Central Characters for Smooth Irreducible Modular
Representations of $GL_2(Q_p)$

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To Francesco Baldassarri, on the occasion of his 60th birthday

ABSTRACT - We prove that every smooth irreducible $\mathbf{F}_p$-linear representation of $GL_2(Q_p)$ admits a central character.

Introduction

Let $\Pi$ be a representation of $GL_2(Q_p)$. We say that $\Pi$ is smooth, if the stabilizer of any $v \in \Pi$ is an open subgroup of $GL_2(Q_p)$. We say that $\Pi$ admits a central character, if every $z \in Z(GL_2(Q_p))$, the center of $GL_2(Q_p)$, acts on $\Pi$ by a scalar. The smooth irreducible representations of $GL_2(Q_p)$ over an algebraically closed field of characteristic $p$, admitting a central character, have been studied by Barthel–Livné in [BL94, BL95] and by Breuil in [Bre03]. The purpose of this note is to prove the following theorem.

THEOREM A. If $\Pi$ is a smooth irreducible $\mathbf{F}_p$-linear representation of $GL_2(Q_p)$, then $\Pi$ admits a central character.

The idea of the proof of theorem A is as follows. If $\Pi$ does not admit a central character, and if $f = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$, then for any nonzero polynomial $Q(X) \in \mathbf{F}_p[X]$, the map $Q(f) : \Pi \to \Pi$ is bijective, so that $\Pi$ has the

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structure of a $\overline{F}_p(X)$-vector space. The representation $\Pi$ is therefore a smooth irreducible $\overline{F}_p(X)$-linear representation of $\text{GL}_2(\mathbb{Q}_p)$, which now admits a central character, since $f$ acts by multiplication by $X$. It remains to apply Barthel–Livné and Breuil’s classification, which gives the structure of the components of $\Pi$ after extending scalars to a finite extension $K$ of $\overline{F}_p(X)$. A corollary of this classification is that these components are all “defined” over a subring $R$ of $K$, where $R$ is a finitely generated $\overline{F}_p$-algebra. This can be used to show that $\Pi$ is not of finite length, a contradiction.

Note that it is customary to ask that smooth irreducible representations of $\text{GL}_2(\mathbb{Q}_p)$ also be admissible (meaning that $\Pi^U$ is finite-dimensional for every open compact subgroup $U$ of $G$). A corollary of Barthel–Livné and Breuil’s classification is that every smooth irreducible $\overline{F}_p$-linear representation of $\text{GL}_2(\mathbb{Q}_p)$ that admits a central character is admissible, and hence theorem A implies that every smooth irreducible $\overline{F}_p$-linear representation of $\text{GL}_2(\mathbb{Q}_p)$ is admissible. In particular, such a representation also satisfies Schur’s lemma: every $\text{GL}_2(\mathbb{Q}_p)$-equivariant map is a scalar. Our theorem A can also be seen as a special case of Schur’s lemma, since $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$ is a $\text{GL}_2(\mathbb{Q}_p)$-equivariant map.

There are (at least) two standard ways of proving Schur’s lemma: one way uses admissibility, and the other works for smooth irreducible $E$-linear representations of $\text{GL}_2(\mathbb{Q}_p)$, but only if $E$ is uncountable (see proposition 2.11 of [BZ76]). In order to prove theorem A, we cannot simply extend scalars to an uncountable extension of $\overline{F}_p$, as we do not know whether the resulting representation will still be irreducible.

We finish this introduction by pointing out that a few years ago, Henniart had sketched a different (and more complicated) argument for the proof of theorem A.

1. Barthel–Livné and Breuil’s classification

Let $E$ be a field of characteristic $p$. In this section, we recall the explicit classification of smooth irreducible $E$-linear representations of $\text{GL}_2(\mathbb{Q}_p)$, admitting a central character.

We denote the center of $\text{GL}_2(\mathbb{Q}_p)$ by $Z$. If $r \geq 0$, then $\text{Sym}^r E^2$ is a representation of $\text{GL}_2(F_p)$ which gives rise, by inflation, to a representation of $\text{GL}_2(\mathbb{Z}_p)$. We extend it to $\text{GL}_2(\mathbb{Z}_p)Z$ by letting $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$ act trivially.
Consider the representation
\[ \text{ind}_{\text{GL}_2(Z_p)Z}^{\text{GL}_2(Q_p)} \text{Sym}^r E^2. \]

The Hecke algebra
\[ \text{End}_{E[\text{GL}_2(Q_p)]} \left( \text{ind}_{\text{GL}_2(Z_p)Z}^{\text{GL}_2(Q_p)} \text{Sym}^r E^2 \right) \]
is isomorphic to $E[T]$ where $T$ is a Hecke operator, which corresponds to the double class $\text{GL}_2(Z_p)Z \cdot \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \cdot \text{GL}_2(Z_p)$. If $\chi : Q_p^* \to E^*$ is a smooth character, and if $\lambda \in E$, then let
\[ \pi(r, \lambda, \chi) = \frac{\text{ind}_{\text{GL}_2(Z_p)Z}^{\text{GL}_2(Q_p)} \text{Sym}^r E^2}{T - \lambda} \otimes (\chi \circ \det). \]

This is a smooth representation of $\text{GL}_2(Q_p)$, with central character $\omega r \bar{\chi}^2$ (where $\omega : Q_p^* \to F_p^*$ is given by $p^n x_0 \mapsto \overline{x_0}$, with $x_0 \in Z_p^*$. Let $\mu_\lambda : Q_p^* \to E^*$ be given by $\mu_\lambda|Z_p^* = 1$, and $\mu_\lambda(p) = \lambda$. If $\lambda = \pm 1$, then we have two exact sequences:
\[ 0 \to \text{Sp}_E \otimes (\chi \mu_\lambda \circ \det) \to \pi(0, \lambda, \chi) \to \chi \mu_\lambda \circ \det \to 0, \]
\[ 0 \to \chi \mu_\lambda \circ \det \to \pi(p - 1, \lambda, \chi) \to \text{Sp}_E \otimes (\chi \mu_\lambda \circ \det) \to 0, \]
where the representation $\text{Sp}_E$ is the "special" representation with coefficients in $E$.

**Theorem 1.1.** If $E$ is algebraically closed, then the smooth irreducible $E$-linear representations of $\text{GL}_2(Q_p)$, admitting a central character, are as follows:

1. $\chi \circ \det$;
2. $\text{Sp}_E \otimes (\chi \circ \det)$;
3. $\pi(r, \lambda, \chi)$, where $r \in \{0, \ldots, p - 1\}$ and $(r, \lambda) \notin \{(0, \pm 1), (p - 1, \pm 1)\}$.

This theorem is proved in [BL95] and [BL94], which treat the case $\lambda \neq 0$, and in [Bre03], which treats the case $\lambda = 0$.

We now explain what happens if $E$ is not algebraically closed.

**Proposition 1.2.** If $\Pi$ is a smooth irreducible $E$-linear representation of $\text{GL}_2(Q_p)$, admitting a central character, then there exists a finite
extension $K/E$ such that $(\Pi \otimes_E K)^{ss}$ is a direct sum of $K$-linear representations of the type described in theorem 1.1.

PROOF. Barthel and Livné’s methods show (as is observed in § 5.3 of [Paš10]) that $\Pi$ is a quotient of
\[
\Sigma = \frac{\text{ind}^\text{GL}_2(Q_p)}{P(T)} \otimes (\chi \circ \det),
\]
for some integer $r \in \{0, \ldots, p - 1\}$, character $\chi : Q_p^\times \to E^\times$, and polynomial $P(Y) \in E[Y]$. Let $K$ be a splitting field of $P(Y)$, write $P(Y) = (Y - \lambda_1) \cdots (Y - \lambda_{d})$, and let $P_i(Y) = (Y - \lambda_1) \cdots (Y - \lambda_{i})$ for $i = 0, \ldots, d$. The representations $P_{i-1}(T)\Sigma/P_i(T)\Sigma$ are then subquotients of the $\pi(r, \lambda_i, \chi)$, for $i = 1, \ldots, d$. \hfill \square

We finish this section by recalling that if $\lambda \neq 0$, then the representations $\pi(r, \lambda, \chi)$ are parabolic inductions (when $\lambda = 0$, they are called supersingular). Let $B_2(Q_p)$ be the upper triangular Borel subgroup of $\text{GL}_2(Q_p)$, let $\chi_1$ and $\chi_2 : Q_p^\times \to E^\times$ be two smooth characters, and consider the parabolic induction $\text{ind}^\text{GL}_2(Q_p)_{B_2(Q_p)}(\chi_1 \otimes \chi_2)$. The following result is proved in [BL94] and [BL95].

**Theorem 1.3.** If $\lambda \in E \setminus \{0; \pm 1\}$, and if $r \in \{0, \ldots, p - 1\}$, then $\pi(r, \lambda, \chi)$ is isomorphic to $\text{ind}^\text{GL}_2(Q_p)_{B_2(Q_p)}(\chi_{\mu_1/\lambda} \otimes \chi_{\omega^r \mu_1}).$

2. Proof of the theorem

We now give the proof of theorem A. Let $\Pi$ be a smooth irreducible $\overline{F}_p$-linear representation of $\text{GL}_2(Q_p)$. We have $\Pi^{(1+pZ_p)\cdot \text{Id}} \neq 0$ (since a $p$-group acting on a $F_p$-vector space always has nontrivial fixed points), so that if $\Pi$ is irreducible, then $(1 + pZ_p) \cdot \text{Id}$ acts trivially on $\Pi$. If $g \in Z_p^\times \cdot \text{Id}$, then $g^{p-1} = \text{Id}$ on $\Pi$, so that $\Pi = \oplus_{\omega \in F_p^\times} \Pi^{g=\omega \cdot \text{Id}}$. Since $\Pi$ is irreducible, this implies that the elements of $Z_p^\times \cdot \text{Id}$ act by scalars.

If $f = \left( \begin{array}{cc} p & 0 \\ 0 & p \end{array} \right)$, then for any nonzero polynomial $Q(X) \in \overline{F}_p[X]$, the kernel and image of the map $Q(f) : \Pi \to \Pi$ are subrepresentations of $\Pi$. If $Q(f) = 0$ on a nontrivial subspace of $\Pi$, then $f$ admits an eigenvector for an eigenvalue $\lambda \in \overline{F}_p^\times$. This implies that $\Pi = \Pi^{f=\lambda \cdot \text{Id}}$, so that $\Pi$ does admit a central character. If this is not
the case, then $Q(f)$ is bijective for every nonzero polynomial $Q(X) \in \overline{F}_p[X]$, so that $\Pi$ has the structure of a $\overline{F}_p(X)$-vector space, and is a $\overline{F}_p(X)$-linear smooth irreducible representation of $GL_2(\mathbb{Q}_p)$, admitting a central character.

Let $E = \overline{F}_p(X)$. Proposition 1.2 gives us a finite extension $K$ of $E$, such that $(\Pi \otimes_E K)^{ss}$ is a direct sum of $K$-linear representations of the type described in theorem 1.1. The $\overline{F}_p$-linear representation underlying $(\Pi \otimes_E K)^{ss}$ is isomorphic to $\Pi^{[K:E]}$, and hence of length $[K:E]$. We now prove that none of the $K$-linear representations of the type described in theorem 1.1 are of finite length, when viewed as $\overline{F}_p$-linear representations.

Let $\Sigma$ be one such representation, and let $\lambda \in K$ be the corresponding Hecke eigenvalue. We now construct a subring $R$ of $K$, which is a finitely generated $\overline{F}_p$-algebra, and an $R$-linear representation $\Sigma_R$ of $GL_2(\mathbb{Q}_p)$, such that $\Sigma = \Sigma_R \otimes_K K$.

If $\lambda \in \overline{F}_p$, then theorem 1.1 shows that

$$\Sigma = \frac{\text{ind}_{GL_2(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)} \text{Sym}^2 \overline{F}_p^2}{T - \lambda} \otimes_{\overline{F}_p} K(\chi \circ \det), \text{or } \text{Sp}_{\overline{F}_p} \otimes_{\overline{F}_p} K(\chi \circ \det), \text{or } K(\chi \circ \det).$$

We can then take $R = \overline{F}_p[\chi(p)^{\pm 1}]$, and $\Sigma_R = (\text{ind}_{GL_2(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)} \text{Sym}^2 \overline{F}_p^2/(T - \lambda)) \otimes_{\overline{F}_p} R(\chi \circ \det), \text{or } \text{Sp}_{\overline{F}_p} \otimes_{\overline{F}_p} R(\chi \circ \det), \text{or } R(\chi \circ \det)$, respectively.

If $\lambda \notin \overline{F}_p$, then by theorem 1.3, we have

$$\Sigma = \text{ind}_{B_{2}(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)}(\chi_{\mu_2}^{\pm 1}, \chi_{\mu_2}^{\pm 1}).$$

We can take $R = \overline{F}_p[\chi(p)^{\pm 1}, \chi(p)^{\pm 1}]$, and let $\Sigma_R$ be the set of functions $f \in \Sigma$ with values in $R$.

In the first case, $\Sigma_R$ is a free $R$-module, while in the second case, $\Sigma_R$ is isomorphic as an $R$-module to $C^\infty(P^1(\mathbb{Q}_p), R)$ and hence also free. In either case, if $f \in R$ is nonzero and not a unit and $j \in \mathbb{Z}$, then $f^{j+1} \cdot \Sigma_R$ is a proper $\overline{F}_p$-linear subrepresentation of $f^j \cdot \Sigma_R$, so that the underlying $\overline{F}_p$-linear representation of $\Sigma_R$ is not of finite length. Since $\Sigma_R \subset \Sigma$, the underlying $\overline{F}_p$-linear representation of $\Sigma$ is not of finite length, which is a contradiction. This finishes the proof of theorem A.

REFERENCES


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