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ON THE DIFFERENTIABILITY OF THE NORM IN TRACE CLASSES $S_p$

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This text provides an outline of the proof of the differentiability of the norm in the trace classes $S_p$.

Let $H$ be a real Hilbert space. By $K(H)$ we denote the space of all compact operators from $H$ to $H$ endowed with the operator norm $\| \cdot \|$. If $A \in K(H)$, then $A^*$ denotes the adjoint of $A$. We define the sequence $(S_n(A))_{n=1}^{\infty}$ of $s$-numbers of the operator $A$ by

$$S_n(A) = \lambda_n \quad n = 1, 2, ...$$

where $\lambda_1 \geq \lambda_2 \geq ...$ is the decreasing sequence of non-zero eigenvalues of the operator $(A^*A)^{1/2}$, each repeated the number of times equal to its multiplicity.

Let $1 \leq p \leq \infty$. We put

$$S_p = \{ A \in K(H) : \|A\|_p = (\sum_{n=1}^{\infty} s_n^p(A))^{1/p} < \infty \}$$

It is well known that $S_p$ is a Banach space under the norm $\| \cdot \|_p$ and that

$$\|A\|_p = (\text{tr}(A^*A)^{p/2})^{1/p}$$

Let $E$ and $F$ be Banach spaces. For an arbitrary natural $K$, $S^K(E,F)$ denotes the Banach space of continuous $K$-linear operators $v : E \times ... \times E \to F$ equipped with the norm

$$\|v\| = \sup_{\|x_1\| = \cdots = \|x_K\| = 1} \|v(x_1, \ldots, x_K)\|$$

Let $O$ be an open set in $E$. A mapping $f : O \to F$ is said to be differentiable at $x \in O$ if there exists a linear operator $f'(x) \in S^1(E,F)$ such that

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - f'(x)h\|}{\|h\|^{1-1}} = 0.$$  

This $f'(x)$, which is unique, is called the derivative of $f$ at $x$. The higher-order derivatives $f^{(K)} : O \to S^K(E,F)$ are defined in the usual manner by
induction. It is well known that the mapping \( f : \mathcal{O} \to F \) is \( n \)-times continuously differentiable (is class \( C^n \), for short) if and only if for every \( x \in \mathcal{O} \) there exist a convex neighbourhood \( x \in U \subset \mathcal{O} \), mappings \( L_k : U \to S^k(E,F) \) \( (K = 1, 2, \ldots, n) \) and a function \( R : U \times E \to F \) such that for every \( h \) with \( x + h \in U \)

\[
f(x+h) = f(x) + L_1(x;h) + \ldots + L_n(x;h) + R(x,h)
\]

where \( \lim_{h \to 0} \|R(x;h)\| = 0 \), uniformly on \( U \).

The differentiability of the norm in the space \( L^p(\Omega,\mu) \) \((1 \leq p < \infty)\) was considered by Bonic and Frampton in [1]. This property can be formulated as follows:

**Theorem 1**: Let \( 1 < p < \infty \). Then

1°) \( p \) is an even integer then the norm in \( L^p(\Omega,\mu) \) is class \( C_\infty \) away from zero;

2°) if \( p \) is an odd integer, then the norm in \( L^p(\Omega,\mu) \) is class \( C_{p-1} \) away from zero and is not class \( C_p \);

3°) if \( p \) is not an integer and \([p]\) denotes the integral part of \( p \), then the norm in \( L^p(\Omega,\mu) \) is class \( C_{[p]} \) away from zero and is not class \( C_{[p]+1} \);

4°) in the space \( c_0 \) there exists an equivalent norm \( |.| \) which is class \( C_\infty \) away from zero.

Part 4°) of this theorem has been observed by Kuiper (see [1]). For our considerations we need only the information that this smooth norm in \( c_0 \) locally depends only on (the absolute values of) a finite number of coordinates (away from zero).

In the case of the trace classes \( S^p \) we have exactly the same result as in the case of \( L^p \), but the proofs are a good deal more complicated.
Theorem 2: Let $1 < p < \infty$. Then

1°) if $p$ is an even integer then the norm in $S_p$ is class $C^\infty$ away from zero;

2°) if $p$ is an odd integer then the norm in $S_p$ is class $C^{p-1}$ away from zero and is not class $C_p$;

3°) if $p$ is not an integer then the norm in $S_p$ is class $C^{[p]}$ away from zero and is not class $C^{[p]+1}$;

4°) in $K(H)$ there exists an equivalent norm $||.||$ which is class $C^\infty$ away from zero.

We begin with some general considerations on orthogonal projections on finite-dimensional subspaces spanned by eigenvectors of a compact operator. In the book by Gohberg and Krein [2] one can find the following useful lemma:

Lemma 3: Let $X \neq 0$ be a compact operator acting in a complex Hilbert space with eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ and eigenvectors $\{x_n\}_{n=1}^{\infty}$. Let $D$ be a circle $D = \{ z \in \mathbb{C} : |z - z_0| < r \}$ where $|z_0| > r$, and $\Gamma$ be its boundary. $\Gamma = \{ z \in \mathbb{C} : |z - z_0| = r \}$ with positive orientation. Assume that $\lambda_m \in D$ for $m \in \mathbb{N}$, $\lambda_k \notin D$ for $k \notin \mathbb{N}$ and $\lambda_n \notin \Gamma$ for $n = 1, 2, \ldots$. Then the integral

$$\frac{1}{2\pi i} \int_{\Gamma} (X - \lambda I)^{-1} \, d\lambda$$

is the orthogonal projection onto the subspace $E_U = \text{span}(x_m)_{m \in \mathbb{N}^+}$.

Now let $\mathbb{N}$ be a finite set of natural numbers. Let $\mathcal{O}_{\mathbb{N}} \subset K(H)$ be the set of all compact operators $A$ such that $s_m(A) \neq 0$ for $m \in \mathbb{N}$. It follows from the continuity of $s$-numbers that $\mathcal{O}_{\mathbb{N}}$ is open. Let $P_{\mathbb{N}}^A$ denote the orthogonal
projection on the finite dimensional subspace spanned by the eigenvectors of $A^*A$ corresponding to the $s$-numbers $s_m(A)$, $m \in \mathbb{N}$. The crucial proposition can be formulated as follows:

**Proposition 4**: The mapping $P_{m}^{\mathbb{C}} : \mathbb{C}_{m} \rightarrow K(H)$ is class $C_{\infty}$.

**Proof**: Let $A_0 \neq 0$ be a compact operator $A_0 \in \mathbb{C}_{m}$. We shall prove that the mapping $P_{m}^{\mathbb{C}}$ is infinitely many times differentiable at $A_0$. For this let us pick a positive number $\varepsilon > 0$ and a complex number $z_0 \in \mathbb{C}$ such that $|s_m^2(A_0) - z_0| < \varepsilon$ for $m \in \mathbb{N}$ and $|s_k^2(A_0) - z_0| > \varepsilon$ for $K \notin \mathbb{N}$. From the continuity of $s$-numbers it follows that there is a $\delta > 0$ such that if $B$ is an arbitrary compact operator with $\|B\| < \delta$ then we have also

$$
|s_m^2(A_0 + B) - z_0| < \varepsilon \quad \text{for } m \in \mathbb{N}
$$

$$
|s_k^2(A_0 + B) - z_0| > \varepsilon \quad \text{for } K \notin \mathbb{N}
$$

Put $\Gamma = \{z \in \mathbb{C} : |z - z_0| = \varepsilon\}$. By Lemma 3 for every compact operator $B$ with $\|B\| < \delta$ the orthogonal projection $P_{m}^{\mathbb{C}}$ considered as an operator acting in associated complex Hilbert space, can be represented in the form

$$
P_{m}^{\mathbb{C}} = \frac{1}{2\pi i} \int_{\Gamma} \left( \frac{(A_0^* + B^*)(A_0 + B) - \lambda I}{(A_0^* + B^*)(A_0 + B) - \lambda I} \right)^{-1} d\lambda
$$

where $(A_0^* + B^*)(A_0 + B)$ is meant as the operator acting in the complex Hilbert space.

At first we shall show that the operator $((A_0^* + B^*)(A_0 + B) - \lambda I)^{-1}$ has an expansion in a Taylor's series at $A_0$, next we obtain the required result integrating this expansion over $\Gamma$.

Observe that for all operators $X$ and $Y$ (in real or complex Hilbert space), if $X$ is invertible and $\|X\| \|X^{-1}\| < 1$, then

$$
(X + Y)^{-1} = X^{-1}[I + \sum_{\nu=1}^{\infty} (-Y X^{-1})^\nu]
$$

Indeed, our assumption on $Y$ implies that the series on the right-hand side
is absolutely convergent and we can verify this equality by multiplying it by \((X+Y)\).

Now substitute in the above formula \(X = A^*o A_o - \lambda I, Y = A^*o B + B^*o A_o + B^*B\). Since for every \(\lambda \in \Gamma\) the operator \((A^*o A_o - \lambda I)\) is invertible, we get

\[
((A^*o + B^*)(A_o + B) - \lambda I)^{-1} = (A^*o A_o - \lambda I)^{-1}[I + \sum_{\nu=1}^{\infty} (- (A^*o B + B^*o A_o + B^*B)(A^*o A_o - \lambda I)^{-1})^\nu]
\]

for all \(B\) such that \(\|A^*o B + B^*o A_o + B^*B\| \max_{\lambda \in \Gamma} \|A^*o A_o - \lambda I\|^{-1} < 1\), i.e. for all \(B\) with \(\|B\| < \delta'\). Rearranging the terms according to the powers of \(B\) and of \(B^*\) we can obtain the desired expansion in a Taylor's series. Finally, by integration this expansion over \(\Gamma\) we get the 'real' Taylor's formula for \(P_{A_o + B}\) as an operator acting in a real Hilbert space, with a good estimate for the remainder. This proves that the mapping \(P_{A_o}^\mu : \mathcal{K}(H) \to \mathcal{K}(H)\) is infinitely many times differentiable at \(A_o\).

The easiest way to see the idea of the proof of Theorem 2 is to consider the case 4°). Therefore we begin with it

Case 4°) : we define a new norm on \(\mathcal{K}(H)\) as follows

\[
\|\|A\|\| = \|\{s_n(A)\}\|
\]

where \(\{s_n(A)\}_{n=1}^{\infty}\) is the sequence of s-numbers of the operator \(A\) and \(\|\cdot\|\) is the Kuiper's norm from Theorem 1, 4°). This norm is obviously equivalent to the usual operator norm in \(\mathcal{K}(H)\). We shall show that this norm is infinitely many times differentiable away from zero. For this we take any compact operator \(A_o \neq 0\).

As it was observed, the norm \(\|\cdot\|\) locally depends on a finite number of coordinates away from zero. Hence there exist a natural number \(N\), a convex neighbourhood \(V\) of \(A_o\) and mappings \(L_K : V \to \mathbb{B}^K(c_o, F)\) \((K = 1, 2, \ldots)\) such that for every \(A \in V\) the \(K\)-linear form \(L_K(A)\) depends only on the first \(N\) coordinates and that for every compact operator \(B\), with \(A + B \in V\) and every natural \(\mu\) we
have

\[ \|s_n(A+B)\| = \|s_n(A)\| + \sum_{k=1}^{\mu} \sum_{\alpha} \Sigma^* a_\alpha(A) \left[ \sum_{\pi} \Sigma^{**} (s_1(A+B) - s_1(A))^{\pi(1)} \ldots \right. \]

\[ \ldots (s_m(A+B) - s_m(A))^{\alpha(m)} \left. \right] \cdot (s_{m+1}(A+B) - s_{m+1}(A))^{\alpha(m+1)} \ldots \]

\[ \ldots (s_N(A+B) - s_N(A))^{\alpha(N)} + R(A; \{s_n(A+B) - s_n(A)\}) \]

where \(S\) is extended over all sequences \((\alpha_i)_{i=1}^N\) of non-negative integers with \(\sum_{1}^{\alpha_i} \geq \ldots \geq \alpha_m\) and \(\sum_{1}^{\alpha_i} = K\), \(\Sigma^{**}\) is extended over all permutations \(\pi\) of the set \(\{1, 2, \ldots, n\}\).

The above formula can be rewritten in the form

\[ \|A+B\| = \|A\| + \sum_{k=1}^{\mu} \sum_{\alpha} \Sigma^* b_\alpha(A) \left[ \sum_{\pi} \Sigma^{**} s_1(A+B)^{\pi(1)} \ldots s_m(A+B)^{\pi(m)} \right] \]

\[ s_{m+1}(A+B)^{\alpha(m+1)} \ldots s_N(A+B)^{\alpha(N)} + R'(A; B) \]

where \(R': V \times K(H) \to \mathbb{R}\) is a mapping satisfying \(\lim_{B \to 0} R'(A; B).\|B\|^{-\mu} = 0\), uniformly on \(V\).
The case where there are more s-numbers of multiplicity greater than 1 can be handled analogously.

Thus the complete the proof it is enough to show the following fact:

**Lemma 5:** Let $A_0 \neq 0$ be a compact operator and $s_{i+1}(A_0)$ be an s-number of multiplicity $m$, i.e. $s_{i+1}(A_0)\ldots=s_{i+m}(A_0) > s_{i+m+1}(A_0)$. Then for every sequence $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_m$ the mapping

$$\varphi(C) = \sum_{\pi}^{\alpha} s_{i+1}(C)^{\alpha_{i+1}} \ldots s_{i+m}(C)^{\alpha_{i+m}}$$

is infinitely many times differentiable at $A_0$.

**Proof of Lemma** Let us take some sequence $\alpha_1 \geq \ldots \geq \alpha_m$ and define the function $\overline{\varphi} : \mathbb{R}^m \to \mathbb{R}$ by

$$\overline{\varphi}(x_1 \ldots x_m) = \sum_{\pi}^{\alpha} x_1^{\alpha_{i+1}} \ldots x_m^{\alpha_{i+m}}$$

furthermore for every natural $\nu = 1, 2, \ldots$ and every natural $j = 1, \ldots, m$ let us define the functions $g_{\nu}, h_j : \mathbb{R}^m \to \mathbb{R}$ by

$$g_{\nu}(x_1 \ldots x_m) = \sum_{n=1}^{m} x_n^{2\nu}$$

$$h_j(x_1 \ldots x_m) = \sum_{1 \leq n_1 < \ldots < n_j \leq m} x_{n_1} \ldots x_{n_j}$$

It is easy to show that if $x^0 = (x_1^0 \ldots x_m^0) \in \mathbb{R}^m$ satisfies $x_n^0 \neq 0$ for $n = 1 \ldots m$, then every function $h_j$ can be expressed as an infinitely many times differentiable function of the $g_{\nu}$ ($\nu = 1 \ldots m$) in some neighbourhood of $(g_{\nu}(x^0), g_m(x_0)) \in \mathbb{R}^m$.

Moreover the function $\overline{\varphi}$ can be expressed as an infinitely many times differentiable function of $g_{\nu}$ ($\nu = 1, 2, \ldots$) and $h_j$ ($j = 1, \ldots, m$) in some neighbourhood of $(h_1(x^0), \ldots, h_m(x^0), g_{\nu}(x^0))$. Thus, $\overline{\varphi}$ is an infinitely many times differentiable function of $g_{\nu}$ ($\nu = 1, 2, \ldots$) in some neighbourhood of $(g_{\nu}(x^0), \ldots)$. 
This implies that to complete the proof of the differentiability of $\varphi$ at $A_0$ it is sufficient to show that the mapping

$$g_{\varphi}(C) = \sum_{i=1}^{n} s_{i}^{2\varphi}(C)$$

for every natural $\nu$ is class $C_\infty$ at $A_0$. But this follows immediately from Proposition 4. This completes the proof of the case 4°).

Case 1°) is obvious.

The proof of 2°) and 3°) starts with showing that the mapping $\| \|_p$ is class $C_q$ for $q = p - 1$ or $q = \lceil p \rceil$ respectively. It is done using the formula mentioned in the proof of Proposition 4. We need the exact form of the Taylor's series for $P_A$ since the norm $\| \|_p$ in $\ell_p$ does not have the "localization property" of the Kuiper's norm $\| \|$ that we have used before. The corresponding computations and estimates are therefore more complicated, thus we omit them.

The fact that the norm $\| \|$ is not of class $C_q$ is obvious because the space $S_p$ contains a subspace isometric to $\ell_p$.

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BIBLIOGRAPHIE
