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**On a result of Olevskiĭ : a uniformly bounded orthonormal
sequence is not a basis for $C[0, 1]$**

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ON A RESULT OF OLEVSKIĪ : A UNIFORMLY BOUNDED
ORTHONORMAL SEQUENCE IS NOT A BASIS FOR $C[0,1]$

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The purpose of this lecture is to present a result of Olevskii^Y (Izvestia Akad Nauk SSSR, volume 30 (1966), 387 - 432) that there is no uniformly bounded orthonormal system which is a basis for the space C of all continuous functions on the interval $[0,1]$ and to explain the relation of this result with the conjecture of the non existence of normalized Besselian bases in C .

Definition : A biorthogonal system (e_n, μ_n) in a Banach space E , in particular a basis, is said to be Besselian (Hilbertian) if there exists a constant $K > 0$ such that for each $x \in E$

$$\|x\| \geq K \left(\sum_{n=1}^{\infty} (\mu_n(x) \|e_n\|)^2 \right)^{1/2}$$

$$\text{(resp. } \|x\| \leq K \left(\sum_{n=1}^{\infty} (\mu_n(x) \|e_n\|)^2 \right)^{1/2} \text{)} .$$

Example : A uniformly bounded in L^∞ -norm orthonormal system in L^2 is a Besselian biorthogonal system in L^∞ .

Indeed if (φ_n) is an orthonormal system in L^2 such that $\|\varphi_n\|_\infty \leq M$ for all n then for $x \in L^\infty$ we have

$$\|x\|_\infty \geq \|x\|_2 \geq \sqrt{\sum_n \left[\int x(t) \varphi_n(t) dt \right]^2} \geq \frac{1}{M} \sqrt{\sum_n \left(\left[\int x(t) \varphi_n(t) dt \right] \|\varphi_n\|_\infty \right)^2}$$

Conjecture : There is no Besselian basis in $C[0,1]$ as well as there is no Hilbertian basis in L^1 .

The following result of Olevskii^Y (1966) strongly support this conjecture.

Theorem 1 : Let ν be a probability measure on $[0,1]$. Let (φ_n) be a uniformly bounded orthonormal (with respect to ν) system in $C[0,1]$. Then (φ_n) is not a basis for $C[0,1]$.

Before passing to the proof of Theorem 1 we shall establish a general fact on Besselian biorthogonal systems in $C[0,1]$ which is trivial for orthonormal systems but which shows some relation between the theorem and the conjecture.

Proposition 1 : Let (e_n, μ_n) be a Besselian biorthogonal system in C . Assume that (μ_n) is total, i.e. $\mu_n(x) = 0$ for all n implies $x = 0$, and let for some $M > 0$, $M^{-1} \leq \|e_n\| \leq M$ for all n .

Then there exists a probability measure ν on $[0,1]$ and $g_n \in L^2(\nu)$ such that $\mu_n(x) = \int_0^1 x(t) g_n(t) d\nu$ for all $x \in C$ and for all n and

$$M^{-1} \leq \int_0^1 |g_n(t)| d\nu \leq \sqrt{\int_0^1 |g_n(t)|^2 d\nu} \leq K_G K M$$

and $(\int_0^1 |e_n(t)|^2 d\nu)^{1/2} \geq \frac{1}{K_G K M}$ for all n where K is the constant appearing in the definition of a Besselian basis and K_G is a universal (Grothendieck) constant.

Proof : Define $T : C \rightarrow l^2$ by $T(x) = (\mu_n(x))$. Since (e_n, μ_n) is a Besselian biorthogonal system and $\|e_n\| \geq M^{-1}$ for all n , T is continuous and $\|T\| \leq KM$. Thus, by a result of Grothendieck, T is 2-integral. Hence, by Grothendieck Pietsch factorization theorem, there exists a probability measure ν on $[0,1]$ such that $T = A I_\nu$ where $I_\nu : C \rightarrow L^2(\nu)$ is the natural embedding and $A : L^2(\nu) \rightarrow l^2$ is a bounded linear operator with $\|A\| \leq K_G K M$.

Let δ_n denote the n -th unit vector in l^2 . We have for $x \in C[0,1]$

$$\mu_n(x) = \langle Tx, \delta_n \rangle_{l^2} = \langle I_\nu x, A^* \delta_n \rangle_{L^2(\nu)} = \int_0^1 x(t) y_n(t) d\nu$$

where $g_n = A^* \delta_n \in L^2(\nu)$. Hence $\mu_n = g_n d\nu$.

Thus

$$\frac{1}{M} = \frac{1}{M} \mu_n(e_n) \leq \|\mu_n\| = \int |g_n(t)| d\psi \leq \sqrt{\int |g_n(t)|^2 d\psi} \leq \|A\| .$$

Since $I_\psi C$ is dense in $L^2(\psi)$ and T is one to one because (μ_n) is total, A is one to one. Hence $I_\psi e_n = A^{-1} \delta_n$. Thus

$$\sqrt{\int_0^1 |e_n(t)|^2 d\psi} \geq \frac{1}{\|A\|} T(e_n) = \frac{1}{\|A\|} .$$

Our next proposition indicates the strategy of the proof of Theorem 1.

Proposition 2 : Let $M > 0$. Let (e_n, μ_n) be a Besselian biorthogonal system in C with $M^{-1} \leq \|e_n\| \leq M$ for all n .

Assume that

(*) for every sequence (c_n) of scalars the condition

$$\sup_N \left\| \sum_{n=1}^N c_n \mu_n \right\| < \infty \quad \text{implies} \quad \lim_N \frac{1}{N} \sum_{n=1}^N c_n^2 = 0 .$$

Then (e_n) is not a basis for C .

Proof : If (μ_n) is not total then (e_n) is not a basis. Assume now that (μ_n) is total. Let ψ and g_n have the same meaning as in Proposition 1. Assume to the contrary that (e_n) is a basis. Then

$$\sup_N \sup_{\|x\|=1} \left\| \sum_{n=1}^N \mu_n(x) e_n \right\| = L < \infty .$$

Hence for all $t \in [0, 1]$:

$$\sup_N \sup_{\|x\|=1} \left| \sum_{n=1}^N \mu_n(x) e_n(t) \right| \leq L .$$

Let us note that

$$\begin{aligned} \sup_{\|x\|=1} \left| \sum_{n=1}^N \mu_n(x) e_n(t) \right| &= \sup_{\|x\|=1} \left| \left(\sum_{n=1}^N e_n(t) \mu_n \right)(x) \right| \\ &= \left\| \sum_{n=1}^N e_n(t) \mu_n \right\|. \end{aligned}$$

Thus for all $t \in [0, 1]$

$$\sup_N \left\| \sum_{n=1}^N e_n(t) \mu_n \right\| \leq L.$$

Hence, by (*),

$$\lim_N \frac{1}{N} \sum_{n=1}^N [e_n(t)]^2 = 0$$

for all $t \in [0, 1]$.

Thus, by the Lebesgue theorem,

$$0 = \lim_N \int_0^1 \frac{1}{N} \sum_{n=1}^N e_n^2(t) d\psi = \lim_N \sum_{n=1}^N \int_0^1 e_n^2(t) d\psi.$$

On the other hand, by Proposition 1, there exists $\delta > 0$ such that

$\int_0^1 e_n^2(t) d\psi \geq \delta$ for all n . Thus

$$\frac{1}{N} \sum_{n=1}^N \int_0^1 e_n^2(t) dt \geq \delta,$$

a contradiction.

Note that if (φ_n) is an orthonormal (in $L^2(\psi)$) system then

$$\mu_n(x) = \int_0^1 x(t) \overline{\varphi_n(t)} d\psi$$

and

$$\left\| \sum_{n=1}^N c_n \mu_n \right\| = \int_0^1 \left| \sum_{n=1}^N c_n \varphi_n(t) \right| d\psi.$$

Hence Theorem 1 is an immediate consequence of Proposition 1 and the following crucial result.

Theorem 2 : Let ψ be a probability measure on $[0,1]$. Let (φ_n) be a uniformly bounded (in $L^\infty(\psi)$) orthonormal (in $L^2(\psi)$) system. Then for every sequence of scalars (c_n) the condition

$$\sup_N \int_0^1 \left| \sum_{n=1}^N c_n \varphi_n(t) \right| d\psi = L < \infty$$

implies

$$\lim_N \frac{1}{N} \sum_{n=1}^N |c_n|^2 = 0.$$

For the proof of Theorem 2 we shall need two lemmas.

Lemma 1 : Let (a_n) be a sequence of real numbers such that $0 < a_n < K$ for all n and

$$\overline{\lim}_n \frac{1}{n} \sum_{j=1}^n a_j > \alpha > 0.$$

Then for every N there exists indices m and k such that

$$\frac{1}{k} \sum_{j=m+k(r-1)+1}^{m+kr} a_j > \frac{\alpha}{2} \quad \text{for } r = 1, 2, \dots, N.$$

Proof : Let ρ be an integer. Pick M so that $K N^\rho < M \frac{\alpha}{4}$ and $\frac{1}{M} \sum_{j=1}^M a_j > \alpha$.

Let $M = N^\rho q + r$ with $0 \leq r < N^\rho$.

Then

$$\frac{1}{N^{\rho}q} \sum_{j=1}^{N^{\rho}q} a_j = \frac{M}{N^{\rho}q} \frac{1}{M} \sum_{j=1}^M a_j - \frac{1}{N^{\rho}q} \sum_{j=N^{\rho}q+1}^M a_j$$

$$\geq \frac{M}{N^{\rho}q} \alpha - \frac{1}{N^{\rho}q} K N^{\rho}$$

$$> \frac{M}{N^{\rho}q} \alpha - \frac{M}{N^{\rho}q} \frac{\alpha}{4}$$

$$\geq \frac{3}{4} \alpha .$$

Now consider the "blocks" $B_{\psi}^1 = (a_j)_{N^{\rho-1}q(\psi-1)+1 \leq j < N^{\rho-1}q \psi}$ for $1 \leq \psi \leq N$ and let $|B_{\psi}^1| = \sum_{j=N^{\rho-1}q(\psi-1)+1}^{N^{\rho-1}q\psi} a_j$.

If for all ψ with $1 \leq \psi \leq N$, $|B_{\psi}^1| > N^{\rho-1}q \frac{\alpha}{2}$ we put $k = N^{\rho-1}q$ and $m = 1$ and we have N consecutive blocks satisfying the assertion of the lemma.

If not the inequality $\frac{1}{N} \sum_{\psi=1}^N |B_{\psi}^1| = \frac{1}{N^{\rho}q} \sum_{j=1}^{N^{\rho}q} a_j > \frac{3}{4} \alpha$ yields the existence of an index ψ_1 with $1 \leq \psi_1 \leq N$ such that

$$|B_{\psi_1}^1| > \frac{3}{4} \alpha \frac{N - \frac{2}{3}}{N - 1} N^{\rho-1}q .$$

We divide the block $B_{\psi_1}^1$ into N consecutive blocks each of length $N^{\rho-3}q$, say $B_1^2, B_2^2, \dots, B_N^2$.

If $|B_{\psi}^2| > \frac{1}{2} \alpha \frac{N - \frac{2}{3}}{N - 1} > \frac{\alpha}{2}$ then we already have the desired division into N consecutive blocks. If not we infer that there exists an index ψ_2 such

that $|B_{\Psi_2}^2| > \frac{3}{4} \alpha \left(\frac{N-\frac{2}{3}}{N-1}\right)^2 N^{\rho-2} q$ and we repeat the same procedure.

If we repeat the procedure ρ times we finally get a block $B_{\Psi_\rho}^\rho$ of length

q such that $|B_{\Psi_\rho}^\rho| > \frac{3}{4} \alpha \left(\frac{N-\frac{2}{3}}{N-1}\right)^\rho q$. Since (for any block of length q)

we have $|B_{\Psi_\rho}^\rho| \leq Kq$ we infer that $K > \frac{3}{4} \alpha \left(\frac{N-\frac{2}{3}}{N-1}\right)^\rho$ which for ρ large

enough is impossible. That means that in some earlier step we must get the desired division into N consecutive blocks.

Lemma 2: Let a measurable function f satisfies the conditions

$$|f(t)| \leq cn \quad \text{for } t \in [0,1]$$

$$\int_0^1 |f(t)|^2 d\psi \geq \frac{n}{c}$$

$$\int_0^1 |f(t)| d\psi \leq c$$

Then $\psi \left\{ |f| > \frac{n}{c^3} \right\} \geq \frac{1}{n} \frac{c-1}{c^4}$.

Proof :

$$\begin{aligned} \int |f|^2 d\psi &= \int_{\left\{ |f| \geq \frac{n}{c^3} \right\}} |f|^2 d\psi + \int_{\left\{ |f| < \frac{n}{c^3} \right\}} |f|^2 d\psi \\ &\leq n^2 c^2 \psi \left\{ |f| > \frac{n}{c^3} \right\} + \frac{n}{c^3} \int_{\left\{ |f| < \frac{n}{c^3} \right\}} |f| d\psi \\ &\leq n c^2 \psi \left\{ |f| \geq \frac{n}{c^3} \right\} + \frac{n}{c^3} c \end{aligned}$$

Hence

$$\left(\frac{n}{c} - \frac{n}{c^2}\right) \frac{1}{n^2 c^2} = \frac{c-1}{n c^4} \leq \psi \left\{ |f| \geq \frac{n}{c^3} \right\}.$$

Proof of Theorem 2 : Assume to the contrary that there exists a sequence of scalars (c_j) such that

$$\sup_n \int_0^1 \left| \sum_{j=1}^n c_j \varphi_j(t) \right| d\psi = M_1 < +\infty \quad \text{and} \quad \overline{\lim}_n \frac{1}{n} \sum_{j=1}^n |c_j|^2 = \alpha > 0 .$$

Let $\sup_j \|\varphi_j\|_\infty = M_2$. Then $1 = \int_0^1 |\varphi_j(t)|^2 d\psi \leq M_2 \int_0^1 |\varphi_j(t)| d\psi$.

Hence for all j $|c_j| \int_0^1 |\varphi_j(t)| d\psi \leq 2M_1$ and $\sup_j |c_j| \leq 2M_1 M_2$.

Now fix a constant C so large that

$$C\alpha > 2 , \quad C > 2 , \quad C > 2M , \quad C > (2M_1 M_2)^2 .$$

Take v large enough and let $N = v^v$. By lemma 2, there exist m and k such that

$$\sum_{j=m+k(r-1)+1}^{m+kr} |c_j|^2 \geq \frac{\alpha}{2} k \quad \text{for } 1 \leq j \leq v^v .$$

We shall define by induction the sequence (i_s) of the indices $1 \leq s \leq v$ such that if

$$f_s = \sum_{j=m+1}^{m+ki_s} c_j \varphi_j , \quad E_s = \left\{ |f_s| \geq \frac{k v^{v-s}}{2 C^3} \right\}$$

then the following conditions are satisfied

$$(1) \quad 1 \leq i_s \leq \frac{v^v - v^{v-s}}{v-1}$$

$$(2) \quad \int_{E_s} |f_s(t)| d\psi \geq s^{1/2} \beta \quad \text{where } \beta = \frac{C-1}{16C^7} \text{ for } s = 1, 2, \dots, v .$$

Clearly having done this we get a contradiction because (2) in particular

implies that $\int_0^1 |f_v(t)| d\psi \geq v^{1/2}\beta$ while

$$\int_0^1 |f_v(t)| d\psi \leq \int_0^1 \left| \sum_{j=1}^m c_j \varphi_j(t) \right| d\psi + \int_0^1 \left| \sum_{j=1}^{m+k} c_j \varphi_j(t) \right| d\psi < C .$$

Hence $v < \left(\frac{2C}{\beta}\right)^2$ which for v large enough is impossible.

The construction of $(i_s)_{1 \leq s \leq v}$: Let us set $i_1 = v^{v-1}$.

Then

$$\begin{aligned} \int_0^1 |f_1(t)|^2 d\psi &= \sum_{j=m+1}^{m+k} |c_j|^{i_1} = \sum_{j=1}^{i_1} |c_j|^{m+kr} \\ &\geq \frac{\alpha}{2} k i_1 = \frac{\alpha}{2} k v^{v-1} > \frac{1}{C} k v^{v-1} . \end{aligned}$$

We also have

$$\int_0^1 |f_1(t)| d\psi < C ,$$

and

$$\sup_{t \in [0,1]} |f_1(t)| \leq k v^{v-1} (2M_1 M_2)^2 \leq C k v^{v-1} .$$

Thus, by lemma 2,

$$\psi(|f_1| > \frac{k v^{v-1}}{C^3}) \geq \frac{C-1}{k v^{v-1} C^4} .$$

Thus

$$\int_{\{|f_1| > \frac{k v^{v-1}}{C^3}\}} |f_1| d\psi \geq \frac{k v^{v-1}}{C^3} \frac{C-1}{k v^{v-1} C^4} = \frac{C-1}{C^7} > \beta .$$

Since $E_1 \subset \{|f_1| > \frac{k v^{v-1}}{C^3}\}$, we get $\int_{E_1} |f_1| d\psi > \beta$.

This completes the first step of induction.

Now assume that for some $s \leq v-1$ the index i_s has been defined to satisfy the conditions (1) and (2). Let us set

$$\mathcal{U}_s = \left\{ \frac{kv^{v-s-1}}{2C^3} \leq |f_s| < \frac{kv^{v-s}}{2C^3} \right\}, \quad \int_{\mathcal{U}_s} f_s(t) d\psi = \delta_s.$$

We put

$$i_{s+1} = \begin{cases} i_s & \text{if } \delta_s \geq \beta \\ i_s + v^{v-s-1} & \text{if } \delta_s < \beta \end{cases}.$$

$$\text{Clearly } 1 \leq i_{s+1} \leq i_s + v^{v-s-1} \leq \frac{v-v^{v-s}}{v-1} + v^{v-s-1} = \frac{v-v^{v-s-1}}{v-1}.$$

To complete the proof we have to check (2). Let us consider separately two cases:

1) $\delta_s \geq \beta$. Then $f_{s+1} = f_s$ and $E_{s+1} = E_s \cup \mathcal{U}_s$. Since $\mathcal{U}_s \cap E_s = \emptyset$, we get (by inductive hypothesis)

$$\int_{E_{s+1}} |f_{s+1}| d\psi = \int_{E_s} |f_s| d\psi + \int_{\mathcal{U}_s} |f_s| d\psi = s\frac{1}{2}\beta + \beta > (s+1)\frac{1}{2}\beta.$$

2) $\delta_s < \beta$. Let us set

$$F_s = \sum_{j=m+i_s k+1}^{m+i_{s+1} k} c_j \varphi_j.$$

Then

$$\begin{aligned} \int_0^1 |F_s|^2 d\psi &= \sum_{j=m+i_s k+1}^{m+i_{s+1} k} |c_j|^2 = \sum_{r=i_s+1}^{i_{s+1}} \sum_{j=m+k(r-1)+1}^{m+kr} |c_j|^2 \\ &\geq \frac{\alpha}{2} k (i_{s+1} - i_s) = \frac{\alpha}{2} k v^{v-s-1} > C k v^{v-s-1}. \end{aligned}$$

We also have

$$\int_0^1 |F_s| dv < C$$

and

$$\sup_{t \in [0,1]} |F_s(t)| \leq k v^{v-s-1} (2M_1 M_2)^2 \leq C k v^{v-s-1}.$$

$$\text{Let } V_s = \left\{ |F_s| \geq \frac{k v^{v-s-1}}{C^3} \right\}.$$

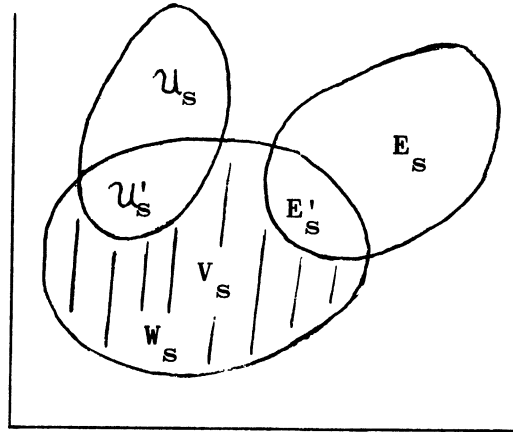
Then, by lemma 2,

$$\nu(V_s) \geq \frac{C-1}{C^4 k v^{v-s-1}}.$$

$$\text{Let us set } E'_s = V_s \cap E_s$$

$$U'_s = V_s \cap U_s$$

$$W_s = V_s \setminus (E'_s \cup U'_s)$$



Picture 1

Clearly $U_s \cap E_s = \emptyset$, $W_s \cap E_s = \emptyset$, $W_s \cap U_s = \emptyset$.

We first show that $E_s \cup W_s \subset E_{s+1}$.

If a) $t \in E_s$ then

$$\begin{aligned} |f_{s+1}(t)| &\geq |f_s(t)| - |F_s(t)| \geq \frac{k v^{v-s}}{2 C^3} - C k v^{v-s-1} = \frac{k v^{v-s-1}}{2 C^3} (v - 2 C^4) \\ &> \frac{k v^{v-s-1}}{2 C^3} \end{aligned}$$

for v large enough ($v > C^5$).

b) $t \in W_s$ then $t \notin E_s$ and $t \notin U_s$, that means that $t \in \left\{ |f_s| < \frac{k v^{v-s-1}}{2 C^3} \right\}$.

On the other hand $t \in V_s = \left\{ |F_s| \geq \frac{k v^{v-s-1}}{C^3} \right\}$.

Thus

$$|f_{s+1}(t)| \leq |F_s(t)| - |f_s(t)| \geq \frac{k v^{v-s-1}}{2 C^3} .$$

Now we separately estimate from below the integrals $\int_{E_s} |f_{s+1}| d\psi$ and

$\int_{W_s} |f_{s+1}| d\psi$. We have for $t \in E_s$,

$$\begin{aligned} |f_{s+1}(t)| &\geq |f_s(t)| - |F_s(t)| \geq |f_s(t)| - C k v^{v-s-1} \\ &= |f_s(t)| - \frac{k v^{v-s}}{2 C^3} \frac{2 C^4}{v} \geq |f_s(t)| \left(1 - \frac{2 C^4}{v}\right) . \end{aligned}$$

Thus using the inductive hypothesis we get

$$\int_{E_s} |f_s(t)| d\psi \geq \left[1 - \frac{2 C^4}{v}\right] \int_{E_s} |f_s(t)| d\psi \geq \beta s^{1/2} \left(1 - \frac{2 C^4}{v}\right) .$$

Since $s < v$, for v large enough (precisely for $v > C^{10} > (2 C^4)^2$) we have

$$\beta s^{1/2} \left(1 - \frac{2 C^4}{v}\right) \geq \beta (s^{1/2} - 1) .$$

Hence

$$\int_{E_s} |f_s(t)| d\psi \geq \beta (s^{1/2} - 1) .$$

Now we estimate the second integral $\int_{W_s} |f_s(t)| d\psi$.

The inclusion $W_s \subset E_{s+1}$ yields

$$\int_{W_s} |f_{s+1}(t)| d\psi \geq \frac{k v^{v-s-1}}{2 C^3} \psi(W_s) .$$

Thus our last aim is to estimate from below the measure of W_s . We have

$$\begin{aligned} \psi(W_s) &\geq \psi(V_s) - \psi(E'_s) - \psi(\mathcal{U}'_s) \\ &\geq \psi(V_s) - \psi(E_s) - \psi(\mathcal{U}_s) \\ &\geq \frac{C-1}{C^4 k v^{v-s-1}} - \psi(E_s) - \psi(\mathcal{U}_s). \end{aligned}$$

We have

$$C \geq \int_0^1 |f_s(t)| d\psi \geq \int_{E_s} |f_s(t)| d\psi \geq \psi(E_s) \frac{k v^{v-s}}{2C^3},$$

thus

$$\psi(E_s) \geq \frac{2C^4}{k v^{v-s}}.$$

Similarly

$$\delta_s = \int_{\mathcal{U}_s} |f_s(t)| d\psi \geq \psi(\mathcal{U}_s) \frac{k v^{v-s-1}}{2C^3},$$

thus using the assumption that $\delta_s < \beta$, we have

$$\psi(\mathcal{U}_s) \leq \frac{\delta_s}{\frac{k v^{v-s-1}}{2C^3}} < \frac{2C^3 \beta}{k v^{v-s-1}}.$$

Therefore

$$\psi(W_s) \geq \frac{1}{k v^{v-s-1}} \left(\frac{C-1}{C^4} - \frac{2C^4}{v} - 2\beta C^3 \right).$$

Hence

$$\int_{W_s} |f_{s+1}(t)| d\psi \geq \frac{1}{2C^3} \left(\frac{C-1}{C^4} - \frac{2C^4}{v} - 2\beta C^3 \right)$$

Thus for v large enough (remembering that $\beta = \frac{C-1}{16 C^7}$) we get

$$\int_{W_s} |f_{s+1}(t)| d\psi \geq 2\beta .$$

Hence

$$\begin{aligned} \int_{E_{s+1}} |f_{s+1}(t)| d\psi &\geq \int_{E_s} |f_{s+1}(t)| d\psi + \int_{W_s} |f_{s+1}(t)| d\psi \geq s^{1/2}\beta + \beta \\ &\geq (s+1)^{1/2}\beta . \end{aligned}$$
