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Finite Groups with Weakly s-Semipermutable Subgroups

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ABSTRACT - Suppose G is a finite group and H is a subgroup of G. H is said to be s-semipermutable in G if $HG_p = G_pH$ for any Sylow p-subgroup G_p of G with $(p, |H|) = 1$; H is called weakly s-semipermutable in G if there is a subgroup T of G such that $G = HT$ and $H \cap T$ is s-semipermutable in G. We investigate the influence of weakly s-semipermutable subgroups on the structure of finite groups. Some recent results are generalized and unified.

1. Introduction.

All groups considered in this paper will be finite. We use conventional notions and notation, as in Huppert [1]. G denotes always a group, $|G|$ is the order of G, $\pi(G)$ denotes the set of all primes dividing $|G|$ and G_p is a Sylow p-subgroup of G for some $p \in \pi(G)$. Let $\mathcal{F}$ be a class of groups. We call $\mathcal{F}$ a formation provided that (i) if $G \in \mathcal{F}$ and $H \triangleleft G$, then $G/H \in \mathcal{F}$, and (ii) if G/M and G/N are in $\mathcal{F}$, then $G/(M \cap N)$ is in $\mathcal{F}$ for any normal subgroups M, N of G. A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$. In this paper, $\mathcal{U}$, $\mathcal{N}$ will denote the class of all supersolvable groups and the class of all nilpotent groups, respectively. As well-known results, $\mathcal{U}$, $\mathcal{N}$ are saturated formations.

Two subgroups H and K of G are said to be permutable if $HK = KH$. A subgroup H of G is said to be s-permutable (or s-quasinormal, π-quasinormal) in G if H permutes with every Sylow subgroup of G [2]. Asaad, Ramadan and Shaalan proved in [3]: Suppose G is a solvable group with a normal subgroup H such that G/H is supersolvable. If all

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maximal subgroups of any Sylow subgroup of $F(H)$ are s-permutable in G, then G is supersolvable. Later Asaad in [4] extended the result using formation theory: Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a solvable group with a normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are s-permutable in G, then $G \in \mathcal{F}$. As a generalization of s-permutable subgroup, the concept of s-semipermutable subgroup is introduced. A subgroup H of G is said to be s-semipermutable in G if $HG_p = G_pH$ for any Sylow p-subgroup G_p of G with $(p, |H|) = 1$. Q. Zhang and L. Wang [5] obtained the following: Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are s-semipermutable in G, then $G \in \mathcal{F}$. In recent years, it has been of interest to use supplementation properties of subgroups to characterize properties of a group. For example, Y. Wang [6] introduced the concept of c-supplemented subgroups and obtained the similar result in [7]: Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are c-supplemented in G, then $G \in \mathcal{F}$.

There is no obvious general relationship between s-semipermutable subgroup and c-supplemented subgroup. Hence it is meaningful to unify and generalize the two concepts and relate results. Recall that H is c-supplemented in G if there exists a subgroup K_1 such that $G = HK_1$ and $H \cap K_1 \leq H_G$, where H_G is the maximal normal subgroup of G contained in H. In this case, writing $K = H_GK_1$ we have $G = HK$ and $H \cap K = H_G$; of course, $H \cap K$ is s-semipermutable in G. On the basis of this observation, we introduce a new embedding property:

DEFINITION 1.1. *A subgroup H of a group G is called weakly s-semipermutable in G if there is a subgroup T of G such that $G = HT$ and $H \cap T$ is s-semipermutable in G.*

In the present paper, we study the influence of weakly s-semipermutable subgroups on the structure of some groups. In particular, we give some new characterizations of supersolvability and p-nilpotency of a group (and, more general, a group belonging to a given formation of finite groups) by using the weakly s-semipermutability of some primary subgroups. As application, we unify and generalize a series of known results.

2. Preliminaries.

LEMMA 2.1. *Suppose that H is an s-semipermutable subgroup of a group G and N is a normal subgroup of G. Then*

(a) *H is s-semipermutable in K whenever $H \leq K \leq G$.*

(b) *If H is p-group for some prime $p \in \pi(G)$, then HN/N is s-semipermutable in G/N.*

(c) *If $H \leq O_p(G)$, then H is s-permutable in G.*

PROOF. (a) is [5, Property 1], (b) is [5, Property 2], and (c) is [5, Lemma 3].

LEMMA 2.2. *Let U be a weakly s-semipermutable subgroup of a group G and N a normal subgroup of G. Then*

(a) *If $U \leq H \leq G$, then H is weakly s-semipermutable in H.*

(b) *Suppose that U is a p-group for some prime p. If $N \leq U$, then U/N is weakly s-semipermutable in G/N.*

(c) *Suppose U is a p-group for some prime p and N is a p'-subgroup, then UN/N is weakly s-semipermutable in G/N.*

PROOF. By the hypotheses, there is a subgroup K of G such that $G = UK$ and $U \cap K$ is s-semipermutable in G.

(a) $H = H \cap UK = U(H \cap K)$ and $U \cap (H \cap K) = U \cap K$ is s-semipermutable in H by Lemma 2.1(a). Hence U is weakly s-semipermutable in H.

(b) $G/N = UK/N = U/N \cdot NK/N$ and $(U/N) \cap (KN/N) = (U \cap KN)/N = (U \cap K)N/N$ is s-semipermutable in G/H by Lemma 2.1(b). Hence U/N is weakly s-semipermutable in G/N.

(c) Since $(|G : K|, |N|) = 1$, $N \leq K$. It is easy to see that $G/N = UN/N \cdot KN/N = UN/N \cdot K/N$ and $(UN/N) \cap (K/N) = (UN \cap K)/N = (U \cap K)N/N$ is s-semipermutable in G/N by Lemma 2.1(b). Hence UN/N is weakly s-semipermutable in G/N.

LEMMA 2.3 ([8], Lemma 2.6). *Let H be a solvable normal subgroup of a group $G(H \neq 1)$. If every minimal normal subgroup of G which is contained in H is not contained in $\Phi(G)$, then the Fitting subgroup $F(H)$ of H is the direct product of minimal normal subgroups of G which are contained in H.*

LEMMA 2.4 ([7], Lemma 2.8). *Let M be a maximal subgroup of G, P a normal p-subgroup of G such that $G = PM$, where p a prime. Then $P \cap M$ is a normal subgroup of G.*

LEMMA 2.5 ([9], Lemma 2.16). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup N such that $G/N \in \mathcal{F}$. If N is cyclic, then $G \in \mathcal{F}$.*

LEMMA 2.6 ([9], Lemma 2.20). *Let A be a p'-automorphisms of a p-group P, where p is an odd prime. Assume that every subgroup of P with prime order is A-invariant. Then A is cyclic.*

LEMMA 2.7 ([1], III, 5.2 and IV, 5.4). *Suppose G is a group which is not nilpotent but whose proper subgroups are all nilpotent. Then*

(a) *G has a normal Sylow p-subgroup P for some prime p and $G = PQ$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$.*

(b) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.*

(c) *If P is non-abelian and $p > 2$, then the exponent of P is p; If P is non-abelian and $p = 2$, then the exponent of P is 4.*

(d) *If P is abelian, then the exponent of P is p.*

(e) *$\Phi(P) \leq Z(P)$.*

LEMMA 2.8 ([8], Lemma 3.12). *Let P be a Sylow p-subgroup of a group G, where p is the smallest prime dividing $|G|$. If G is A_4-free and $|P| \leqslant p^2$, then G is p-nilpotent.*

3. Results.

THEOREM 3.1. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. A group $G \in \mathcal{F}$ if and only if there is a normal subgroup E of G such that $G/E \in \mathcal{F}$ and every cyclic subgroup $\langle x \rangle$ of any noncyclic Sylow subgroup of E with prime order or order 4 (if the Sylow 2-subgroup is non-abelian) is weakly s-semipermutable in G.*

PROOF. We need only to prove the sufficiency part since the necessity part is evident. Suppose that the assertion is false and let (G, E) be a counterexample for which $|G||E|$ is minimal. Then

(1) E is solvable.

Let K be any proper subgroup of E. Then $|K| < |G|$ and $K/K \in \mathcal{U}$. Let $\langle x \rangle$ be any cyclic subgroup of any noncyclic Sylow subgroup of K with prime order or order 4 (if the Sylow 2- subgroup is non-abelian). It is clear that $\langle x \rangle$ is also a cyclic subgroup of a noncyclic Sylow subgroup of E with prime order or order 4. By the hypothesis, $\langle x \rangle$ is weakly s-semipermutable in G. By Lemma 2.2, $\langle x\rangle$ is weakly s-semipermutable in K. This shows that the hypothesis still holds for $(\mathcal{U}, K)$. By the choice of G, K is supersolvable. By [10, Theorem 3.11.9], E is solvable.

(2) $G^{\mathcal{F}}$ is a p-group,where $G^{\mathcal{F}}$ is the $\mathcal{F}$-residual of G. $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G and $\exp(G^{\mathcal{F}}) = p$ or $\exp(G^{\mathcal{F}}) = 4$ (if $p = 2$ and $G^{\mathcal{F}}$ is non-abelian).

Since $G/E \in \mathcal{F}$, $G^{\mathcal{F}} \leq E$. Let M be a maximal subgroup of G such that $G^{\mathcal{F}} \nsubseteq M$ (that is, M is an $\mathcal{F}$-abnormal maximal subgroup of G). Then $G = ME$. We claim that the hypothesis holds for $(\mathcal{F}, M)$. In fact, $M/M\cap E \cong ME/E = G/E \in \mathcal{F}$ and by the similar argument as above, we can prove that the hypothesis holds for $(\mathcal{F}, M)$. By the choice of G, $M \in \mathcal{F}$. Thus (2) holds by [10, Theorem 3.4.2].

(3) $\langle x\rangle$ is s-permutable in G for any element $x \in G^{\mathcal{F}}$.

Let $x \in G^{\mathcal{F}}$. Then the order of x is p or 4 by step (2). By the hypothesis, $\langle x\rangle$ is weakly s-semipermutable in G. Then there is a group T of G such that $G = \langle x\rangle T$ and $\langle x\rangle \cap T$ is s-semipermutable in G. It follows that $G = \langle x\rangle T$ and $G^{\mathcal{F}} = G^{\mathcal{F}} \cap G = G^{\mathcal{F}} \cap \langle x\rangle T = \langle x\rangle(G^{\mathcal{F}} \cap T)$. Since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is abelian, $(G^{\mathcal{F}} \cap T)\Phi(G^{\mathcal{F}})/\Phi(G^{\mathcal{F}}) \lhd G/\Phi(G^{\mathcal{F}})$. Since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G, $G^{\mathcal{F}} \cap T \leq \Phi(G^{\mathcal{F}})$ or $G^{\mathcal{F}} = (G^{\mathcal{F}} \cap T)\Phi(G^{\mathcal{F}}) = G^{\mathcal{F}} \cap T$. If $G^{\mathcal{F}} \cap T \leq \Phi(G^{\mathcal{F}})$, $\langle x\rangle = G^{\mathcal{F}} \lhd G$. In this case, $\langle x\rangle$ is s-permutable in G. Now assume that $G^{\mathcal{F}} = G^{\mathcal{F}} \cap T$. Then $T = G$ and $\langle x\rangle = \langle x\rangle \cap T$ is s-semipermutable in G. Since $\langle x\rangle \leq G^{\mathcal{F}} \leq O_p(G)$, $\langle x\rangle$ is s-permutable in G by Lemma 2.1.

(4) $|G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})| = p$.

Assume that $|G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})| \neq p$ and let $L/\Phi(G^{\mathcal{F}})$ be any cyclic subgroup of $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$. Let $x \in L\backslash\Phi(G^{\mathcal{F}})$. Then $L = \langle x\rangle\Phi(G^{\mathcal{F}})$. Since $\langle x\rangle$ is s-permutable in G by step (3), $L/\Phi(G^{\mathcal{F}})$ is s-permutable in $G/\Phi(G^{\mathcal{F}})$. It follows from [9, Lemma 2.11] that $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ has a maximal subgroup which is normal in $G/\Phi(G^{\mathcal{F}})$. But this is impossible since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G. Thus $|G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})| = p$.

(5) The final contradiction.

Since $(G/\Phi(G^{\mathcal{F}}))/(G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})) \cong G/G^{\mathcal{F}} \in \mathcal{F}$, $G/\Phi(G^{\mathcal{F}}) \in \mathcal{F}$ by Lemma 2.5. As $\Phi(G^{\mathcal{F}}) \leq \Phi(G)$ and $\mathcal{F}$ is a saturated formation, we have $G \in \mathcal{F}$. The final contradiction completes the proof.

COROLLARY 3.2 [14], Theorem 3.4). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups. If every cyclic subgroup of $G^{\mathcal{F}}$ with prime order or order 4 is c-normal in G, then $G \in \mathcal{F}$.*

COROLLARY 3.3 ([15], Theorem 4.2). *If every cyclic subgroup of a group G with prime order or order 4 is c-normal in G, then G is supersolvable.*

Corollary 3.4 ([16], Theorem 4.1). *If every cyclic subgroup of $G^{\mathcal{U}}$ with prime order or order* 4 *is c-supplemented in G, then G is supersolvable.*

Corollary 3.5 ([17], Theorem 1). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups. If there is a normal subgroup H of G such that $G/H \in \mathcal{F}$ and every cyclic subgroup of H with prime order or order* 4 *is s-permutable in G, then $G \in \mathcal{F}$.*

Corollary 3.6 ([21], Theorem 3.9). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups. Then $G \in \mathcal{F}$ if and only if there is a normal subgroup H of G such that $G/H \in \mathcal{F}$ and the subgroups prime order or order* 4 *of H with are c-normal in G.*

Corollary 3.7 ([19], Theorem 3.1). *Let G be a group and N a normal subgroup of a group G such that G/N is supersolvable. If every minimal subgroup of E is c-supplemented in G and if every cyclic subgroup of order* 4 *of N is c-normal in G, then G is supersolvable.*

Theorem 3.8. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. A group $G \in \mathcal{F}$ if and only if there is a solvable normal subgroup H of G such that $G/H \in \mathcal{F}$ and every cyclic subgroup $\langle x\rangle$ of any noncyclic Sylow subgroup of $F(H)$ with prime order or order* 4 *(if the Sylow* 2*-subgroup is non-abelian) is weakly s-semipermutable in G.*

Proof. It is clear that the condition is necessary. We only need to prove that it is sufficient. Suppose that the assertion is false and let (G, H) be a counterexample for which $|G||H|$ is minimal. Let p be the smallest prime divisor of $|F(H)|$ and P the Sylow p-subgroup of $F(H)$. Then $P \lhd G$. Now we proceed with our proof as follows:

(1) $F(H) \neq H$ and $C_H(F(H)) \leq F(H)$.

If $F(H) = H$, then $G \in \mathcal{F}$ by Theorem 3.1, a contradiction. Obviously, $C_H(F(H)) \leq F(H)$ since H is solvable.

(2) Let $V/P = F(H/P)$ and Q be a Sylow q-subgroup of V, where $q||V/P|$. Then $q \neq p$ and either $Q \leq F(H)$ or $p > q$ and $C_Q(P) = 1$.

Since V/P is nilpotent, QP/P char V/P and so $QP \lhd H$. Then, it is easy to see that $p \neq q$. By Theorem 3.1, PQ is supersolvable. If $q > p$, then $Q \lhd PQ$ and so $Q \leq F(H)$. Now assume that $p > q$. Then $p > 2$. Since p is the minimal prime divisor of $|F(H)|$, $F(H)$ is a q'-group. Let R be a Sylow r-

subgroup of $F(H)$ where $r \neq p$. Then $r \neq q$ and so $[R, Q] \leq P$. Assume that for some $x \in Q$, we have $x \in C_H(P)$. Since V/P is nilpotent, $[R, \langle x\rangle] = [R, \langle x\rangle, \langle x\rangle] = 1$ by [11, Chapter 5, Theorem 3.6]. Hence $x \in C_H(F(H))$. By (1), $C_H(F(H)) \leq F(H)$ and so $C_Q(P) = 1$.

(3) $p > 2$. If $p = 2$, then by (2), we see that $F(H/P) = F(H)/P$ and $2 \nmid |F(H/P)|$. This implies that if $\langle x\rangle P/P$ is an arbitrary minimal subgroup of $F(H)/P$, then $|x| = r$, where $r \neq 2$. By Lemma 2.2, every minimal subgroup of $F(H/P)$ is weakly s-semipermutable in G/P. Hence $(G/P, H/P)$ satisfies the hypothesis. The minimal choice of (G, H) implies that $G/P \in \mathcal{F}$. Hence by Theorem 3.1, $G \in \mathcal{F}$, a contradiction. Thus, (3) holds.

(4) Final contradiction. Let $V/P = F(H/P)$ and Q be a Sylow q-subgroup of V, where $q \| V/P|$. Then by (2), either $Q \leq F(H)$ or $p > q$ and $C_Q(P) = 1$. In the second case, Q is cyclic by (3) and Lemma 2.6. Hence every Sylow subgroup of $F(H/P)$ either is cyclic or is contained in $F(H)$. Moreover by (2), $p \nmid |F(H/P)|$. Let K/P be a cyclic subgroup of a non-cyclic Sylow subgroup of $F(H/P)$ with prime order. Then it is easy to see that $K/P = \langle x\rangle P/P$, where $\langle x\rangle$ is a cyclic subgroup of some non-cyclic Sylow subgroup of $F(H)$ with prime order. By hypothesis, $\langle x\rangle$ is weakly s-semipermutable in G. Hence $\langle x\rangle P/P$ is weakly s-semipermutable in G/P by Lemma 2.2. This shows that $(G/P, H/P)$ satisfies the hypothesis. The minimal choice of (G, H) implies that $G/P \in \mathcal{F}$. Therefore, $G \in \mathcal{F}$ by Theorem 3.1. The final contradiction completes the proof.

Corollary 3.9 ([22], Theorem 3). *Let G be a group and E a solvable normal subgroup of G such that G/E is supersolvable. If all minimal subgroups and all cyclic subgroups with order 4 of $F(E)$ are c-normal in G, then G is supersolvable.*

Corollary 3.10 ([23], Theorem 2). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If all minimal subgroups and all cyclic subgroups with order 4 of $F(H)$ is c-normal in G, then $G \in \mathcal{F}$.*

Corollary 3.11 ([24], Theorem 3). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. A group $G \in \mathcal{F}$ if and only if there is a solvable normal subgroup H of G such that $G/H \in \mathcal{F}$ and the subgroups of prime order or order 4 of $F(H)$ is c-normal in G.*

Corollary 3.12 ([7], Theorem 4.1). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a solvable normal subgroup*

H such that $G/H \in \mathcal{F}$. If all minimal subgroups and all cyclic subgroups with order 4 of $F(H)$ is c-supplemented in G, then $G \in \mathcal{F}$.

Corollary 3.13 ([5], Theorem 4). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If all minimal subgroups and all cyclic subgroups with order 4 of $F(H)$ is s-semipermutable in G, then $G \in \mathcal{F}$.*

Corollary 3.14 ([27], Corollary 1). *Suppose G is a solvable group with a normal subgroup H such that G/H is supersolvable. If every subgroup of $F(H)$ of prime order or order 4 is s-permutable in G, then G is supersolvable.*

Corollary 3.15 ([27], Theorem 1). *A group $G \in \mathcal{F}$ if and only if there is a solvable normal subgroup H of G such that $G/H \in \mathcal{F}$ and the subgroups of prime order or order 4 of $F(H)$ is s-permutable in G.*

Theorem 3.16. *Suppose G is a group. If every subgroup of G with prime order is contained in $Z_\infty(G)$ and every cyclic subgroup of G with order 4 is weakly s-semipermutable in G or lies in $Z_\infty(G)$, then G is nilpotent.*

Proof. Suppose that the theorem is false, and let G be a counterexample of minimal order. Let H be an arbitrary proper subgroup of G and $\langle x\rangle$ be a cyclic subgroup of H with prime order or order 4, then $\langle x\rangle \le Z_\infty(G) \cap H \le Z_\infty(H)$. By Lemma 2.2, $\langle x\rangle$ is weakly s-semipermutable in H. Thus H satisfies the hypotheses of the theorem in any case. The minimal choice of G implies that H is nilpotent, thus G is a group which is not nilpotent but whose proper subgroups are all nilpotent. By Lemmas 2.7, $G = PQ$, where P is normal in G for some $p \in \pi(G)$ and Q is non-normal cyclic. Then we have:

(1) $p = 2$ and every element with order 4 is weakly s-semipermutable in G.

If $p > 2$, by Lemma 2.7, $\exp(P) = p$. Thus $P \le Z_\infty(G)$ by hypotheses. Therefore, $G/Z_\infty(G)$ is nilpotent. It follows that G is nilpotent, a contradiction. If every element with order 4 of G lies in $Z_\infty(G)$, then $P \le Z_\infty(G)$, we have the same contradiction. Thus (1) holds.

(2) For every $x \in P\backslash\Phi(P)$, we have $\circ(x) = 4$.

If not, there exists $x \in P\backslash\Phi(P)$ and $\circ(x) = 2$. Denote $M = \langle x\rangle^G \le P$. Then $M\Phi(P)/\Phi(P) \lhd G/\Phi(P)$, we have that $P = M\Phi(P) = M \le Z(G)$ as $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$ by Lemma 2.7, a contradiction.

(3) Final contradiction.

By (2), every element x in $P \backslash \Phi(P)$ is order 4. Then x is weakly s-semipermutable in G. Thus there is a subgroup T of G such that $G = \langle x \rangle T$ and $\langle x \rangle \cap T$ is semipermutable in G. Hence $P = P \cap G = P \cap \langle x \rangle T = \langle x \rangle (P \cap T)$. Since $P/\Phi(P)$ is abelian, we have $(P \cap T)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$. Since $P/\Phi(P)$ is the minimal normal subgroup of $G/\Phi(P)$, $P \cap T \leq \Phi(P)$ or $P = (P \cap T)\Phi(P) = P \cap T$. If $P \cap T \leq \Phi(P)$, then $\langle x \rangle = P$, a contraction with Lemma 2.7(d). Therefore $P = P \cap T$. Then $T = G$ and so $\langle x \rangle$ is s-semipermutable in G. Since $\langle x \rangle \leq O_p(G) = P$, $\langle x \rangle$ is permutable in G by Lemma 2.1. We have $\langle x \rangle Q$ is a proper subgroup of G and so $\langle x \rangle Q = \langle x \rangle \times Q$. Therefore $x \in N_G(Q)$, and it follows that $P \leq N_G(Q)$ and $G = P \times Q$, the final contradiction.

THEOREM 3.17. *Let $\mathcal{F}$ be a saturated formation such that $\mathcal{N} \subseteq \mathcal{F}$. Let G be a group such that every cyclic subgroup of $G^{\mathcal{F}}$ with order 4 is weakly s-semipermutable in G. Then $G \in \mathcal{F}$ if and only if every subgroup of $G^{\mathcal{F}}$ with prime order lies in the $\mathcal{F}$-hypercenter $Z_{\mathcal{F}}(G)$ of G.*

PROOF. If $G \in \mathcal{F}$, then $Z_{\mathcal{F}}(G) = G$ and we are done. So we only need to prove that the converse is true. Assume the converse is false and let G be a counterexample of minimal order. Then $G \notin \mathcal{F}$. Let x be an element with prime order of $G^{\mathcal{F}}$. Then $x \in Z_{\mathcal{F}}(G) \cap G^{\mathcal{F}}$ which is contained in $Z(G^{\mathcal{F}})$ by [12, IV, 6.10]. By Lemma 2.2, every cyclic subgroup of $G^{\mathcal{F}}$ with order 4 is weakly s-semipermutable in $G^{\mathcal{F}}$. Theorem 3.16 implies that $G^{\mathcal{F}}$ is nilpotent. If $G^{\mathcal{F}} \leq \Phi(G)$, then $G/\Phi(G) \in \mathcal{F}$, hence $G \in \mathcal{F}$ since $\mathcal{F}$ is saturated. This is a contradiction. So there exists a maximal subgroup of G, say M, such that $G = MG^{\mathcal{F}} = MF(G)$. By [13, Theorem 3.5], we may choose M to be an $\mathcal{F}$-critical maximal subgroup. Since $G/M_G \notin \mathcal{F}$, it follows that $Z_{\mathcal{F}}(G) \leq M$. Moreover, a G-chief factor A/B below $Z_{\mathcal{F}}(G)$ is actually an M-chief factor and $\mathrm{Aut}_M(A/B)$ is isomorphic to $\mathrm{Aut}_G(A/B)$ because $F(G)$ centralizes A/B. Consequently $Z_{\mathcal{F}}(G)$ is contained in $Z_{\mathcal{F}}(M)$. By [12, IV, 1.17], $M^{\mathcal{F}} \leq G^{\mathcal{F}}$. Hence M satisfies the hypotheses of the theorem. The minimal choice of G implies that $M \in \mathcal{F}$. By [10, Theorem 3.4.2], G has the following properties:

(a) $G^{\mathcal{F}}$ is a p-group, for some prime p.
(b) $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a minimal normal subgroup of $G/\Phi(G^{\mathcal{F}})$.
(c) If $G^{\mathcal{F}}$ is abelian, then $G^{\mathcal{F}}$ is an elementary abelian p-group.
(d) If $p > 2$, then $\exp(G^{\mathcal{F}}) = p$; if $p = 2$, then $\exp(G^{\mathcal{F}}) = 2$ or 4.

If $G^{\mathcal{F}}$ is abelian, then $G^{\mathcal{F}}$ is an elementary abelian subgroup by (c). Hence, by hypothesis, we have that $G^{\mathcal{F}} \leq Z_{\mathcal{F}}(G)$. It follows that $G \in \mathcal{F}$.

This contradiction shows that $G^{\mathcal{F}}$ is nonabelian. If $\exp(G^{\mathcal{F}}) = p$, then $G^{\mathcal{F}} \leq Z_{\mathcal{F}}(G)$ by hypothesis and consequently $G \in \mathcal{F}$, a contradiction again. Thus, $G^{\mathcal{F}}$ is a non-abelian 2-group and $\exp(G^{\mathcal{F}}) = 4$.

Let x be an arbitrary element of $G^{\mathcal{F}}\backslash\Phi(G^{\mathcal{F}})$. Then $|x| = 4$. Indeed, suppose that there exists an element $x \in G^{\mathcal{F}}\backslash\Phi(G^{\mathcal{F}})$ such that $|x| = 2$. Let $T = \langle x\rangle^G$. Then $T \leq G^{\mathcal{F}}$ and $T\Phi(G^{\mathcal{F}})/\Phi(G^{\mathcal{F}})$ is normal in $G/\Phi(G^{\mathcal{F}})$. Since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G, $G^{\mathcal{F}} = T$, which contradicts the fact that $\exp(G^{\mathcal{F}}) = 4$. Now we will prove $\langle x\rangle$ is s-permutable in G. By hypothesis, $\langle x\rangle$ is weakly s-semipermutable in G. Hence there exists a subgroup K of G such that $G = \langle x\rangle K$ and $\langle x\rangle \cap K$ is s-semipermutable in G. It follows that $G^{\mathcal{F}} = G^{\mathcal{F}} \cap G = G^{\mathcal{F}} \cap \langle x\rangle K = \langle x\rangle(G^{\mathcal{F}} \cap K)$. Since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is abelian, $(G^{\mathcal{F}} \cap K)\Phi(G^{\mathcal{F}})/\Phi(G^{\mathcal{F}}) \lhd G/\Phi(G^{\mathcal{F}})$. Since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G, $G^{\mathcal{F}} \cap K \leq \Phi(G^{\mathcal{F}})$ or $G^{\mathcal{F}} = (G^{\mathcal{F}} \cap K)\Phi(G^{\mathcal{F}}) = G^{\mathcal{F}} \cap K$. If $G^{\mathcal{F}} \cap K \leq \Phi(G^{\mathcal{F}})$, $\langle x\rangle = G^{\mathcal{F}} \lhd G$. In this case, $\langle x\rangle$ is s-permutable in G. Now assume that $G^{\mathcal{F}} = G^{\mathcal{F}} \cap K$. Then $K = G$ and $\langle x\rangle$ is s-semipermutable in G. Since $\langle x\rangle \leq G^{\mathcal{F}} \leq O_p(G)$, $\langle x\rangle$ is s-permutable in G by Lemma 2.1.

Thus for any $q \in \pi(G)$, $q \neq 2$, $\langle x\rangle$ is normalized by any Sylow q-subgroup Q of M. So Q acts on $\langle x\rangle$ by conjugation. But the automorphism group of a cyclic group of order 4 is a cyclic group of order 2, so Q acts trivially on $\langle x\rangle$ and Q centralizes $\langle x\rangle$. Thus $\langle x\rangle$ is centralized by $O^2(M)$, it implies that $G^{\mathcal{F}}$ is centralized by $O^2(M)$. Hence $O^2(M) \lhd G$ as $G = MG^{\mathcal{F}}$. It follows that G/M_G is a 2-group. Therefore, $G/M_G \in \mathcal{F}$ since $\mathcal{N} \subseteq \mathcal{F}$, a final contradiction. This completes the proof of Theorem 3.17.

Corollary 3.18 ([14], Theorem 3.2). *Let $\mathcal{F}$ be a saturated formation such that $\mathcal{N} \subseteq \mathcal{F}$. Let G be a group such that every cyclic subgroup of $G^{\mathcal{F}}$ with order 4 is c-normal in G. Then $G \in \mathcal{F}$ if and only if every subgroup of $G^{\mathcal{F}}$ with prime order lies in the $\mathcal{F}$-hypercenter $Z_{\mathcal{F}}(G)$ of G.*

Corollary 3.19 ([18], Theorem 4.4). *Let $\mathcal{F}$ be a saturated formation such that $\mathcal{N} \subseteq \mathcal{F}$. Let G be a group such that every cyclic subgroup of $G^{\mathcal{F}}$ with order 4 is c-supplemented in G. Then $G \in \mathcal{F}$ if and only if every subgroup of $G^{\mathcal{F}}$ with prime order lies in the $\mathcal{F}$-hypercenter $Z_{\mathcal{F}}(G)$ of G.*

Corollary 3.20 ([19], Theorem 2.5). *Suppose that p is a prime and $K = G^{\mathcal{N}}$ be the nilpotent residual of G. Then G is p-nilpotent if every minimal subgroup of K is contained in $Z_\infty(G)$ and every cyclic $\langle x\rangle$ of K with order 4 is c-supplemented in G.*

Corollary 3.21 ([20], Theorem 2.4). *Let G be a finite group and $K = G^{\mathcal{N}}$ be the nilpotent residual of G. Then G is nilpotent if and only if* *every minimal subgroup $\langle x\rangle$ of K lies in the hypercenter $Z_\infty(G)$ of G and every cyclic element of P with order 4 is c-normal in G.*

THEOREM 3.22. *Let p be the smallest prime dividing the order of a group G and N a normal subgroup of G such that G/N is p-nilpotent. If G is A_4-free and every subgroup of N with order p^2 is weakly s-semipermutable in G, then G is p-nilpotent.*

PROOF. Assume that the Theorem is false and let G be a counterexample of minimal order. Then:

(1) Every proper subgroup of G is p-nilpotent.

By Lemma 2.8, we see that $|N|_p > p^2$. Let L be a proper subgroup of G. Since $L/(L\cap N)\cong LN/N\le G/N$, $L/(L\cap N)$ is p-nilpotent. If $|L\cap N|_p\leqslant p^2$, then L is p-nilpotent by Lemma 2.8. If $|L\cap N|_p > p^2$, then every subgroup of $L\cap N$ of order p^2 is weakly s-semipermutable in L by Lemma 2.2. Hence L is p-nilpotent by the choice of G. This shows that G is a minimal non-p-nilpotent group.

(2) G has the following properties: (i) $G=PQ$, where $P=G^{\mathcal{N}}$ is a normal Sylow p-subgroup of G and Q is a non-normal cyclic Sylow q-subgroup of G; (ii) $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$; (iii) If $p>2$, then the exponent of P is p; if $p=2$, then the exponent of P is 2 or 4; (iv) $\Phi(P)\le Z(P)$; (v) p^3 dividing the order of P; (vi) $P\le N$.

By Step (1) and [1, Theorem IV. 5.4], G is a minimal non-nilpotent group. Hence (i)-(iv) follow directly from Lemma 2.7. (v) follows from Lemma 2.8. (vi) is clear since $P=G^{\mathcal{N}}$ is the p-nilpotent residual of G and G/N is p-nilpotent.

(3) If H is a subgroup of P of order p^2, then H is s-permutable in G.

Let H be a subgroup of P of order p^2. By the hypothesis, H is weakly s-semipermutable in G. Then there is a subgroup T of G such that $G=HT$ and $H\cap T$ is s-semipermutable in G. Hence $P=P\cap G=P\cap HT=H(P\cap T)$. Since $P/\Phi(P)$ is abelian, we have $(P\cap T)\Phi(P)/\Phi(P)\lhd G/\Phi(P)$. By step (2) (ii), $P\cap T\le \Phi(P)$ or $P=(P\cap T)\Phi(P)=P\cap T$. If $P\cap T\le\Phi(P)$, then $H=P\lhd G$. In this case, H is s-permutable in G. If $P=P\cap T$, then $T=G$ and so H is s-semipermutable in G. Since $H\le P=O_p(G)$, H is s-permutable in G by Lemma 2.1.

(4) There exists a subgroup H of P such that $|H|=p^2$ which is not contained in $\Phi(P)$.

If $\Phi(P)=1$, then it is clear. Hence we may assume that $\Phi(P)\ne 1$. If $|P|=p^3$, then clearly P has a maximal subgroup of order p^2. Since P is not cyclic by Burnside's Theorem [11, Theorem 4.3, P.252], P has at least two different maximal subgroups P_1 and P_2. If P_1 and P_2 are all contained in $\Phi(P)$, then $P = P_1P_2 \leq \Phi(P)$, a contradiction. Hence, we can assume that $|P| > p^3$. Let $x \in P \setminus \Phi(P)$ and $a \in \Phi(P)$ where $|a| = p$. Since $\Phi(P) \leq Z(P)$, $\langle x\rangle\langle a\rangle \leq G$. By Step (2), we see that $|x| = p$ or 4. If $|x| = 4$, we can choose $H = \langle x\rangle$. If $|x| = p$, then $|\langle x\rangle\langle a\rangle| \leqslant p^2$. If $|\langle x\rangle\langle a\rangle| = p$, then $\langle x\rangle = \langle a\rangle$, a contradiction. Hence $|\langle x\rangle\langle a\rangle| = p^2$. Therefore (4) holds.

(5) Final contradiction.

By Step (2), $G = [P]Q$. By Step (4), there exists a subgroup H of P with order p^2 such that $H \nleq \Phi(P)$. Then by (3), H is s-permutable in G. Hence $HQ = QH$. Then $H = H(Q \cap P) = HQ \cap P \lhd HQ$. It follows that $Q \leq N_G(H)$. On the other hand, since $P/\Phi(P)$ is abelian, $H\Phi(P)/\Phi(P) \lhd P/\Phi(P)$. This implies that $H\Phi(P)/\Phi(P) \lhd G/\Phi(P)$. However, since $P/\Phi(P)$ is chief factor of G, we obtain that $H\Phi(P) = P$ and consequently $H = P$, a contradiction.

THEOREM 3.23. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are weakly s-semipermutable in G, then $G \in \mathcal{F}$.*

PROOF. Assume that the assertion is false and let (G, H) be a counter example with $|G||H|$ is minimal. Let P be an arbitrary Sylow p-subgroup of $F(H)$. Clearly $P \lhd G$. We proceed the proof by the following steps.

(1) $P \cap \Phi(G) = 1$.

If $P \cap \Phi(G) \neq 1$, then $P \cap \Phi(G) = R \lhd G$. Obviously, $(G/R)/(H/R) \cong G/H \in \mathcal{F}$ and $F(H/R) = F(H)/R$. Let P_1/R be a maximal subgroup of the Sylow p-subgroup P/R. Then P_1 is a maximal subgroup of the Sylow p-subgroup P. By hypothesis, P_1 is weakly s-semipermutable in G. Hence P_1/R is weakly s-semipermutable in G/R by Lemma 2.2. Let M_1/R be a maximal subgroup of the Sylow q-subgroup of $F(H)/R$, where $p \neq q$. It is clear that $M_1 = Q_1R$, where Q_1 is a maximal subgroup of the Sylow q-subgroup of $F(H)$. Then Q_1 is weakly s-semipermutable in G by hypothesis. Hence M_1/R is weakly s-semipermutable in G/R by Lemma 2.2. Now we have proved that $(G/R, H/R)$ satisfies the hypotheses of the theorem. Therefore $G/R \in \mathcal{F}$ by minimal choice of (G, H). Since $R \leq \Phi(G)$ and $\mathcal{F}$ is a saturated formation, we have that $G \in \mathcal{F}$, a contradiction. Thus (1) holds.

(2) $P = R_1 \times R_2 \times \cdots \times R_m$, where $R_i (i = 1, 2, \cdots, m)$ is some normal subgroup of G of order p.

Since $P \lhd G$ and $P \cap \Phi(G) = 1$, $P = R_1 \times R_2 \times \cdots \times R_m$, where $R_i(i = 1, 2, \cdots, m$ is an abelian minimal normal subgroup of G by Lemma 2.3. We now prove that $|R_i| = p$. Since $R_i \nsubseteq \Phi(G)$, there exists a maximal subgroup M of G such that $G = R_iM$ and $R_i \cap M = 1$. Let M_p be a Sylow p-subgroup of M and $G_p = M_pR_i$. Then G_p is a Sylow p-subgroup of G. Let G_1 be a maximal subgroup of G_p containing M_p and $P_1 = G_1 \cap P$. Then $|P : P_1| = |P : G_1 \cap P| = |PG_1 : G_1| = |G_p : G_1| = p$ and so P_1 is a maximal subgroup of P. We also have that $P_1M_p = (G_1 \cap P)M_p = G_1 \cap PM_p = G_1 \cap G_p = G_1$ and $P_1 \cap M_p = P \cap G_1 \cap M = P \cap M_p$. By hypothesis, P_1 is weakly s-semipermutable in G. Hence there exists a subgroup T of G such that $G = P_1T$ and $P_1 \cap T$ is s-semipermutable in G. By Lemma 2.1(c), $P_1 \cap T$ is s-permutable in G. Then, for an arbitrary Sylow q-subgroup G_q of G with $q \neq p$, $(P_1 \cap T)G_q = G_q(P_1 \cap T)$. Hence $P_1 \cap T = (P_1 \cap T)(P \cap G_q) = P \cap (P_1 \cap T)G_q \lhd (P_1 \cap T)G_q$. It follows that $G_q \leq N_G(P_1 \cap T)$. On the other hand, $P \cap T \lhd T$ and $P \cap T \lhd P$ since P is abelian. Hence $P \cap T \lhd PT = G$ and consequently $P_1 \cap T = G_1 \cap P \cap T \lhd G_1$. It follows that $P_1 \cap T \lhd G_1P = G_p$. This shows that both G_p and G_q are contained in $N_G(P_1 \cap T)$. The arbitrary choice of q implies that $P_1 \cap T \lhd G$ and so $P_1 \cap T \leq (P_1)_G$. Assume that $P_1 \cap T < (P_1)_G$ and let $N = (P_1)_GT$. Then $G = P_1T = P_1(P_1)_GT = P_1N$ and $P_1 \cap N = P_1 \cap (P_1)_GT = (P_1)_G(P_1 \cap T) = (P_1)_G$. This shows that there always exists a subgroup K of G such that $G = P_1K$ and $P_1 \cap K = (P_1)_G$.

Since P is abelian, $P_1(P \cap M) \lhd P$. Thus $P_1(P \cap M) = P$ or $P_1(P \cap M) = P_1$. If $P_1(P \cap M) = P$, then $G = PM = P_1(P \cap M)M = P_1M$ and so $P = P \cap P_1M = P_1(P \cap M) = P_1(P \cap G_1 \cap M) = P_1(P_1 \cap M) = P_1$, a contradiction. Hence $P_1(P \cap M) = P_1$ and so $P \cap M \leq P_1$. Since $P \cap M \lhd G$ by Lemma 2.4, $P \cap M \leq (P_1)_G = P_1 \cap K$.

Assume that $K < G$. Let K_1 be a maximal subgroup of G containing K. Then $P \cap K_1 \lhd G$ by Lemma 2.4. Hence $(P \cap K_1)M$ is a subgroup of G. Since $M < G$, $(P \cap K_1)M = G$ or $(P \cap K_1)M = M$. If $(P \cap K_1)M = G = PM$, then $P = P \cap (P \cap K_1)M = (P \cap K_1)(P \cap M) = P \cap K_1$ since $P \cap M \leq (P_1)_G = P_1 \cap K \leq P \cap K_1$. It follows that $P \leq K_1$ and hence $G = PK \leq PK_1 = K_1$, a contradiction. If $(P \cap K_1)M = M$, then $P \cap K_1 \leq M$ and so $P_1 \cap K \leq P \cap K \leq P \cap K_1 = P \cap K_1 \cap M \leq P \cap M \leq P_1 \cap K$. Hence $P_1 \cap K = P \cap K$. Since $G = PK = P_1K$, $|G : P| = |PK : P| = |K : (P \cap K)| = |K : (P_1 \cap K)| = |P_1K : P_1| = |G : P_1|$, which is impossible. Thus $K = G$. It follows that $P_1 \cap K = P_1 = (P_1)_G \lhd G$. Consequently, $P_1 \cap R_i \lhd G$. But since $G_p = R_iM_p = R_iG_1$ and G_1 is a maximal subgroup of G_p containing M_p, we have $R_i \nleq P_1 = G_1 \cap P$. The minimal normality of R_i implies that $P_1 \cap R_i = 1$. Hence $|R_i| = |R_i : (P_1 \cap R_i)| = |R_iP_1 : P_1| = |R_i(P \cap G_1) : P_1| = |(P \cap R_iG_1) : P_1| = |P \cap G_p : P_1| = |P : P_1| = p$. Therefore R_i is a cyclic group of order p.

(3) Final contradiction.

Let $R_i \subseteq H$ and $C_0 = C_H(R_i)$. We claim that the hypothesis holds for $(G/R_i, C_0/R_i)$. Indeed, since $G/C_G(R_i) \leq$ Aut(R_i) is abelian, $G/C_G(R_i) \in \mathcal{F}$. Consequently, $G/C_0 = G/(H \cap C_G(R_i)) \in \mathcal{F}$. Besides, since $R_i \leq Z(C_0)$ and $F(H) \leq C_0$, we have $F(H) = F(C_0)$. Thus $F(C_0/R_i) = F(H)/R_i$. Let P/R_i be a Sylow p-subgroup of $F(H)/R_i$, where P is a Sylow p-subgroup of $F(H)$ and G_1/R_i is a maximal subgroup of P/R_i. Then P_1 is a maximal subgroup of P. By hypothesis, P_1 is weakly s-semipermutable in G. Hence P_1/R_i is weakly s-semipermutable in G/R_i by Lemma 2.2. Now assume that QR_i/R_i is the Sylow q-subgroup of $F(H)/R_i$, where $q \neq p$ and Q is the Sylow q-subgroup of $F(H)$. Then every maximal subgroup of QR_i/R_i is of the form of Q_1R_i/R_i, where Q_1 is a maximal subgroup of Q. By hypothesis and Lemma 2.2, we see that Q_1R_i/R_i is weakly s-semipermutable in G/R_i. This shows that $(G/R_i, C_0/R_i)$ satisfies the condition of the theorem. The minimal choice of (G, H) implies that $G \in \mathcal{F}$ by Lemma 2.5. The final contradiction completes the proof.

Corollary 3.24 ([5], Theorem 2). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are s-semipermutable in G, then $G \in \mathcal{F}$.*

Corollary 3.25 ([4], Theorem 1.4). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a solvable group with a normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are s-permutable in G, then $G \in \mathcal{F}$.*

Corollary 3.26 ([23], Theorem 1). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are c-normal in G, then $G \in \mathcal{F}$.*

Corollary 3.27 ([7], Theorem 4.5). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are c-supplemented in G, then $G \in \mathcal{F}$.*

COROLLARY 3.28 ([25], Theorem 1.6). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are complemented in G, then $G \in \mathcal{F}$.*

COROLLARY 3.29 ([22], Theorem 2). *Let G be a group and E a solvable normal subgroup of G such that G/E is supersolvable. If all maximal subgroups of the Sylow subgroups of $F(E)$ are c-normal in G, then G is supersolvable.*

COROLLARY 3.30 ([28], Theorem 1.2). *Suppose that G is a solvable group with a normal subgroup H such that G/H is supersolvable. If all maximal subgroups of every Sylow subgroup of $F(H)$ are complement in G, then G is supersolvable.*

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