On the Components of the Push-out Space with Certain Indices

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Abstract - Given an immersion of a connected, $m$-dimensional manifold $M$ without boundary into the Euclidean $(m + k)$-dimensional space, the idea of the push-out space of the immersion under the assumption that immersion has flat normal bundle is introduced in [3]. It is known that the push-out space has finitely many path-connected components and each path-connected component can be assigned an integer called the index of the component. In this study, when $M$ is compact, we give some new results on the push-out space. Especially it is proved that if the push-out space has a component with index 1, then the Euler number of $M$ is 0 and if the immersion has a co-dimension 2, then the number of path-connected components of the push-out space with index $(m - 1)$ is at most 2.

1. Introduction

Throughout we assume $M$ (or $M^m$) is an $m$-dimensional connected smooth ($C^\infty$) manifold without boundary. The tangent space of $M$ at a point $p$ will be denoted by $T_pM$.

$f : M^m \to \mathbb{R}^{m+k}$ will be assumed a smooth immersion or embedding into Euclidean $m + k$ space, i.e. $f$ has co-dimension $k$. In this case

$$df_p : T_pM^m \to T_{f(p)}(\mathbb{R}^{m+k}) = \{f(p)\} \times \mathbb{R}^{m+k} \cong \mathbb{R}^{m+k}$$

is an injection. We identify $T_pM$ with $Im\, df_p$, $\forall p \in M$. In this way, we can assume that $f$ is an isometric immersion. There is a standard inner product $\langle , \rangle$ on $\mathbb{R}^{m+k}$. So we can define the normal space at $p$ as the normal complement of $Im\, df_p$. Let $\nu_p(f)$ denote the $k-$plane which is normal to $f(M)$ at

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\( f(p) \). The total space of the normal bundle is defined by

\[
N(f) = \{(p, v) \in M \times \mathbb{R}^{m+k} : f(p) + v \in v_p(f)\}.
\]

Note that \( N(f) \) is an \((m + k)\)-dimensional smooth manifold.

A normal field on \( M \) for \( f \) is a smooth map \( \xi : M^m \to \mathbb{R}^{m+k} \) where \( f(p) + \xi(p) \in v_p(f) \) for all \( p \in M \).

With this notation, the endpoint map \( E : N(f) \to \mathbb{R}^{m+k} \) is defined by \( E(p, v) = f(p) + v \), and \( E \) is known to be a smooth map.

1.1 – Immersions of manifolds and focal points

**Definition 1.** A point \( x \in \mathbb{R}^{m+k} \) is a focal point of \( f(M) \) with base \( p \) if \( E \) is singular at \( (p, x - f(p)) \), i.e. \((p, x - f(p))\) is a critical point of \( E \). The focal point has multiplicity \( \mu > 0 \) if rank (Jacobian \( E \)) = \( m + k - \mu \).

The set of focal points of \( f \) (or \( f(M) \) ) with base \( p \) will be denoted by \( F_p(f) \). This is an algebraic variety, that is, it is a set of zeros of a polynomial with degree at most \( m \) in \( k \) variables in \( v_p(f) \) and in general it can be quite complicated [8]. In this study, we will be considering the simplest case. We remark that by [6], \( x \in F_p(f) \) iff \( x \in v_p(f) \) and \( x = f(p) + \frac{1}{\lambda} \xi(p) \) where \( \xi(p) = \frac{x - f(p)}{||x - f(p)||} \) and \( \lambda \) is an eigenvalue of the shape operator \( A_{\xi(p)} : T_pM \to T_pM \), i.e. \( \lambda \) is a principal curvature of \( f \) at \( f(p) \) in the normal direction \( \xi(p) \).

For \( x \in \mathbb{R}^{m+k} \) the distance function for \( f \), \( L_x : M^m \to \mathbb{R} \) is defined by \( L_x(p) = ||x - f(p)||^2 \). Using [6], the point \( p \in M \) is a critical point of \( L_x \) if and only if \( x \in v_p(f) \) and further \( p \) is a non-degenerate critical point of \( L_x \) if and only if \( x \) is not a focal point of \( f \) with base \( p \). So,

\[
F_p(f) = \{x \in \mathbb{R}^{m+k} : p \text{ is a degenerate critical point of } L_x\}.
\]

We use this characterisation of \( F_p(f) \) to calculate focal points with base \( p \). Further, using [6] again, the index of \( L_x \) at a non-degenerate critical point \( p \in M \) is equal to the number of focal points of \( f \) with base \( p \) which lie on the line segment from \( f(p) \) to \( x \), each focal point being counted with its multiplicity.

1.2 – Parallel immersions to a given immersion

**Definition 2.** Let \( f : M^m \to \mathbb{R}^{m+k} \) be an immersion and \( \{\eta_1, \eta_2, \ldots, \eta_k\} \) be an orthonormal set of normal fields for \( f \) in a neigh-
bourhood of some point \( p \in M \). A normal field \( \xi \) for \( f \) is said to be a parallel normal field, if \( \left\langle \frac{\partial \xi}{\partial p_i}, \eta_j \right\rangle = 0 \) for all \( p \in M \), where \( i = 1, \ldots, m \), \( j = 1, \ldots, k \) and \( p_1, \ldots, p_m \) is a coordinate system in a neighbourhood of \( p \in M \).

Since we assume \( M \) is connected, note that a parallel normal field on \( M \) has constant length.

Let \( f : M^m \to \mathbb{R}^{m+k} \) be an immersion and assume \( \xi : M^m \to \mathbb{R}^{m+k} \) is a parallel normal field for \( f \). The map \( f_\xi : M^m \to \mathbb{R}^{m+k} \) is defined by

\[
f_\xi(p) = f(p) + \xi(p).
\]

If \( f_\xi \) is an immersion, it is called a parallel immersion to \( f \) and \( \xi \) is said to be immersive. We remark that, for all \( p \in M \), the normal planes of \( f \) and \( f_\xi \) at each \( p \in M \) are the same.

If \( f_\xi \) is an immersion, then the index of \( f_\xi \), \( \text{ind} f_\xi \), is defined to be the total multiplicity of the focal points of \( f \) with base \( p \) on the line segment between \( f(p) \) and \( f_\xi(p) \), this index is shown to be constant over \( M \) by the following well-known fact. We call this number as the index of the immersive parallel normal field \( \xi \) as well.

**Lemma 1.** Let \( f : M^m \to \mathbb{R}^{m+k} \) be an immersion and let \( \xi : M^m \to \mathbb{R}^{m+k} \) be a parallel normal field for \( f \). Then the following are satisfied.

(i) \( f_\xi \) is an immersion if and only if for all \( p \in M \), \( f_\xi(p) \) is not a focal point of \( f \) with base \( p \)

(ii) \( x \in \mathbb{R}^{m+k} \) is a focal point of \( f_\xi \) with base \( p \) if and only if \( x \) is a focal point of \( f \) with base \( p \). So, \( F_p(f_\xi) = F_p(f) \) for all \( p \in M \).

1.3 – The push-out space of immersions with flat normal bundle

Let \( M \) be a connected, \( m \)-dimensional manifold and \( f : M^m \to \mathbb{R}^{m+k} \) be an immersion. If for all \( p \in M \), there exists a neighbourhood \( U \subset M \) of \( p \) and a parallel normal frame field for \( f \) on \( U \), then it is said that \( f \) has locally flat normal bundle. The normal bundle \( N(f) \) is flat (or globally flat) if there exists a global parallel normal frame on \( M \).

If the immersion \( f \) has locally flat normal bundle, then at each base
point \( p \in M \), the focal set on \( v_p(f) \) is a union of at most \( m \) hyperplanes (which is the simplest set that can occur as focal set, if non empty) where each plane is counted with its proper multiplicity [8, pp. 69-70]. A generalisation and the converse of this result can be derived from [4].

First, we assume that the normal bundle of \( f \) is globally flat. So there exists an orthonormal set of parallel normal fields \( \xi_1, \ldots, \xi_k : M^m \to \mathbb{R}^{m+k} \) for \( f \). For each \( p \in M \), a map \( \varphi_p : v_p(f) \to \mathbb{R}^k \) can be defined by

\[
\varphi_p \left( f(p) + \sum_{i=1}^{k} a_i \xi_i(p) \right) = (a_1, \ldots, a_k).
\]

For each \( p \in M \), we denote \( \Omega_p = \mathbb{R}^k \setminus \varphi_p(F_p(f)) \). Then, the push-out space of the immersion \( f \) is defined by

\[
\Omega(f) = \bigcap_{p \in M} \Omega_p.
\]

This set is essentially defined and many properties of it are studied in [3]. For example, \( \Omega(f) \) has finitely many path-connected components with each component convex and each component can be assigned an integer called as index. Further, if \( M \) is compact, each component is open. The definition of \( \Omega(f) \) depends on the choice of \( \xi_1, \ldots, \xi_k \), but, it is shown in [3] that different choices produces an isometric set. We are going to study some properties of \( \Omega(f) \) which are related to number of path-connected components of \( \Omega(f) \) with certain indices and some relations with the Euler characteristic of \( M \) (when \( M \) is compact).

As pointed out in [3] we can next consider an immersion \( f \) of \( m \)-dimensional manifold \( M \) which has locally flat normal bundle but the normal holonomy group is nontrivial. In this case we can take the simply connected covering space \( \tilde{M} \) of \( M \) with covering map \( \pi : \tilde{M}^m \to M^m \) and work with the immersion \( \tilde{f} = f \circ \pi : \tilde{M}^m \to \mathbb{R}^{m+k} \) which has globally flat normal bundle with trivial normal holonomy. We know that \( f \) and \( \tilde{f} \) have the same focal set:

**Proposition 1.** With the notation above, \( F_p(f) = F_{\tilde{p}}(\tilde{f}) \) for all \( p \in M \) and \( \tilde{p} \in \tilde{M} \) with \( \pi(\tilde{p}) = p \), where \( \pi : \tilde{M} \to M \) is the covering map.

**Proof.** Let \( x \in \mathbb{R}^{m+k} \) and define \( \tilde{L}_x : \tilde{M}^m \to \mathbb{R} \) (distance function for the immersion \( \tilde{f} \)) by

\[
\tilde{L}_x(\tilde{p}) = \|x - \tilde{f}(\tilde{p})\|^2 = L_x \circ \pi(\tilde{p}),
\]

where \( L_x : M^m \to \mathbb{R} \) is the usual distance function for \( f \). Since \( \pi \) is an im-
mersion, $\tilde{p}$ is a degenerate critical point of $\tilde{L}_x$ if and only if $\pi(\tilde{p})$ is a degenerate critical point of $L_x$. Therefore $F_{\tilde{p}}(f) = F_{\tilde{p}}(\tilde{f})$ for all $\tilde{p} \in \tilde{M}$ and $p \in M$ with $\pi(\tilde{p}) = p$. □

So this result allows $\Omega(f)$ to be defined by $\Omega(f) = \Omega(f \circ \pi) = \Omega(\tilde{f})$. This is useful especially when we have an immersion of a nonorientable manifold with locally flat normal bundle where obviously the normal holonomy group is nontrivial. Consequently, by replacing $f$ with $\tilde{f}$ if necessary, we may assume that $f$ has globally flat normal bundle with trivial normal holonomy group. Remark that $\tilde{M}$ may fail to be compact again even $M$ is compact, but we can use critical point theory of distance function thorough the immersion of $M$ to deduce some results on $\Omega(\tilde{f})$.

Let $a = (a_1, a_2, \ldots, a_k) \in \Omega(f)$. As in [3], define $\zeta(a) : M^m \to \mathbb{R}^{m+k}$ by $\zeta(a)(p) = \sum_{i=1}^{k} a_i \xi_i(p)$, where $\xi_1, \xi_2, \ldots, \xi_k$ are unit parallel normal fields on $M$ forming a basis for the normal $k$-plane at $f(p)$ for all $p \in M$. Then it is easy to check that $\zeta(a)$ is an immersive parallel normal field for $f$ on $M$. With this notation $\Omega(f)$ can be defined as

$$\Omega(f) = \{ a \in \mathbb{R}^k : f(p) + \zeta(a)(p) \text{ is not a focal point of } f \text{ with base } p, \forall p \in M \}.$$

**Definition 3.** Let $a \in \Omega(f)$. The index of $a$, $\text{ind } a$, is defined to be the index of the immersion $f_{\zeta(a)}$.

We know by [3] that if $A$ is a path-connected component of $\Omega(f)$ and if $a, b \in A$, then $\text{ind } a = \text{ind } b$. Then the index of $A$ is defined to be $\text{ind } a$ for some $a \in A$ which is constant over $A$. So each path-connected component of $\Omega(f)$ can be assigned a number, called its index. We will denote the union of the path-connected components of $\Omega(f)$ with index $\mu$ by $\Omega^\mu$. So $\Omega(f) = \Omega^0 \cup \Omega^1 \cup \cdots \cup \Omega^m$. Note that $\Omega^0$ is always non empty and the others may be empty or not.

2. Path-connected components and their respective indices.

In this section, firstly, we give an example to illustrate the $\Omega(f)$ for a given embedding $f$ with flat normal bundle and then we prove some general results on $\Omega(f)$.
EXAMPLE 1. Let \( \tilde{f} : \mathbb{R} \times \mathbb{R} \rightarrow S^3 \subset \mathbb{R}^{2+2} \) be given by

\[
\tilde{f}(\theta, \phi) = \frac{1}{\sqrt{2}} (\cos \theta, \sin \theta, \cos \phi, \sin \phi),
\]

then \( \tilde{f} \) induces an embedding \( f \) of \( S^1 \times S^1 \) into \( S^3 \subset \mathbb{R}^{2+2} \) by taking \( \theta \mod 2\pi, \phi \mod 2\pi \) and also \( \Omega(f) = \Omega(\tilde{f}) \). Now,

\[
\xi_1(\theta, \phi) = f(\theta, \phi) = \frac{1}{\sqrt{2}} (\cos \theta, \sin \theta, \cos \phi, \sin \phi),
\]

\[
\xi_2(\theta, \phi) = \frac{1}{\sqrt{2}} (-\cos \theta, -\sin \theta, \cos \phi, \sin \phi)
\]

are unit parallel normal fields to \( \tilde{f} \) and form a basis for the normal planes for all \( (\theta, \phi) \in \mathbb{R} \times \mathbb{R} \). Put \( \zeta(\theta, \phi) = t\xi_1(\theta, \phi) + s\xi_2(\theta, \phi), \) for some \( t, s \in \mathbb{R} \), then

\[
\tilde{f}_\xi(\theta, \phi) = \tilde{f}(\theta, \phi) + t\xi_1(\theta, \phi) + s\xi_2(\theta, \phi).
\]

Using the distance function \( L_x(\theta, \phi) = ||x - \tilde{f}(\theta, \phi)||^2 \) for \( x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \), we get

\[
\frac{\partial L_x}{\partial \theta} = \frac{2}{\sqrt{2}} (x_1 \sin \theta - x_2 \cos \theta), \quad \frac{\partial L_x}{\partial \phi} = \frac{2}{\sqrt{2}} (x_3 \sin \phi - x_4 \cos \phi),
\]

\[
\frac{\partial^2 L_x}{\partial \theta^2} = \frac{2}{\sqrt{2}} (x_1 \cos \theta + x_2 \sin \theta), \quad \frac{\partial^2 L_x}{\partial \phi^2} = \frac{2}{\sqrt{2}} (x_3 \cos \phi + x_4 \sin \phi),
\]

\[
\frac{\partial^2 L_x}{\partial \phi \partial \theta} = \frac{\partial^2 L_x}{\partial \theta \partial \phi} = 0.
\]

Then

\[
Hess(L_x) = H = \begin{bmatrix}
\frac{\partial^2 L_x}{\partial \theta^2} & 0 \\
0 & \frac{\partial^2 L_x}{\partial \phi^2}
\end{bmatrix}
\]

So \( \tilde{f}_\xi(\theta, \phi) \) is a focal point of \( \tilde{f} \) at \( (\theta, \phi) \iff (\theta, \phi) \) is a degenerate critical point of \( L_x \). From the equations \( \frac{\partial L_x}{\partial \theta} = 0 = \frac{\partial L_x}{\partial \phi} \), we obtain, for each \( (\theta, \phi) \in \mathbb{R}^2 \),

\( x = \tilde{f}_\xi(\theta, \phi) \) for some \( t, s \in \mathbb{R} \). By replacing \( x \) by \( \tilde{f}_\xi(\theta, \phi) \) and using \( \det H = 0 \)
we get
\[
\det \begin{bmatrix}
\frac{2}{\sqrt{2}}(1 + t - s) & 0 \\
0 & \frac{2}{\sqrt{2}}(1 + t + s)
\end{bmatrix} = 0 \iff (1 + t + s)(1 + t - s) = 0
\]
\[
\iff (1 + t)^2 - s^2 = 0
\]
\[
\iff s = \pm (1 + t).
\]
Therefore the focal set of \( \tilde{f} \) with base \((\theta, \phi) \in \mathbb{R} \times \mathbb{R}\) is a pair of lines perpendicular to one another which is the same for all base points \((\theta, \phi)\). Consequently
\[
\Omega(\tilde{f}) = \Omega_{(\theta, \phi)}(\tilde{f}) = \Omega_{(\theta', \phi')}(\tilde{f}), \quad \forall (\theta, \phi), (\theta', \phi') \in \mathbb{R} \times \mathbb{R}.
\]
Then, \(\Omega(\tilde{f})\) has four path-connected components since each \(\Omega_{(\theta, \phi)}(\tilde{f})\) has four path-connected components; one of index 0, two of index 1, and one of index 2. Hence the same is true for \(\Omega(f)\), as \(\Omega(f) = \Omega(\tilde{f})\), see the Figure 1. Then \(\Omega(f) = \Omega^0 \cup \Omega^1 \cup \Omega^2\) and in the Figure 1, we put \(\Omega^1 = A \cup B\).

![Figure 1](image)

We know by [3], \(\Omega(f)\) can have at most one component with index \(m\) and if there is such a component, it is unbounded. Note that, we have examples of immersions such that \(\Omega^0\) is bounded.
THEOREM 1. Let $f : M^m \to \mathbb{R}^{m+k}$ be an immersion of a compact, $m$-dimensional manifold such that $f$ has flat normal bundle. If $\Omega(f)$ has a component with index $m$, then $\Omega^0$ is unbounded.

PROOF. Let $a \in \Omega^m$ and take the immersive parallel normal field $\zeta(a)$. Then $\varphi_p^{-1}(a) = f_{\zeta(a)}(p), \forall p \in M$ and index $f_{\zeta(a)} = m$. So for all $p \in M$, the total multiplicity of focal points with base $p$ on the line segment from $f(p)$ to $f_{\zeta(a)}(p)$ is $m$. Therefore there are no focal points on the rays

$$R_p = \{ f(p) + t\zeta(a)(p) \in v_p(f) : t \geq 1 \}$$

$$Q_p = \{ f(p) + t\zeta(a)(p) \in v_p(f) : t \leq 0 \}.$$

Also $\forall p \in M$, $\varphi_p(Q_p) = \{ ta : t \leq 0 \} \subset \Omega_p$ and so

$$\{ ta : t \leq 0 \} \subset \bigcap \{ \Omega_p : p \in M \} = \Omega(f).$$

Hence $\{ ta : t \leq 0 \} \subset \Omega^0$ and $\Omega^0$ is unbounded. \hfill $\square$

PROPOSITION 2. Let $f : M^m \to \mathbb{R}^{m+k}$ be an immersion with flat normal bundle and assume $\zeta, \eta : M^m \to \mathbb{R}^{m+k}$ are immersive parallel normal fields for $f$ with indices $\lambda$ and $\mu$ respectively. Then, the number of focal points with base $p$ on the line segment from $f_{\zeta}(p)$ to $f_{\eta}(p)$ is constant for all $p \in M$ and it is $\lambda + \mu - 2l$ for some $l \in \mathbb{N}$ where $\max\{ 0, \lambda + \mu - m \} \leq l \leq \min\{ \lambda, \mu \}$.

PROOF. As in Lemma 4.4 of [3], since $f_{\zeta} + (\eta - \zeta) = f_\eta$, so $\eta - \zeta$ is an immersive parallel normal field for the immersion $f_\zeta$ and we need to find its index for $f_\zeta$ which is constant. Here we try to formulate this constant.

Let $p \in M$ and put $x = f_{\zeta}(p)$ and $y = f_{\eta}(p)$. If $f(p), x, y$ are collinear, then the number of focal points with base $p$ on the line segment from $f_{\zeta}(p)$ to $f_{\eta}(p)$ is $\lambda + \mu$ or $|\lambda - \mu|$ with respect to positioning of $f(p)$ and we can take $l = 0$ or $l = \lambda$ or $l = \mu$. Otherwise, take the triangle on $v_p(f)$ with vertices $f(p), x, y$, and consider the 2-plane $Q(p)$ which contains this triangle. We know that $Q(p) \cap F_p(f)$ is a union of at most $m$ lines if it is non empty, since $F_p(f)$ is a union of at most $m$ hyperplanes on $v_p(f)$ [8].

If $u, v \in v_p(f)$, then the notation $\overline{uv}$ denotes the line segment from $u$ to $v$. We know the total multiplicity of focal points on $\overline{f(p)x}$ is $\lambda$ and on $\overline{f(p)y}$ is $\mu$. Now, let $l(p) \geq 0$ be an integer and assume that $l(p)$ lines(counting multiplicities) meet both of the edges $\overline{f(p)x}$ and $\overline{f(p)y}$. Clearly $0 \leq l(p) \leq \min\{ \lambda, \mu \}$. Then the remaining $\lambda - l(p)$ lines intersecting $\overline{f(p)x}$ must intersect $\overline{xy}$. And similarly the remaining $\mu - l(p)$ lines intersecting $\overline{f(p)y}$ must intersect $\overline{xy}$. So we get the total multiplicity on $\overline{xy}$ is exactly
\[ \lambda - l(p) + \mu - l(p) = \lambda + \mu - 2l(p). \] So we deduce the total multiplicity of focal points on the line segment from \( f_\xi(p) \) to \( f_\eta(p) \) is \( \lambda + \mu - 2l(p) \). But the index of the parallel immersion \( (f_\xi)_\eta \) to \( f_\xi \) is a constant number, so \( l(p) \) is constant for all \( p \in M \).

Put \( l = l(p) \). Since there exist at most \( m \) lines on \( Q(p) \), then \( \lambda + \mu - m \leq l \). Then max\{0, \lambda + \mu - m\} \leq l \leq \min\{\lambda, \mu\}. \square

**Corollary 1.** If \( \lambda = \mu = 1 \) in Proposition 2, then, for all \( p \in M \), the number of focal points with base \( p \) on the line segment from \( f_\xi(p) \) to \( f_\eta(p) \) is 2 (where \( l = 0 \)).

**Theorem 2.** Let \( f : M^m \to \mathbb{R}^{m+k} \) be an immersion with flat normal bundle and assume \( \xi : M^m \to \mathbb{R}^{m+k} \) is an immersive parallel normal field for \( f \).

(i) There exists a \( w \in \mathbb{R}^k \) such that \( \Omega(f_\xi) = \Omega(f) - w \), where \( \Omega(f) - w = \{a - w : a \in \Omega(f)\} \).

(ii) If \( A \) is a path-connected component of \( \Omega(f) \) with index \( \mu \) and if the index of \( f_\xi \) is \( \lambda \), then there exists an \( l \in \mathbb{N} \) such that \( A - w \) is a path-connected component of \( \Omega(f_\xi) \) with index \( \lambda + \mu - 2l \), where \( \max\{0, \lambda + \mu - m\} \leq l \leq \min\{\lambda, \mu\} \).

**Proof.** (i) Since \( f \) has flat normal bundle, there exists a set of orthonormal parallel normal fields \( \{\xi_1, \xi_2, \cdots, \xi_k\} \) forming a basis of the normal space at each base point \( p \in M \). We will use this basis to define \( \Omega(f) \) and \( \Omega(f_\xi) \). We can write

\[ \xi = w_1\xi_1 + \cdots + w_k\xi_k \]

for some constants \( w_1, \ldots, w_k \in \mathbb{R} \) and put \( w = (w_1, \ldots, w_k) \in \mathbb{R}^k \). Then we can easily see that \( a \in \Omega(f) \iff a - w \in \Omega(f_\xi) \). In fact, let \( a = (a_1, \ldots, a_k) \in \Omega(f) \). We know \( F_p(f) = F_p(f_\xi) \) for all \( p \in M \) by Lemma 1 (ii). Then, for all \( p \in M \)

\[ f(p) + a_1\xi_1 + \cdots + a_k\xi_k \notin F_p(f) \iff f + \xi + a_1\xi_1 + \cdots + a_k\xi_k - \xi \]

\[ = f_\xi + (a_1 - w_1)\xi_1 + \cdots + (a_k - w_k)\xi_k \notin F_p(f_\xi). \]

So \( a - w \in \Omega(f_\xi) \) and therefore \( \Omega(f_\xi) = \Omega(f) - w \).

(ii) Let \( a \in A \), then clearly \( a - w \in \Omega(f_\xi) \) by Theorem 2 (i), hence \( A - w \) is a path-connected component of \( \Omega(f_\xi) \). Since \( A \) is a path-connected component of \( \Omega(f) \) with index \( \mu \), there exists an immersive parallel normal field \( \eta \) for \( f \) with index \( \mu \) and \( \varphi_p(f(p) + \eta(p)) = a \) for all \( p \in M \). As in
Proposition 2, $f_{\xi} + (\eta - \zeta) = f_\mu$, so $\eta - \zeta$ is an immersive parallel normal field for $f_\xi$ and its index for $f_\xi$ is $\lambda + \mu - 2l$ for some $l \in \mathbb{N}$ where $\max\{0, \lambda + \mu - m\} \leq l \leq \min\{\lambda, \mu\}$. 

The following result concerns the positioning of the path-connected components of $\Omega(f)$ in $\mathbb{R}^k$.

**Theorem 3.** Let $f : M^m \to \mathbb{R}^{m+k}$ be an immersion with flat normal bundle. Let $A, B$ be path-connected components of $\Omega(f)$ with index $\lambda$ and $\mu$ respectively. If $\lambda + \mu > m$, then there exists a hyperplane in $\mathbb{R}^k$ such that $A$ and $B$ lie on one side of the hyperplane and $\Omega^0$ lies on the opposite side of the hyperplane.

**Proof.** Let $A, B$ be path-connected components of $\Omega(f)$ with index $\lambda$ and $\mu$ respectively and $a \in A, b \in B$. Then there exist immersive parallel normal fields $\xi, \eta$ for $f$ such that index $f_{\xi} = \lambda$, index $f_{\eta} = \mu$ and also for all $p \in M$, $\varphi_p^{-1}(a) = f_{\xi}(p)$ and $\varphi_p^{-1}(b) = f_{\eta}(p)$. Now consider the normal plane $v_p(f)$ for a fixed $p \in M$ and the focal hyperplanes $\Pi_1, \ldots, \Pi_s$ on $v_p(f)$ with their respective multiplicity $w_i$ where $1 \leq i \leq s$, $s \leq m$ and $w_1 + \cdots + w_s \leq m$.

Since $f_{\xi}$ has index $\lambda$, the line segment joining $f(p)$ to $f_{\xi}(p)$ must cross $\Pi_{\beta_1}, \ldots, \Pi_{\beta_d}$ where $d \leq \mu$, $w_{\beta_1} + \cdots + w_{\beta_d} = \mu$.

Here, $\Pi_{\beta_1}, \ldots, \Pi_{\beta_1}, \Pi_{\beta_1}, \ldots, \Pi_{\beta_d}$ are not all distinct since $\lambda + \mu > m$. So let $\Pi \in \{\Pi_{\beta_1}, \ldots, \Pi_{\beta_1}\} \cap \{\Pi_{\beta_1}, \ldots, \Pi_{\beta_d}\}$. Then we claim that $A, B$ stay on one side of the hyperplane $\varphi_{\Pi}(\Pi) = A$ in $\mathbb{R}^k$. Set $\varphi_{\Pi}(\Pi_i) = A_i$, $1 \leq i \leq s$. Since each $A_i$ divides $\mathbb{R}^k$ into two open connected regions, we identify them by writing $A_i^-$ for the region including the origin and $A_i^+$ for the other part.

Then, $\Omega^0 \subset A_i^-$ for all $1 \leq i \leq s$, $A \subset A_{\beta_1}^+ \cap \cdots \cap A_{\beta_d}^+$ and $B \subset A_{\beta_1}^+ \cap \cdots \cap A_{\beta_d}^+$. Therefore $A$ and $B$ stay in $A^+$, and hence $A$ is the hyperplane we are seeking. 

3. Number of path-connected components of $\Omega(f)$ with certain indices

It is interesting to know the number of path-connected components of $\Omega(f)$ with their respective indices for an immersion $f$ of $M$ as it includes some information on the geometry and the topology of the $m$-dimensional compact manifold $M$. Here, we prove that if we have a path-connected
component of $\Omega(f)$ with index 1, then the Euler characteristic of $M$ is 0. Secondly, we prove that the number of path-connected components of $\Omega(f)$ with index $(m - 1)$ is at most 2 for a co-dimension 2 immersion.

**Theorem 4.** Let $f : M^m \to \mathbb{R}^{m+k}$ be an immersion of compact manifold with flat normal bundle and let $\chi(M) \neq 0$ where $m$ is an even number. Then $\Omega^1 = \emptyset$.

**Proof.** If $\Omega^1 \neq \emptyset$, then there exists a unit parallel normal field $\xi$ for $f$ such that $f_\xi = f + s\xi$ is an immersion with index 1 for some $s > 0$. So $\forall \ p \in M$, there exists only one focal point $c(p)$ of multiplicity 1 on the line segment from $f(p)$ to $f_\xi(p)$ such that $c : M^m \to \mathbb{R}^{m+k}$, $p \to c(p)$ is continuous. Define $\lambda : M^m \to \mathbb{R}$ by

$$\lambda(p) = \frac{1}{\|f(p) - c(p)\|}.$$ 

Then $\lambda$ is continuous as it is the principal curvature function of $f$ in the unit normal direction $\xi$. Also $\lambda$ is smooth since it is of constant multiplicity 1 on $M$ [7]. So the principal direction corresponding to the principal curvature $\lambda(p)$ defines a nonzero smooth tangent vector field on $M$ which has no zeros. So considering that $M$ is compact, $\chi(M) = 0$ by the Poincaré-Hopf Theorem in [5]. But this contradicts $\chi(M) \neq 0$. Therefore $\Omega^1 = \emptyset$. 

A generalisation of this theorem to any odd indexed component is proved in [1] by a different method. Present method here may not be generalized, because respective vector field can fail to be smooth.

**Definition 4.** Let $f : M^m \to \mathbb{R}^{m+k}$ be an immersion with flat normal bundle, then $d(f)$ is defined to be the total number of the path-connected components of $\Omega(f)$.

It was proved in [3] that $d(f) \leq \alpha(m, k)$ where $\alpha(m, k)$ is the number of path-connected regions in the complement of $m$ hyperplanes in general position in $\mathbb{R}^k$ as

$$\alpha(m, k) = \begin{cases} 2^m & \text{if } m \leq k \\ \sum_{i=0}^{k} {m \choose i} & \text{if } m > k \end{cases}.$$
COROLLARY 2. Let \( f : M^2 \rightarrow \mathbb{R}^{2+k} \) be an immersion with flat (or locally flat) normal bundle of a compact surface for some \( k \geq 1 \) and let \( \chi(M) \neq 0 \). Then \( \Omega^1 = \emptyset \) and so \( d(f) \leq 2 \).

**Proof.** By Theorem 4, \( \Omega^1 = \emptyset \) and also \( \Omega^0 \), \( \Omega^2 \) are connected [3], hence \( d(f) \leq 2 \). Of course \( \Omega^2 \) can occur, definitely when \( f \) is spherical, [2]. \( \square \)

**Example 2.** For the homology groups of real projective space \( \mathbb{R}P^m \), we know that \( H_i(\mathbb{R}P^m, \mathbb{Z}_2) = \mathbb{Z}_2 \) for all \( i = 1, 2, \ldots, m \). Then

\[
\chi(\mathbb{R}P^m) = \begin{cases} 
0 , & \text{if } m \text{ is odd} \\
1 , & \text{if } m \text{ is even}
\end{cases}
\]

So, by Theorem 4, if \( f : \mathbb{R}P^m \rightarrow \mathbb{R}^{m+k} \) is any immersion with locally flat normal bundle, we have \( \Omega^1 = \emptyset \) for \( m \) is even.

Let \( f : M^2 \rightarrow \mathbb{R}^{2+k} \) be an immersion of a 2-dimensional manifold \( M \) with flat normal bundle. Then for any point \( p \in M \), \( F_p(f) \) is a union of at most 2 hyperplanes in \( v_p(f) \). So \( F_p(f) \) can divide \( v_p(f) \) into at most 4 path-connected regions, and the number of path-connected components of \( \Omega(f) \) with index 1 can be at most 2 for any \( k \geq 2 \).

In the following theorems we generalize this and prove a result concerning the number of path-connected components of \( \Omega(f) \) with index \((m - 1)\) where \( m \geq 3 \).

**Theorem 5.** Let \( m \geq 2 \) and \( f : M^m \rightarrow \mathbb{R}^{m+k} \) be an immersion with flat normal bundle. Assume \( A, B \) are two different path-connected components of \( \Omega(f) \) both with index \((m - 1)\) and \( a \in A, b \in B \). Then for each \( p \in M \), all the focal hyperplanes in \( v_p(f) \) meet the triangle \( \triangle \) with vertices \( f(p), \varphi^{-1}_p(a), \varphi^{-1}_p(b) \), and moreover the total number of focal points on the line segment from \( \varphi^{-1}_p(a) \) to \( \varphi^{-1}_p(b) \) is exactly 2.

**Proof.** Let \( a \in A, b \in B \). Then there are corresponding parallel normal fields \( \zeta = \zeta(a) \) and \( \eta = \zeta(b) \) say, such that index \( f_\zeta = \text{index} \) \( f_\eta = m - 1 \). By Proposition 2, for all \( p \in M \), we have total number of focal points between \( f_\zeta(p) = \varphi^{-1}_p(a) \) and \( f_\eta(p) = \varphi^{-1}_p(b) \) is \( 2(m - 1) - 2l \) for some \( l \in \mathbb{N} \) where \( m - 2 \leq l \leq m - 1 \). Since \( a \) and \( b \) are in different components, there must be at least one focal point between \( f_\zeta(p) \) and \( f_\eta(p) \) for all \( p \in M \). So \( l = m - 2 \).
Let $Q \subset v_p(f)$ be the plane including the triangle $\triangle$ with vertices $f(p)$, $f_1(p)$, $f_0(p)$. Since $l = m - 2$, we have proved that the total multiplicity of focal points on $f_1(p)f_0(p)$ is exactly 2 for all $p \in M$ and hence there are exactly $m$ focal lines meeting with the triangle $\triangle$ as required. Since there are $m$ lines in $Q$, this implies that all focal hyperplanes on $v_p(f)$ meet with the triangle $\triangle$, for all $p \in M$.

**Theorem 6.** Let $f : M^m \to \mathbb{R}^{m+2}$ be an immersion of a compact manifold such that $f$ has flat normal bundle, where $m \geq 3$. Then the number of path-connected components of $\Omega f$ with index $(m - 1)$ is at most 2.

**Proof.** Assume there exist at least three path-connected components of $\Omega f$ with index $(m - 1)$, say $A, B, C$. Take $a \in A$, $b \in B$, $c \in C$. Let $p \in M$ be an arbitrary point and consider the points $x = \varphi_p^{-1}(a)$, $y = \varphi_p^{-1}(b)$, $z = \varphi_p^{-1}(c)$ on $v_p(f)$. Clearly $x, y, z$ are nonfocal distinct points, since $a, b, c$ are in different components.

Since $a, b, c$ are in different components there is at least one focal point on each line segment $xy, yz, zx$. So we can check that the points $x, y, f(p)$ cannot be collinear. Assume they lie on a line $\ell$ say. If $f(p)$ is on $xy$ then the total multiplicity of focal points on $\ell$ is at least $(2m - 2)$ which is not possible for $m \geq 3$, since $2m - 2 > m$. If $f(p)$ is not on $xy$ we get the total multiplicity of focal points on $f(p)x$ or $f(p)y$ is at least $m$ depending on the positioning of $f(p)$ on $\ell$ with respect to the points $x, y$. This contradicts the hypothesis that this number is $(m - 1)$. By a similar discussion we get the points $x, z, f(p)$ or $y, z, f(p)$ or $x, y, z, f(p)$ cannot be collinear.

By Theorem 5, all of the focal lines must meet the triangle with vertices $x, y, f(p)$ and further the total multiplicity of focal points on $xy$ is exactly 2. Similarly we get the same result considering the triangles with vertices $y, z, f(p)$ and $x, z, f(p)$.

We next show $x, y, z$ are not collinear. For if $x, y, z$ all lie on a line then by the above argument the total multiplicity of focal points on each line segment $xy, yz, zx$ is exactly 2. Without loss of generality we can assume $y$ is on $zx$. Then we obtain the total multiplicity of focal points on $zx$ is $2 + 2 = 4$ which is a contradiction.

Now consider the triangle with vertices $x, y, z$. There are 3 cases to be considered.

**Case 1.** Assume $f(p)$ is in the region I bounded by the triangle with vertices $x, y, z$ as shown in Figure 2. By Theorem 5 there exists at least one focal line meeting with $xf(p)$ and $zf(p)$ considering the triangle with ver-
vertices $x, f(p), z$. Similarly there exists one focal line meeting with $\overline{x f(p)}$ and $\overline{y f(p)}$ considering the triangle with vertices $x, f(p), y$. And also there exists one focal line meeting with $\overline{y f(p)}$ and $\overline{z f(p)}$ considering the triangle with vertices $y, f(p), z$. These focal lines are necessarily all different and together bound $f(p)$. This implies that $f(p)$ is in a bounded region of the complement of the focal lines on $v_p(f)$.

**Case 2.** Assume $f(p)$ is in the region II as shown in Figure 2. Consider the triangle with vertices $z, f(p), y$. By Theorem 5, there must be a focal line meeting with $\overline{f(p)z}$ and $\overline{z y}$ and this line must necessarily meet $\overline{x f(p)}$ and $\overline{x y}$. Similarly by considering the triangle with vertices $x, f(p), z$, there must be a focal line meeting with $\overline{f(p)z}$ and $\overline{x z}$ and this line must necessarily meet $\overline{f(p)y}$ and $\overline{x y}$. Now we get at least 2 focal points on $\overline{x y}$. But again by Theorem 5 and considering the triangle with vertices $x, f(p), y$, it is exactly 2. So there are no more focal lines meeting with $\overline{x y}$. So far we have one focal line meeting both $\overline{x f(p)}$ and $\overline{z f(p)}$. By Theorem 5 and considering the triangle with vertices $x, f(p), z$, we need $(m - 3)$ more focal lines meeting with $\overline{x f(p)}$ and $\overline{z f(p)}$ which must necessarily meet with $\overline{y f(p)}$. And one more focal line meeting both $\overline{x f(p)}$ and $\overline{x z}$ which must necessarily meet with $\overline{x y}$ or $\overline{z y}$. We know there are no more focal lines meeting with $\overline{x y}$. So the focal line meeting both $\overline{x f(p)}$ and $\overline{x z}$ must necessarily meet with $\overline{x y}$. This implies that for all $p \in M$, $z = f_{\xi_a}(p)$ is bounded by focal lines on $v_p(f)$ where the immersive parallel normal field $\xi(a)$ is corresponding to $a$.

**Case 3.** Now assume $f(p)$ is in the region III as shown in Figure 2. Then we know every focal line must meet the triangle with vertices $f(p), x, y$. But there must be a focal line meeting with $\overline{f(p)z}$ and $\overline{z x}$ simultaneously. So this line cannot meet the triangle with vertices $f(p), x, y$. This gives a contradiction by Theorem 5. So we deduce that Case 3 cannot occur.

Since $p$ is an arbitrary point in $M$ and $\varphi_p^{-1}$ is an isometry, then either Case 1 holds for all $p \in M$ or Case 2 holds for all $p \in M$ i.e. either $f(p)$ is bounded by focal lines on $v_p(f)$ or $f_{\xi_a}(p)$ is bounded by focal lines on $v_p(f)$ for all $p \in M$.

Now, for some $w \in \mathbb{R}^{m+2}$, take the distance function $L_w$ for $f$. Since $M$ is compact, there is a critical point of $L_w$ with index $m$. So the total number of focal points with base $p$ on the line segment from $w$ to $f(p)$ is $m$ and so there is no focal point with base $p$ on the ray $\{f(p) + t(w - f(p)) \mid t \leq 0\} \subset v_p(f)$. This implies that for some $p \in M$, $f(p)$ is not bounded by focal hyperplanes on $v_p(f)$ and a similar statement is true for $f_{\xi_a}(q)$ considering the immersion.
for some $q \in M$. So there cannot be such path-connected components $A, B, C$ of $\Omega(f)$. Therefore $\Omega(f)$ can have at most two path-connected components with index $(m - 1)$.

![Figure 2](image.png)

**Remark 1.** Theorem 6 is not true for an immersion with co-dimension $k > 2$. We can see that by taking product immersions. In Example 1, we have an embedding $f$ of $T^2$ into $S^3 \subset R^4$ such that $\Omega(f)$ has two path-connected components with index 1. Now take

$$f \times f : T^2 \times T^2 \to R^{4+4}$$

by $(f \times f)(p, q) = (f(p), f(q))$ where $p, q \in T^2$. Note that, by Theorem 4.2 of [3], $f \times f$ has flat normal bundle and $\Omega(f \times f) = \Omega(f) \times \Omega(f)$, since $f$ has flat normal bundle. Then, we can easily check that $\Omega(f \times f)$ has 4 path-connected components with index 3.

Consequently, for $m > 2$ and $k > 2$, it is a considerable question to ask what is the maximum number of path-connected components of $\Omega(f)$ with index $(m - 1)$. This might be at most $k$.

**References**


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