsilon\)

iii) \(K\) does not contain an interpolating subsequence.

The preceding lemmas give following corollary (\(\mathbb{N}\) denotes the class of \(f \in L^1\) in the quotient \(L^1/\mathbb{N}\)).

**Corollary:** If \(f_n\) is a bounded sequence of positive elements then \(f_n\) is a weakly relatively compact in \(L^1(m)\) if and only if \(f_n\) is weakly relatively compact in \(L^1(m)/\mathbb{N}\).

**Proof:** If \(\mu\) is adherent to \(f_n\) in \((L^\infty)^*, \sigma((L^\infty)^*, L^\infty)\) and \(\mu = \mu_a + \mu_s\) is the Hewitt-Yosida decomposition then \(\mu_s\) is positive. It follows that \(\mu_s = 0\) if and only if \(\mu_s\) is orthogonal to \(H^\infty\) i.e. if and only if \(f_n\) does not contain an interpolating subsequence.
THEOREM: If $K \subset L^1(m)/N$ is weakly compact then there is $K'$ in $L^1(m)$ such that the quotient $L^1(m) \rightarrow L^1(m)/N$ maps $K'$ onto $K$.

Proof: Let $\mu_\phi \in L^1(m)$ such that $\|\mu_\phi\| = \|\phi\|$. By the first lemma $\mu_\phi \in L^1(m)$.

Let $d\mu_\phi = g \, d|\mu_\phi|$ be the polar decomposition of $\mu_\phi$. It is well known that $|\phi| = 1$, $|\mu_\phi|$ a.e.. Since $\phi \in L^1(m)$ there is $h_\phi \in H^\infty$, $\|h_\phi\|_\infty = 1$ such that $|\phi| = \phi(h_\phi)$.

So $\|\phi\| = \int h_\phi \, d\mu_\phi = \int h_\phi \, g \, d|\mu_\phi| = \|u\| = \int d|\mu_\phi|$ and hence $g = h_\phi$, $|u_\phi|$ almost everywhere and $d\mu_\phi = h_\phi \, d|\mu_\phi|$.

We now claim that $K_1' = \{|u_\phi| \mid \phi \in K\}$ is weakly relatively compact in $L^1(m)$. By the corollary we only have to prove that the image of $K_1'$ in $L^1/N$ is weakly relatively compact.

So let $\varepsilon > 0$ and take $\delta > 0$ such that $\sup_{\phi \in K} |\phi(f)| \leq \varepsilon$ as soon as $f \in H^\infty$, $\|f\|_\infty \leq 1$ and $\|f\|_1 \leq \delta$. But if $f$ is a function satisfying these inequalities then $f \cdot h_\phi$ also satisfies these inequalities and hence

$$\sup_{\phi \in K} \int f \, d|u_\phi| = \sup_{\phi \in K} \int f \cdot h_\phi \, d|u_\phi| = \sup_{\phi \in K} \int f h_\phi \, d\mu_\phi \leq \varepsilon.$$ 

The lemma above implies now that $K_1'$ is relatively weakly compact and hence is equally integrable in $L^1(m)$ ([3] p. 294).

Let now $K_2' = \{u_\phi \mid \phi \in K\}$ then $K_2'$ is obtained from $K_1'$ by multiplying the elements of $K_1'$ by functions bounded by 1. It is then obvious that $K_2'$ is also equally integrable and hence weakly relatively compact ([3] p. 294). Define $K_3'$ as the weak closure of $K_2'$ in $L^1(m)$ and let $K' = K_3' \cap q^{-1}(K)$ where $q$ is the quotient map $q : L^1(m) \rightarrow L^1(m)/N$. 


