On the Components of the Push-out Space with Certain Indices

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ABSTRACT - Given an immersion of a connected, $m$-dimensional manifold $M$ without boundary into the Euclidean $(m + k)$-dimensional space, the idea of the push-out space of the immersion under the assumption that immersion has flat normal bundle is introduced in [3]. It is known that the push-out space has finitely many path-connected components and each path-connected component can be assigned an integer called the index of the component. In this study, when $M$ is compact, we give some new results on the push-out space. Especially it is proved that if the push-out space has a component with index 1, then the Euler number of $M$ is 0 and if the immersion has a co-dimension 2, then the number of path-connected components of the push-out space with index $(m - 1)$ is at most 2.

1. Introduction

Throughout we assume $M$ (or $M^m$) is an $m$-dimensional connected smooth ($C^\infty$) manifold without boundary. The tangent space of $M$ at a point $p$ will be denoted by $T_p M$.

$f : M^m \rightarrow \mathbb{R}^{m+k}$ will be assumed a smooth immersion or embedding into Euclidean $m + k$ space, i.e. $f$ has co-dimension $k$. In this case

$$df_p : T_p M^m \rightarrow T_{f(p)}(\mathbb{R}^{m+k}) = \{f(p)\} \times \mathbb{R}^{m+k} \cong \mathbb{R}^{m+k}$$

is an injection. We identify $T_p M$ with $Im \; df_p$, $\forall p \in M$. In this way, we can assume that $f$ is an isometric immersion. There is a standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{m+k}$. So we can define the normal space at $p$ as the normal complement of $Im \; df_p$. Let $v_p(f)$ denote the $k-$ plane which is normal to $f(M)$ at

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$f(p)$. The total space of the normal bundle is defined by

$$N(f) = \{(p, v) \in M \times \mathbb{R}^{m+k} : f(p) + v \in v_p(f)\}.$$  

Note that $N(f)$ is an $(m + k)$-dimensional smooth manifold.

A normal field on $M$ for $f$ is a smooth map $\xi : M^m \to \mathbb{R}^{m+k}$ where $f(p) + \xi(p) \in v_p(f)$ for all $p \in M$.

With this notation, the endpoint map $E : N(f) \to \mathbb{R}^{m+k}$ is defined by $E(p, v) = f(p) + v$, and $E$ is known to be a smooth map.

1.1 – Immersions of manifolds and focal points

**Definition 1.** A point $x \in \mathbb{R}^{m+k}$ is a focal point of $f(M)$ with base $p$ if $E$ is singular at $(p, x - f(p))$, i.e. $(p, x - f(p))$ is a critical point of $E$. The focal point has multiplicity $\mu > 0$ if rank (Jacobian $E) = m + k - \mu$.

The set of focal points of $f$ (or $f(M)$) with base $p$ will be denoted by $F_p(f)$. This is an algebraic variety, that is, it is a set of zeros of a polynomial with degree at most $m$ in $k$ variables in $v_p(f)$ and in general it can be quite complicated [8]. In this study, we will be considering the simplest case. We remark that by [6], $x \in F_p(f)$ iff $x \in v_p(f)$ and $x = f(p) + \ldots \xi(p)$ where $\xi(p) = \frac{x - f(p)}{\|x - f(p)\|}$ and $\lambda$ is an eigenvalue of the shape operator $A_{x(p)} : T_pM \to T_pM$, i.e. $\lambda$ is a principal curvature of $f$ at $f(p)$ in the normal direction $\xi(p)$.

For $x \in \mathbb{R}^{m+k}$ the distance function for $f$, $L_x : M^m \to \mathbb{R}$ is defined by $L_x(p) = \|x - f(p)\|^2$. Using [6], the point $p \in M$ is a critical point of $L_x$ if and only if $x \in v_p(f)$ and further $p$ is a non-degenerate critical point of $L_x$ if and only if $x$ is not a focal point of $f$ with base $p$. So,

$$F_p(f) = \{x \in \mathbb{R}^{m+k} : p \text{ is a degenerate critical point of } L_x\}.$$  

We use this characterisation of $F_p(f)$ to calculate focal points with base $p$. Further, using [6] again, the index of $L_x$ at a non-degenerate critical point $p \in M$ is equal to the number of focal points of $f$ with base $p$ which lie on the line segment from $f(p)$ to $x$, each focal point being counted with its multiplicity.

1.2 – Parallel immersions to a given immersion

**Definition 2.** Let $f : M^m \to \mathbb{R}^{m+k}$ be an immersion and $\{\eta_1, \eta_2, \ldots, \eta_k\}$ be an orthonormal set of normal fields for $f$ in a neigh-
bourhood of some point \( p \in M \). A normal field \( \xi \) for \( f \) is said to be a parallel normal field, if \( \left\langle \frac{\partial \xi}{\partial p_i}, \eta_j \right\rangle = 0 \) for all \( p \in M \), where \( i = 1, \ldots, m \), \( j = 1, \ldots, k \) and \( p_1, \ldots, p_m \) is a coordinate system in a neighbourhood of \( p \in M \).

Since we assume \( M \) is connected, note that a parallel normal field on \( M \) has constant length.

Let \( f : M^m \to \mathbb{R}^{m+k} \) be an immersion and assume \( \xi : M^m \to \mathbb{R}^{m+k} \) is a parallel normal field for \( f \). The map \( f_\xi : M^m \to \mathbb{R}^{m+k} \) is defined by

\[
    f_\xi(p) = f(p) + \xi(p).
\]

If \( f_\xi \) is an immersion, it is called a parallel immersion to \( f \) and \( \xi \) is said to be immersive. We remark that, for all \( p \in M \), the normal planes of \( f \) and \( f_\xi \) at each \( p \in M \) are the same.

If \( f_\xi \) is an immersion, then the index of \( f_\xi \), \( \text{ind} f_\xi \), is defined to be the total multiplicity of the focal points of \( f \) with base \( p \) on the line segment between \( f(p) \) and \( f_\xi(p) \), this index is shown to be constant over \( M \) by the following well-known fact. We call this number as the index of the immersive parallel normal field \( \xi \) as well.

**Lemma 1.** Let \( f : M^m \to \mathbb{R}^{m+k} \) be an immersion and let \( \xi : M^m \to \mathbb{R}^{m+k} \) be a parallel normal field for \( f \). Then the following are satisfied.

(i) \( f_\xi \) is an immersion if and only if for all \( p \in M \), \( f_\xi(p) \) is not a focal point of \( f \) with base \( p \)

(ii) \( x \in \mathbb{R}^{m+k} \) is a focal point of \( f_\xi \) with base \( p \) if and only if \( x \) is a focal point of \( f \) with base \( p \). So, \( F_p(f_\xi) = F_p(f) \) for all \( p \in M \).

1.3 – The push-out space of immersions with flat normal bundle

Let \( M \) be a connected, \( m \)-dimensional manifold and \( f : M^m \to \mathbb{R}^{m+k} \) be an immersion. If for all \( p \in M \), there exists a neighbourhood \( U \subset M \) of \( p \) and a parallel normal frame field for \( f \) on \( U \), then it is said that \( f \) has locally flat normal bundle. The normal bundle \( N(f) \) is flat (or globally flat) if there exists a global parallel normal frame on \( M \).

If the immersion \( f \) has locally flat normal bundle, then at each base
point \( p \in M \), the focal set on \( v_p(f) \) is a union of at most \( m \) hyperplanes (which is the simplest set that can occur as focal set, if non empty) where each plane is counted with its proper multiplicity [8, pp. 69-70]. A generalisation and the converse of this result can be derived from [4].

First, we assume that the normal bundle of \( f \) is globally flat. So there exists an orthonormal set of parallel normal fields \( \xi_1, \ldots, \xi_k : M^m \to \mathbb{R}^{m+k} \) for \( f \). For each \( p \in M \), a map \( \varphi_p : v_p(f) \to \mathbb{R}^k \) can be defined by 
\[
\varphi_p \left( f(p) + \sum_{i=1}^{k} a_i \xi_i(p) \right) = (a_1, \ldots, a_k).
\]
For each \( p \in M \), we denote \( \Omega_p = \mathbb{R}^k \setminus \varphi_p(F_p(f)) \). Then, the push-out space of the immersion \( f \) is defined by
\[
\Omega(f) = \bigcap_{p \in M} \Omega_p.
\]

This set is essentially defined and many properties of it are studied in [3]. For example, \( \Omega(f) \) has finitely many path-connected components with each component convex and each component can be assigned an integer called as index. Further, if \( M \) is compact, each component is open. The definition of \( \Omega(f) \) depends on the choice of \( \xi_1, \ldots, \xi_k \), but, it is shown in [3] that different choices produces an isometric set. We are going to study some properties of \( \Omega(f) \) which are related to number of path-connected components of \( \Omega(f) \) with certain indices and some relations with the Euler characteristic of \( M \) (when \( M \) is compact).

As pointed out in [3] we can next consider an immersion \( f \) of \( m \)-dimensional manifold \( M \) which has locally flat normal bundle but the normal holonomy group is nontrivial. In this case we can take the simply connected covering space \( \tilde{M} \) of \( M \) with covering map \( \pi : \tilde{M}^m \to M^m \) and work with the immersion \( \tilde{f} = \tilde{f} \circ \pi : \tilde{M}^m \to \mathbb{R}^{m+k} \) which has globally flat normal bundle with trivial normal holonomy. We know that \( f \) and \( \tilde{f} \) have the same focal set:

**Proposition 1.** With the notation above, \( F_p(f) = F_p(\tilde{f}) \) for all \( p \in M \) and \( \tilde{p} \in \tilde{M} \) with \( \pi(\tilde{p}) = p \), where \( \pi : \tilde{M} \to M \) is the covering map.

**Proof.** Let \( x \in \mathbb{R}^{m+k} \) and define \( \tilde{L}_x : \tilde{M}^m \to \mathbb{R} \) (distance function for the immersion \( \tilde{f} \)) by
\[
\tilde{L}_x(\tilde{p}) = \|x - \tilde{f}(\tilde{p})\|^2 = L_x \circ \pi(\tilde{p}),
\]
where \( L_x : M^m \to \mathbb{R} \) is the usual distance function for \( f \). Since \( \pi \) is an im-
mersion, \( \tilde{p} \) is a degenerate critical point of \( \tilde{L}_x \) if and only if \( \pi(\tilde{p}) \) is a degenerate critical point of \( L_x \). Therefore \( F_{\tilde{p}}(f) = F_{\tilde{p}}(\tilde{f}) \) for all \( \tilde{p} \in \tilde{M} \) and \( p \in M \) with \( \pi(\tilde{p}) = p \). \( \square \)

So this result allows \( \Omega(f) \) to be defined by \( \Omega(f) = \Omega(f \circ \pi) = \Omega(\tilde{f}) \). This is useful especially when we have an immersion of a nonorientable manifold with locally flat normal bundle where obviously the normal holonomy group is nontrivial. Consequently, by replacing \( f \) with \( \tilde{f} \) if necessary, we may assume that \( f \) has globally flat normal bundle with trivial normal holonomy group. Remark that \( \tilde{M} \) may fail to be compact again even \( M \) is compact, but we can use critical point theory of distance function thorough the immersion of \( M \) to deduce some results on \( \Omega(\tilde{f}) \).

Let \( a = (a_1, a_2, \ldots, a_k) \in \Omega(f) \). As in [3], define \( \xi(a) : M^m \rightarrow \mathbb{R}^{m+k} \) by

\[
\xi(a)(p) = \sum_{i=1}^{k} a_i \xi_i(p),
\]

where \( \xi_1, \xi_2, \ldots, \xi_k \) are unit parallel normal fields on \( M \) forming a basis for the normal \( k \)-plane at \( f(p) \) for all \( p \in M \). Then it is easy to check that \( \xi(a) \) is an immersive parallel normal field for \( f \) on \( M \). With this notation \( \Omega(f) \) can be defined as

\[
\Omega(f) = \{ a \in \mathbb{R}^k : f(p) + \xi(a)(p) \text{ is not a focal point of } f \text{ with base } p, \forall p \in M \}.
\]

**Definition 3.** Let \( a \in \Omega(f) \). The index of \( a \), \( \text{ind} \ a \), is defined to be the index of the immersion \( f_\xi(a) \).

We know by [3] that if \( A \) is a path-connected component of \( \Omega(f) \) and if \( a, b \in A \), then \( \text{ind} \ a = \text{ind} \ b \). Then the index of \( A \) is defined to be \( \text{ind} \ a \) for some \( a \in A \) which is constant over \( A \). So each path-connected component of \( \Omega(f) \) can be assigned a number, called its index. We will denote the union of the path-connected components of \( \Omega(f) \) with index \( \mu \) by \( \Omega^\mu \). So \( \Omega(f) = \Omega^0 \cup \Omega^1 \cup \cdots \cup \Omega^m \). Note that \( \Omega^0 \) is always non empty and the others may be empty or not.

2. Path-connected components and their respective indices.

In this section, firstly, we give an example to illustrate the \( \Omega(f) \) for a given embedding \( f \) with flat normal bundle and then we prove some general results on \( \Omega(f) \).
**Example 1.** Let \( \tilde{f} : \mathbb{R} \times \mathbb{R} \rightarrow S^3 \subset \mathbb{R}^{2+2} \) be given by

\[
\tilde{f}(\theta, \phi) = \frac{1}{\sqrt{2}} (\cos \theta, \sin \theta, \cos \phi, \sin \phi),
\]

then \( \tilde{f} \) induces an embedding \( f \) of \( S^1 \times S^1 \) into \( S^3 \subset \mathbb{R}^{2+2} \) by taking \( \theta \) mod \( 2\pi \), \( \phi \) mod \( 2\pi \) and also \( \Omega(f) = \Omega(\tilde{f}) \). Now,

\[
\xi_1(\theta, \phi) = \tilde{f}(\theta, \phi) = \frac{1}{\sqrt{2}} (\cos \theta, \sin \theta, \cos \phi, \sin \phi),
\]

\[
\xi_2(\theta, \phi) = \frac{1}{\sqrt{2}} (-\cos \theta, -\sin \theta, \cos \phi, \sin \phi)
\]

are unit parallel normal fields to \( \tilde{f} \) and form a basis for the normal planes for all \((\theta, \phi) \in \mathbb{R} \times \mathbb{R}\). Put \( \xi(\theta, \phi) = t\xi_1(\theta, \phi) + s\xi_2(\theta, \phi) \), for some \( t, s \in \mathbb{R} \), then

\[
\tilde{f}_\xi(\theta, \phi) = \tilde{f}(\theta, \phi) + t\xi_1(\theta, \phi) + s\xi_2(\theta, \phi).
\]

Using the distance function \( L_x(\theta, \phi) = ||x - \tilde{f}(\theta, \phi)||^2 \) for \( x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \), we get

\[
\frac{\partial L_x}{\partial \theta} = \frac{2}{\sqrt{2}} (x_1 \sin \theta - x_2 \cos \theta), \quad \frac{\partial L_x}{\partial \phi} = \frac{2}{\sqrt{2}} (x_3 \sin \phi - x_4 \cos \phi),
\]

\[
\frac{\partial^2 L_x}{\partial \theta^2} = \frac{2}{\sqrt{2}} (x_1 \cos \theta - x_2 \sin \theta), \quad \frac{\partial^2 L_x}{\partial \phi^2} = \frac{2}{\sqrt{2}} (x_3 \cos \phi + x_4 \sin \phi),
\]

\[
\frac{\partial^2 L_x}{\partial \phi \partial \theta} = \frac{\partial^2 L_x}{\partial \theta \partial \phi} = 0.
\]

Then

\[
Hess(L_x) = H = \begin{bmatrix}
0 & \frac{\partial^2 L_x}{\partial \phi^2} \\
0 & \frac{\partial^2 L_x}{\partial \theta^2}
\end{bmatrix}.
\]

So \( \tilde{f}_\xi(\theta, \phi) \) is a focal point of \( \tilde{f} \) at \((\theta, \phi) \iff (\theta, \phi) \) is a degenerate critical point of \( L_x \). From the equations \( \frac{\partial L_x}{\partial \theta} = 0 = \frac{\partial L_x}{\partial \phi} \), we obtain, for each \((\theta, \phi) \in \mathbb{R}^2\), \( x = \tilde{f}_\xi(\theta, \phi) \) for some \( t, s \in \mathbb{R} \). By replacing \( x \) by \( \tilde{f}_\xi(\theta, \phi) \) and using \( \det H = 0 \)
we get
\[
\det \begin{bmatrix}
\frac{2}{\sqrt{2}} (1 + t - s) & 0 \\
0 & \frac{2}{\sqrt{2}} (1 + t + s)
\end{bmatrix} = 0 \iff (1 + t + s)(1 + t - s) = 0
\]
\[
\iff (1 + t)^2 - s^2 = 0
\]
\[
\iff s = \pm (1 + t).
\]
Therefore the focal set of \( \tilde{f} \) with base \((\theta, \phi) \in \mathbb{R} \times \mathbb{R}\) is a pair of lines perpendicular to one another which is the same for all base points \((\theta, \phi)\). Consequently
\[
\Omega(\tilde{f}) = \Omega_{(\theta, \phi)}(\tilde{f}) = \Omega_{(\theta', \phi')}(\tilde{f}), \quad \forall (\theta, \phi), (\theta', \phi') \in \mathbb{R} \times \mathbb{R}.
\]
Then, \( \Omega(\tilde{f}) \) has four path-connected components since each \( \Omega_{(\theta, \phi)}(\tilde{f}) \) has four path-connected components; one of index 0, two of index 1, and one of index 2. Hence the same is true for \( \Omega(f) \), as \( \Omega(f) = \Omega(\tilde{f}) \), see the Figure 1. Then \( \Omega(f) = \Omega^0 \cup \Omega^1 \cup \Omega^2 \) and in the Figure 1, we put \( \Omega^1 = A \cup B \).

![Figure 1](image)

We know by [3], \( \Omega(f) \) can have at most one component with index \( m \) and if there is such a component, it is unbounded. Note that, we have examples of immersions such that \( \Omega^0 \) is bounded.
THEOREM 1. Let \( f : M^m \rightarrow \mathbb{R}^{m+k} \) be an immersion of a compact, \( m \)-dimensional manifold such that \( f \) has flat normal bundle. If \( \Omega(f) \) has a component with index \( m \), then \( \Omega^0 \) is unbounded.

PROOF. Let \( a \in \Omega^m \) and take the immersive parallel normal field \( \zeta(a) \). Then \( \varphi^{-1}(a) = f_{\zeta(a)}(p) , \forall p \in M \) and \( \text{index } f_{\zeta(a)} = m \). So for all \( p \in M \), the total multiplicity of focal points with base \( p \) on the line segment from \( f(p) \) to \( f_{\zeta(a)}(p) \) is \( m \). Therefore there are no focal points on the rays

\[
R_p = \{ f(p) + t\zeta(a)(p) \in v_p(f) : t \geq 1 \} \\
Q_p = \{ f(p) + t\zeta(a)(p) \in v_p(f) : t \leq 0 \}.
\]

Also \( \forall p \in M, \varphi_p(Q_p) = \{ ta : t \leq 0 \} \subset \Omega_p \) and so

\[
\{ ta : t \leq 0 \} \subset \bigcap \{ \Omega_p : p \in M \} = \Omega(f).
\]

Hence \( \{ ta : t \leq 0 \} \subset \Omega^0 \) and \( \Omega^0 \) is unbounded. \( \square \)

PROPOSITION 2. Let \( f : M^m \rightarrow \mathbb{R}^{m+k} \) be an immersion with flat normal bundle and assume \( \zeta, \eta : M^m \rightarrow \mathbb{R}^{m+k} \) are immersive parallel normal fields for \( f \) with indices \( \lambda \) and \( \mu \) respectively. Then, the number of focal points with base \( p \) on the line segment from \( f_\zeta(p) \) to \( f_\eta(p) \) is constant for all \( p \in M \) and it is \( \lambda + \mu - 2l \) for some \( l \in \mathbb{N} \) where \( \max\{0, \lambda + \mu - m\} \leq l \leq \min\{\lambda, \mu\} \).

PROOF. As in Lemma 4.4 of [3], since \( f_{\zeta} + (\eta - \zeta) = f_{\eta} \), so \( \eta - \zeta \) is an immersive parallel normal field for the immersion \( f_{\zeta} \) and we need to find its index for \( f_{\zeta} \) which is constant. Here we try to formulate this constant.

Let \( p \in M \) and put \( x = f_{\zeta}(p) \) and \( y = f_{\eta}(p) \). If \( f(p), x, y \) are collinear, then the number of focal points with base \( p \) on the line segment from \( f_{\zeta}(p) \) to \( f_{\eta}(p) \) is \( \lambda + \mu \) or \( |\lambda - \mu| \) with respect to positioning of \( f(p) \) and we can take \( l = 0 \) or \( l = \lambda \) or \( l = \mu \). Otherwise, take the triangle on \( v_p(f) \) with vertices \( f(p), x, y \), and consider the 2-plane \( Q(p) \) which contains this triangle. We know that \( Q(p) \cap F_p(f) \) is a union of at most \( m \) lines if it is non empty, since \( F_p(f) \) is a union of at most \( m \) hyperplanes on \( v_p(f) \) [8].

If \( u, v \in v_p(f) \), then the notation \( \overline{uv} \) denotes the line segment from \( u \) to \( v \). We know the total multiplicity of focal points on \( \overline{f(p)x} \) is \( \lambda \) and on \( \overline{f(p)y} \) is \( \mu \). Now, let \( l(p) \geq 0 \) be an integer and assume that \( l(p) \) lines (counting multiplicities) meet both of the edges \( \overline{f(p)x} \) and \( \overline{f(p)y} \). Clearly \( 0 \leq l(p) \leq \min\{\lambda, \mu\} \). Then the remaining \( \lambda - l(p) \) lines intersecting \( \overline{f(p)x} \) must intersect \( \overline{xy} \). And similarly the remaining \( \mu - l(p) \) lines intersecting \( \overline{f(p)y} \) must intersect \( \overline{xy} \). So we get the total multiplicity on \( \overline{xy} \) is exactly
\[ \lambda - l(p) + \mu = \lambda + \mu - 2l(p). \] So we deduce the total multiplicity of focal points on the line segment from \( f_\xi(p) \) to \( f_\eta(p) \) is \( \lambda + \mu - 2l(p) \). But the index of the parallel immersion \((f_\xi)_\eta\) to \( f_\xi \) is a constant number, so \( l(p) \) is constant for all \( p \in M \).

Put \( l = l(p) \). Since there exist at most \( m \) lines on \( Q(p) \), then
\[ \lambda + \mu - m \leq l. \] Then \( \max \{0, \lambda + \mu - m\} \leq l \leq \min \{\lambda, \mu\} \).

**Corollary 1.** If \( \lambda = \mu = 1 \) in Proposition 2, then, for all \( p \in M \), the number of focal points with base \( p \) on the line segment from \( f_\xi(p) \) to \( f_\eta(p) \) is 2 (where \( l = 0 \)).

**Theorem 2.** Let \( f : M^m \rightarrow \mathbb{R}^{m+k} \) be an immersion with flat normal bundle and assume \( \xi : M^m \rightarrow \mathbb{R}^{m+k} \) is an immersive parallel normal field for \( f \).

(i) There exists a \( w \in \mathbb{R}^k \) such that \( \Omega(f_\xi) = \Omega(f) - w \), where \( \Omega(f) - w = \{a - w : a \in \Omega(f)\} \),

(ii) if \( A \) is a path-connected component of \( \Omega(f) \) with index \( \mu \) and if the index of \( f_\xi \) is \( \lambda \), then there exists an \( l \in \mathbb{N} \) such that \( A - w \) is a path-connected component of \( \Omega(f_\xi) \) with index \( \lambda + \mu - 2l \), where \( \max \{0, \lambda + \mu - m\} \leq l \leq \min \{\lambda, \mu\} \).

**Proof.** (i) Since \( f \) has flat normal bundle, there exists a set of orthonormal parallel normal fields \( \{\xi_1, \xi_2, \cdots, \xi_k\} \) forming a basis of the normal space at each base point \( p \in M \). We will use this basis to define \( \Omega(f) \) and \( \Omega(f_\xi) \). We can write
\[ \xi = w_1 \xi_1 + \cdots + w_k \xi_k \]
for some constants \( w_1, \ldots, w_k \in \mathbb{R} \) and put \( w = (w_1, \ldots, w_k) \in \mathbb{R}^k \). Then we can easily see that \( a \in \Omega(f) \iff a - w \in \Omega(f_\xi) \). In fact, let \( a = (a_1, \ldots, a_k) \in \Omega(f) \). We know \( F_p(f) = F_p(f_\xi) \) for all \( p \in M \) by Lemma 1 (ii). Then, for all \( p \in M \)
\[ f(p) + a_1 \xi_1 + \cdots + a_k \xi_k \notin F_p(f) \iff f + \xi + a_1 \xi_1 + \cdots + a_k \xi_k - \xi \]
\[ = f_\xi + (a_1 - w_1) \xi_1 + \cdots + (a_k - w_k) \xi_k \notin F_p(f_\xi). \]
So \( a - w \in \Omega(f_\xi) \) and therefore \( \Omega(f_\xi) = \Omega(f) - w \).

(ii) Let \( a \in A \), then clearly \( a - w \in \Omega(f_\xi) \) by Theorem 2 (i), hence \( A - w \) is a path-connected component of \( \Omega(f_\xi) \). Since \( A \) is a path-connected component of \( \Omega(f) \) with index \( \mu \), there exists an immersive parallel normal field \( \eta \) for \( f \) with index \( \mu \) and \( \varphi_p(f(p) + \eta(p)) = a \) for all \( p \in M \). As in
Proposition 2, \( f_\xi + (\eta - \zeta) = f_\eta \), so \( \eta - \zeta \) is an immersive parallel normal field for \( f_\xi \) and its index for \( f_\xi \) is \( \lambda + \mu - 2l \) for some \( l \in \mathbb{N} \) where \( \max\{0, \lambda + \mu - m\} \leq l \leq \min\{\lambda, \mu\} \). \( \square \)

The following result concerns the positioning of the path-connected components of \( \Omega(f) \) in \( \mathbb{R}^k \).

**Theorem 3.** Let \( f : M^m \rightarrow \mathbb{R}^{m+k} \) be an immersion with flat normal bundle. Let \( A, B \) be path-connected components of \( \Omega(f) \) with index \( \lambda \) and \( \mu \) respectively. If \( \lambda + \mu > m \), then there exists a hyperplane in \( \mathbb{R}^k \) such that \( A \) and \( B \) lie on one side of the hyperplane and \( \Omega^0 \) lies on the opposite side of the hyperplane.

**Proof.** Let \( A, B \) be path-connected components of \( \Omega(f) \) with index \( \lambda \) and \( \mu \) respectively and \( a \in A, b \in B \). Then there exist immersive parallel normal fields \( \xi, \eta \) for \( f \) such that index \( f_\xi = \lambda \), index \( f_\eta = \mu \) and also for all \( p \in M \), \( \varphi_p^{-1}(a) = f_\xi(p) \) and \( \varphi_p^{-1}(b) = f_\eta(p) \). Now consider the normal plane \( v_p(f) \) for a fixed \( p \in M \) and the focal hyperplanes \( \Pi_1, \ldots, \Pi_s \) on \( v_p(f) \) with their respective multiplicities \( w_i \) where \( 1 \leq i \leq s \), \( s \leq m \) and \( w_1 + \cdots + w_s \leq m \).

Since \( f_\xi \) has index \( \lambda \), the line segment joining \( f(p) \) to \( f_\xi(p) \) must cross \( \Pi_{2z_1}, \ldots, \Pi_{2z_l} \) where \( l \leq \lambda \), \( w_{2z_1} + \cdots + w_{2z_l} = \lambda \) and similarly the line segment joining \( f(p) \) to \( f_\eta(p) \) must cross \( \Pi_{\beta_1}, \ldots, \Pi_{\beta_d} \) where \( d \leq \mu \), \( w_{\beta_1} + \cdots + w_{\beta_d} = \mu \).

Here, \( \Pi_{2z_1}, \ldots, \Pi_{2z_l}, \Pi_{\beta_1}, \ldots, \Pi_{\beta_d} \) are not all distinct since \( \lambda + \mu > m \). So let \( \Pi \in \{\Pi_{2z_1}, \ldots, \Pi_{2z_l}\} \cap \{\Pi_{\beta_1}, \ldots, \Pi_{\beta_d}\} \). Then we claim that \( A, B \) stay on one side of the hyperplane \( \varphi_p(\Pi) = \Lambda \) in \( \mathbb{R}^k \). Set \( \varphi_p(\Pi_i) = A_i, 1 \leq i \leq s \). Since each \( A_i \) divides \( \mathbb{R}^k \) into two open connected regions, we identify them by writing \( A_i^- \) for the region including the origin and \( A_i^+ \) for the other part.

Then, \( \Lambda^0 \subset A_i^- \) for all \( 1 \leq i \leq s \), \( A \subset A_{2z_1}^+ \cap \cdots \cap A_{2z_l}^+ \) and \( B \subset A_{\beta_1}^+ \cap \cdots \cap A_{\beta_d}^+ \). Therefore \( A \) and \( B \) stay in \( A^+ \), and hence \( A \) is the hyperplane we are seeking. \( \square \)

3. Number of path-connected components of \( \Omega(f) \) with certain indices

It is interesting to know the number of path-connected components of \( \Omega(f) \) with their respective indices for an immersion \( f \) of \( M \) as it includes some information on the geometry and the topology of the \( m \)-dimensional compact manifold \( M \). Here, we prove that if we have a path-connected
component of \( \Omega(f) \) with index 1, then the Euler characteristic of \( M \) is 0. Secondly, we prove that the number of path-connected components of \( \Omega(f) \) with index \((m - 1)\) is at most 2 for a co-dimension 2 immersion.

Theorem 4. Let \( f : M^m \to \mathbb{R}^{m+k} \) be an immersion of compact manifold with flat normal bundle and let \( \chi(M) \neq 0 \) where \( m \) is an even number. Then \( \Omega^1 = \emptyset \).

Proof. If \( \Omega^1 \neq \emptyset \), then there exists a unit parallel normal field \( \xi \) for \( f \) such that \( f_{s\\xi} = f + s\\xi \) is an immersion with index 1 for some \( s > 0 \). So \( \forall \ p \in M \), there exists only one focal point \( c(p) \) of multiplicity 1 on the line segment from \( f(p) \) to \( f_{s\\xi}(p) \) such that \( c : M^m \to \mathbb{R}^{m+k}, \ p \to c(p) \) is continuous. Define \( \hat{\lambda} : M^m \to \mathbb{R} \) by

\[
\hat{\lambda}(p) = \frac{1}{\|f(p) - c(p)\|}.
\]

Then \( \hat{\lambda} \) is continuous as it is the principal curvature function of \( f \) in the unit normal direction \( \xi \). Also \( \hat{\lambda} \) is smooth since it is of constant multiplicity 1 on \( M \) [7]. So the principal direction corresponding to the principal curvature \( \hat{\lambda}(p) \) defines a nonzero smooth tangent vector field on \( M \) which has no zeros. So considering that \( M \) is compact, \( \chi(M) = 0 \) by the Poincaré-Hopf Theorem in [5]. But this contradicts \( \chi(M) \neq 0 \). Therefore \( \Omega^1 = \emptyset \).

A generalisation of this theorem to any odd indexed component is proved in [1] by a different method. Present method here may not be generalized, because respective vector field can fail to be smooth.

Definition 4. Let \( f : M^m \to \mathbb{R}^{m+k} \) be an immersion with flat normal bundle, then \( d(f) \) is defined to be the total number of the path-connected components of \( \Omega(f) \).

It was proved in [3] that \( d(f) \leq \alpha(m, k) \) where \( \alpha(m, k) \) is the number of path-connected regions in the complement of \( m \) hyperplanes in general position in \( \mathbb{R}^k \) as

\[
\alpha(m, k) = \begin{cases} 
2^m & \text{if } m \leq k \\
\sum_{i=0}^{k} \binom{m}{i} & \text{if } m > k
\end{cases}.
\]
Corollary 2. Let $f : M^2 \to \mathbb{R}^{2+k}$ be an immersion with flat (or locally flat) normal bundle of a compact surface for some $k \geq 1$ and let $\chi(M) \neq 0$. Then $\Omega^1 = \emptyset$ and so $d(f) \leq 2$.

Proof. By Theorem 4, $\Omega^1 = \emptyset$ and also $\Omega^0$, $\Omega^2$ are connected [3], hence $d(f) \leq 2$. Of course $\Omega^2$ can occur, definitely when $f$ is spherical, [2]. □

Example 2. For the homology groups of real projective space $\mathbb{R}P^m$, we know that $H_i(\mathbb{R}P^m, \mathbb{Z}_2) = \mathbb{Z}_2$ for all $i = 1, 2, \ldots, m$. Then

$$\chi(\mathbb{R}P^m) = \begin{cases} 0 & \text{if } m \text{ is odd} \\ 1 & \text{if } m \text{ is even} \end{cases}.$$  

So, by Theorem 4, if $f : \mathbb{R}P^m \to \mathbb{R}^{m+k}$ is any immersion with locally flat normal bundle, we have $\Omega^1 = \emptyset$ for $m$ is even.

Let $f : M^2 \to \mathbb{R}^{2+k}$ be an immersion of a 2-dimensional manifold $M$ with flat normal bundle. Then for any point $p \in M$, $F_p(f)$ is a union of at most 2 hyperplanes in $v_p(f)$. So $F_p(f)$ can divide $v_p(f)$ into at most 4 path-connected regions, and the number of path-connected components of $\Omega(f)$ with index 1 can be at most 2 for any $k \geq 2$.

In the following theorems we generalize this and prove a result concerning the number of path-connected components of $\Omega(f)$ with index $(m - 1)$ where $m \geq 3$.

Theorem 5. Let $m \geq 2$ and $f : M^m \to \mathbb{R}^{m+k}$ be an immersion with flat normal bundle. Assume $A, B$ are two different path-connected components of $\Omega(f)$ both with index $(m - 1)$ and $a \in A$, $b \in B$. Then for each $p \in M$, all the focal hyperplanes in $v_p(f)$ meet the triangle $\triangle$ with vertices $f(p)$, $\varphi_p^{-1}(a)$, $\varphi_p^{-1}(b)$, and moreover the total number of focal points on the line segment from $\varphi_p^{-1}(a)$ to $\varphi_p^{-1}(b)$ is exactly 2.

Proof. Let $a \in A$, $b \in B$. Then there are corresponding parallel normal fields $\xi = \xi(a)$ and $\eta = \xi(b)$ say, such that index $f_\xi = \text{index } f_\eta = m - 1$. By Proposition 2, for all $p \in M$, we have total number of focal points between $f_\xi(p)$ and $f_\eta(p)$ is $2(m - 1) - 2l$ for some $l \in \mathbb{N}$ where $m - 2 \leq l \leq m - 1$. Since $a$ and $b$ are in different components, there must be at least one focal point between $f_\xi(p)$ and $f_\eta(p)$ for all $p \in M$. So $l = m - 2$. 

Let \( Q \subset \nu_p(f) \) be the plane including the triangle \( \triangle \) with vertices \( f(p), f_x(p), f_y(p) \). Since \( l = m - 2 \), we have proved that the total multiplicity of focal points on \( f_x(p) f_y(p) \) is exactly 2 for all \( p \in M \) and hence there are exactly \( m \) focal lines meeting with the triangle \( \triangle \) as required. Since there are \( m \) lines in \( Q \), this implies that all focal hyperplanes on \( \nu_p(f) \) meet with the triangle \( \triangle \), for all \( p \in M \).

\[ \square \]

**Theorem 6.** Let \( f : M^m \to \mathbb{R}^{m+2} \) be an immersion of a compact manifold such that \( f \) has flat normal bundle, where \( m \geq 3 \). Then the number of path-connected components of \( \Omega(f) \) with index \( (m - 1) \) is at most 2.

**Proof.** Assume there exist at least three path-connected components of \( \Omega(f) \) with index \( (m - 1) \), say \( A, B, C \). Take \( a \in A, b \in B, c \in C \). Let \( p \in M \) be an arbitrary point and consider the points \( x = \varphi_p^{-1}(a), \ y = \varphi_p^{-1}(b), \ z = \varphi_p^{-1}(c) \) on \( \nu_p(f) \). Clearly \( x, y, z \) are nonfocal distinct points, since \( a, b, c \) are in different components.

Since \( a, b, c \) are in different components there is at least one focal point on each line segment \( \overline{xy}, \overline{yz}, \overline{zx} \). So we can check that the points \( x, y, f(p) \) cannot be collinear. Assume they lie on a line \( \ell \) say. If \( f(p) \) is on \( \overline{xy} \) then the total multiplicity of focal points on \( \ell \) is at least \( (2m - 2) \) which is not possible for \( m \geq 3 \), since \( 2m - 2 > m \). If \( f(p) \) is not on \( \overline{xy} \) we get the total multiplicity of focal points on \( f(p) \) on \( \ell \) with respect to the points \( x, y, z \). This contradicts the hypothesis that this number is \( (m - 1) \). By a similar discussion we get the points \( x, z, f(p) \) or \( y, z, f(p) \) or \( x, y, z, f(p) \) cannot be collinear.

By Theorem 5, all of the focal lines must meet the triangle with vertices \( x, y, f(p) \) and further the total multiplicity of focal points on \( \overline{xy} \) is exactly 2. Similarly we get the same result considering the triangles with vertices \( y, z, f(p) \) and \( x, z, f(p) \).

We next show \( x, y, z \) are not collinear. For if \( x, y, z \) all lie on a line then by the above argument the total multiplicity of focal points on each line segment \( \overline{xy}, \overline{yz}, \overline{zx} \) is exactly 2. Without loss of generality we can assume \( y \) is on \( \overline{zx} \). Then we obtain the total multiplicity of focal points on \( \overline{zx} \) is \( 2 + 2 = 4 \) which is a contradiction.

Now consider the triangle with vertices \( x, y, z \). There are 3 cases to be considered.

**Case 1.** Assume \( f(p) \) is in the region I bounded by the triangle with vertices \( x, y, z \) as shown in Figure 2. By Theorem 5 there exists at least one focal line meeting with \( \overline{xf(p)} \) and \( \overline{zf(p)} \) considering the triangle with ver-
vertices $x, f(p), z$. Similarly there exists one focal line meeting with $xf(p)$ and $yf(p)$ considering the triangle with vertices $x, f(p), y$. And also there exists one focal line meeting with $yf(p)$ and $zf(p)$ considering the triangle with vertices $y, f(p), z$. These focal lines are necessarily all different and together bound $f(p)$. This implies that $f(p)$ is in a bounded region of the complement of the focal lines on $v_p(f)$.

**Case 2.** Assume $f(p)$ is in the region II as shown in Figure 2. Consider the triangle with vertices $x, f(p), y$. By Theorem 5, there must be a focal line meeting with $f(p)z$ and $zx$ and this line must necessarily meet $xf(p)$ and $yf(p)$. Similarly by considering the triangle with vertices $x, f(p), z$, there must be a focal line meeting with $f(p)z$ and $zx$ and this line must necessarily meet $f(p)y$ and $zx$. Now we get at least 2 focal points on $zx$. But again by Theorem 5 and considering the triangle with vertices $x, f(p), y$, it is exactly 2. So there are no more focal lines meeting with $zx$. So far we have one focal line meeting both $xf(p)$ and $zf(p)$. By Theorem 5 and considering the triangle with vertices $x, f(p), z$, we need $(m - 3)$ more focal lines meeting with $xf(p)$ and $zf(p)$ which must necessarily meet with $yf(p)$. And one more focal line meeting both $xf(p)$ and $zx$ which must necessarily meet with $zx$ or $yf(p)$. We know there are no more focal lines meeting with $zx$. So the focal line meeting both $xf(p)$ and $zx$ must necessarily meet with $zx$. This implies that for all $p \in M$, $z = f_{\xi(a)}(p)$ is bounded by focal lines on $v_p(f)$ where the immersive parallel normal field $\xi(a)$ is corresponding to $a$.

**Case 3.** Now assume $f(p)$ is in the region III as shown in Figure 2. Then we know every focal line must meet the triangle with vertices $f(p), x, y$. But there must be a focal line meeting with $f(p)z$ and $zx$ simultaneously. So this line cannot meet the triangle with vertices $f(p), x, y$. This gives a contradiction by Theorem 5. So we deduce that Case 3 cannot occur.

Since $p$ is an arbitrary point in $M$ and $\varphi_p^{-1}$ is an isometry, then either Case 1 holds for all $p \in M$ or Case 2 holds for all $p \in M$ i.e. either $f(p)$ is bounded by focal lines on $v_p(f)$ or $f_{\xi(a)}(p)$ is bounded by focal lines on $v_p(f)$ for all $p \in M$.

Now, for some $w \in \mathbb{R}^{m+2}$, take the distance function $L_w$ for $f$. Since $M$ is compact, there is a critical point of $L_w$ with index $m$. So the total number of focal points with base $p$ on the line segment from $w$ to $f(p)$ is $m$ and so there is no focal point with base $p$ on the ray $\{f(p) + t(w - f(p)) \mid t \leq 0\} \subset v_p(f)$. This implies that for some $p \in M, f(p)$ is not bounded by focal hyperplanes on $v_p(f)$ and a similar statement is true for $f_{\xi(a)}(q)$ considering the immersion.
\( f_{q(a)} \) for some \( q \in M \). So there cannot be such path-connected components \( A, B, C \) of \( \Omega(f) \). Therefore \( \Omega(f) \) can have at most two path-connected components with index \((m - 1)\).

This is illustrated by the diagram below:

![Diagram](image)

Figure 2

\[
f \times f : \mathbb{T}^2 \times \mathbb{T}^2 \longrightarrow \mathbb{R}^{4+4}
\]

by \((f \times f)(p, q) = (f(p), f(q))\) where \(p, q \in \mathbb{T}^2\). Note that, by Theorem 4.2 of [3], \(f \times f\) has flat normal bundle and \(\Omega(f \times f) = \Omega(f) \times \Omega(f)\), since \(f\) has flat normal bundle. Then, we can easily check that \(\Omega(f \times f)\) has 4 path-connected components with index 3.

Consequently, for \(m > 2\) and \(k > 2\), it is a considerable question to ask what is the maximum number of path-connected components of \(\Omega(f)\) with index \((m - 1)\). This might be at most \(k\).

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